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2D TOPOLOGICAL SOLITONS IN THE GAUGED EASY-AXIS HEISENBERG ANTIFERROMAGNET MODEL

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Двумерные топологические солитоны в калибровочно-инвариантной модели легкоосного антиферромагнетика Гейзенберга

Представлены полевые уравнения 3-компонентной $U(1)$ калибровочно-инвариантной сигма-модели («А3М-модели») со спонтанно нарушенной $Z(2)$ симметрией для $(D+1)$-мерного пространства-времени. Численно найдены локализованные решения двумерной А3М-модели, имеющие единичный топологический заряд. Эти топологические солитоны описывают (по крайней мере, при малых значениях безразмерного параметра p, $p < p_0$) *связанные* состояния единичного трехкомпонентного «легкоосного» поля Гейзенберга и поля Максвелла. Обсуждается возможная физическая интерпретация 2-мерных и 3-мерных солитонов А3М-модели.

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2D Topological Solitons in the Gauged Easy-Axis Heisenberg Antiferromagnet Model

Field equations of the 3-component $U(1)$ gauged sigma model («the A3M model») with spontaneously broken $Z(2)$ symmetry are presented for $(D+1)$-dimensional spacetime. Localized solutions of the two-dimensional A3M model with the unit topological charge are found numerically; these topological solitons describe (at least for small values of a parameter p, $p < p_0$) *bound* states of the unit 3-component «easy-axis» Heisenberg field and the Maxwell field. Possible physical implications of the 2D and 3D solitons within the A3M model are discussed.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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1.Investigation of D-dimensional ($D = 1,2,3$) solitons, which are nonanalytic in coupling constants localized particle-like solutions to Lorentz-invariant field equations, is the fruitful approach to exploring nonperturbative effects within the nonlinear models of the field theory in quasiclassical approximation (see, e.g., [1-6]). Moreover, stable 3-dimensional particle-like solutions in Lorentz-invariant nonlinear field models can be regarded as classical images of extended quantum particles with nonzero masses, which can be presently considered to be "structureless" (e.g., electron, μ- and τ- leptons, W- and Z-bozons). It is important to note that in general case the existence of D-dimensional ($D \geq 2$) solitons within models comprising N interacting fields ($N \geq 2$) is not forbidden by the Derrick no-go theorem [7] and its generalizations [3], and one can succeed in finding stable D-dimensional solitons, if the fields considered have different transformation laws under the Lorentz transformations.

In the present paper we investigate localized solutions of the gauged 3-component sigma model ("the A3M model") proposed in [8]. This model is Lorentz- and gauge-invariant, it describes the minimal interaction of the 3-component unit isovector field $s_a(x)$, $s_a s_a = 1$, $a = 1,2,3$, characterized by the easy-axis anisotropy in internal space (named for brevity "the A3-field" [8,9]) with the Maxwell field $A_\mu(x)$ in $(D+1)$-dimensional space-time ($D = 1,2,...$). The Lagrangian of the A3M model possesses both $U(1)$ local symmetry and $Z(2)$ symmetry; the latter is spontaneously broken both on vacuum and soliton solutions. Below we present basic equations of the A3M model in D-dimensional space and investigate its soliton solutions for $D = 2$.

2. The gauge-invariant Lagrangian density of the A3M model reads:

$$\mathcal{L} = \eta^2 \left(\bar{\mathcal{D}}_\mu s_- \mathcal{D}^\mu s_+ + \partial_\mu s_3 \partial^\mu s_3\right) - V(s_a) - \frac{1}{4} F_{\mu\nu}^2, \tag{1}$$

$$\bar{\mathcal{D}}_\mu = \partial_\mu + igA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - igA_\mu,$$

$$s_+ = s_1 + is_2, \quad s_- = s_1 - is_2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(s_a) = \beta^2(1 - s_3^2),$$

where β^2, η^2, are constants, $[\eta^2] = L^{(1-D)}$, $[\beta^2] = L^{-(1+D)}$, g is a coupling constant, $[g^2] = L^{(D-3)}$, $\mu, \nu = 0,1,...,D$, and summation over repeated indices μ, ν is meant. It is straightforward to rewrite this Lagrangian density in the equivalent form

$$\mathcal{L} = \eta^2(\partial_\mu s_a)^2 - V(s_a) - \frac{1}{4}F_{\mu\nu}^2 + 2g\eta^2 A_\mu(s_2\partial^\mu s_1 - s_1 \partial^\mu s_2) + g^2\eta^2(s_1^2 + s_2^2)A_\mu A^\mu. \tag{2}$$

The A3M model can be regarded as a gauge-invariant analog of the classical Heisenberg antiferromagnet model with the easy-axis anisotropy; the Lagrangian density of the latter is

$$\mathcal{L} = \eta^2(\partial_\mu s_a)^2 - V(s_a), \quad s_a s_a = 1, \quad a = 1,2,3. \tag{3}$$

Making rescaling $x_\mu \to g^{-1}\eta^{-1}x_\mu$, $A_\mu \to \eta^{-1}A_\mu$, we obtain the Euler-Lagrange equations of the A3M model in dimensionless form. These equations, governing evolution of the fields $s_a(x)$, $A_\mu(x)$, in $(D+1)$-dimensional space-time, take the simplest form if the Lorentz gauge, $\partial_\mu A^\mu = 0$, is chosen:

$$\partial_\mu\partial^\mu s_i + [\partial_\mu s_a \partial^\mu s_a + 2A_\mu j^\mu + p(s_3^2 - \delta_{i3}) + A_\mu A^\mu (s_1^2 + s_2^2 - \delta_{1i} - \delta_{2i})] s_i$$

$$-2A_\mu(\delta_{2i}\partial^\mu s_1 - \delta_{1i}\partial^\mu s_2) = 0, \quad (4)$$

$$j_\mu = s_2 \partial_\mu s_1 - s_1 \partial_\mu s_2, \quad (5)$$

$$\partial_\mu \partial^\mu A_\nu + 2j_\nu + 2(s_1^2 + s_2^2) A_\nu = 0, \quad (6)$$

$$\mu, \nu = 0, 1, ..., D, \quad i = 1, 2, 3$$

(we denote $p = \beta^2 g^{-2} \eta^{-4}$). Equation (4) can be rewritten using variables $s_\pm = s_1 \pm i s_2$ and s_3:

$$\partial_\mu \partial^\mu s_\pm + [\partial_\mu s_a \partial^\mu s_a + 2A_\mu j^\mu + p s_3^2 - A_\mu A^\mu s_3^2] s_\pm - 2iA_\mu \partial^\mu s_\pm = 0, \quad (7)$$

$$\partial_\mu \partial^\mu s_3 + [\partial_\mu s_a \partial^\mu s_a + 2A_\mu j^\mu - p(1 - s_3^2) + A_\mu A^\mu (1 - s_3^2)] s_3 = 0. \quad (8)$$

It is instructive to present the equations of the A3M model in terms of angular variables θ, ϕ on the unit sphere S^2,

$$s_1 = \sin\theta\cos\phi, \quad s_2 = \sin\theta\sin\phi, \quad s_3 = \cos\theta. \quad (9)$$

Then the Lagrangian density (2) takes the form (in rescaled x_μ, A_μ):

$$g^{-2}\eta^{-4}\mathcal{L} = \partial_\mu\theta\partial^\mu\theta + \sin^2\theta\,[\partial_\mu\phi\partial^\mu\phi - 2A_\mu\partial^\mu\phi + A_\mu A^\mu - p] - \frac{1}{4}F_{\mu\nu}^2. \quad (10)$$

and the Euler-Lagrange equations become:

$$\partial_\mu\partial^\mu\theta + \frac{1}{2}\sin 2\theta\,[p - \partial_\mu\phi\partial^\mu\phi + 2A_\mu\partial^\mu\phi - A_\mu A^\mu] = 0, \quad (11)$$

$$\partial_\mu\left[\sin^2\theta(\partial^\mu\phi - A^\mu)\right] = 0, \quad (12)$$

$$\partial_\mu\partial^\mu A_\nu + 2j_\nu + 2A_\nu \sin^2\theta = 0, \quad j_\nu = -\sin^2\theta\partial_\nu\phi. \quad (13)$$

Note that Eqs. (11)-(13) can be satisfied if

$$\phi(x) = \phi(\mathbf{x}) - \omega t, \quad A_0 = \omega = const, \quad A_k(x) = A_k(\mathbf{x}), \quad \theta(x) = \theta(\mathbf{x}), \quad k = 1, ..., D, \quad (14)$$

where A_k, θ are subject to equations which do not contain ω ($k, m = 1, ..., D$, summation over repeated k):

$$\partial_k^2\theta - \frac{1}{2}\sin 2\theta\left[p + (\partial_k\phi - A_k)^2\right] = 0, \quad (15)$$

$$\partial_k\left[\sin^2\theta(\partial_k\phi - A_k)\right] = 0, \quad (16)$$

$$\partial_k^2 A_m + 2\sin^2\theta(\partial_m\phi - A_m) = 0. \quad (17)$$

3. Below we shall study localized solutions to Eqs. (15)-(17) for $D = 2$ using the "hedgehog" ansatz for the unit isovector field $s_i(\mathbf{x})$, $i = 1, 2, 3$,

$$s_1 = \cos m\chi \sin\theta(r), \quad s_2 = \sin m\chi\sin\theta(r), \quad s_3 = \cos\theta(r), \quad (18)$$

$$\sin\chi = \frac{y}{r}, \quad \cos\chi = \frac{x}{r}, \quad r^2 = x^2 + y^2,$$

where m is an integer number, and the "vortex" ansatz for the Maxwell field $A_\mu(\mathbf{x})$,

$$A_0 = 0, \quad A_1 = A_x = -m\alpha(r)\frac{y}{r^2}, \quad A_2 = A_y = m\alpha(r)\frac{x}{r^2} \quad (19)$$

As a result we obtain equations for $\theta(r)$ and $\alpha(r)$ [8],

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{m^2(\alpha-1)^2}{r^2} + p\right] = 0, \quad (20)$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} + 2(1-\alpha)\sin^2\theta = 0. \quad (21)$$

with boundary conditions

$$\theta(0) = \pi, \quad \theta(\infty) = 0 \quad (22)$$

$$\alpha(0) = 0, \quad \alpha(\infty) = 1. \quad (23)$$

Note that field configurations $s_i(\mathbf{x})$ given by Eqs.(18),(22) correspond to maps from R^2_{comp} to S^2 with integer homotopic indices ("winding numbers")[1,6,10], and we shall refer to solutions of the problem (20)-(23) as topological solitons with the "topological charges" $Q_t = m$.

Using series expansion of $\theta(r)$ and $\alpha(r)$ at $r \to 0$, we find from Eqs. (20) and (21) for $m = 1$

$$\theta(r) = \pi - C_1 r + o(r), \quad (24)$$

$$\alpha(r) = r^2(E_1^2 - \frac{1}{4}C_1^2 r^2) + o(r^4). \quad (25)$$

and for $m = 2$

$$\theta(r) = \pi - C_2 r^2 + o(r^2), \quad (26)$$

$$\alpha(r) = r^2(E_2^2 - \frac{1}{12}C_2^2 r^4) + o(r^6). \quad (27)$$

Equations (24),(26) and (26),(27) are useful when searching for solutions of the problem (20)-(23) with $m = 1$ and $m = 2$, respectively, by the shooting method.

We have found numerically the profile functions $\theta(r)$ and $\alpha(r)$ of solitons with $m = 1$ for p =0.01, 0.03, 0.05 (Fig.1), p =0.1 and 0.15 (Fig.2). Distributions of the energy density

$$\mathcal{H}(r) = \left(\frac{d\theta}{dr}\right)^2 + \sin^2\theta\left[p + \frac{m^2(\alpha-1)^2}{r^2}\right] + \frac{m^2}{2}\left(\frac{1}{r}\frac{d\alpha}{dr}\right)^2. \quad (28)$$

and the magnetic field, $-B(r) = (d\alpha/dr)/r$, are plotted in Figs. 3 and 4, respectively. Note that for small p, $p \leq 0.05$ the asymptotic value of $\alpha(r)$, $\alpha(\infty) = 1$, is reached at smaller values of r, than the asymptotic value of $\theta(r)$, $\theta(\infty)$ is (see Fig.1); the reverse situation is observed for $p \geq 0.1$ (Fig.2). For all p considered the characteristic width of the localized solutions $\alpha(r), \theta(r)$ is rather large. It is important however that distributions of physical quantities (energy density, magnetic field) of the solitons are

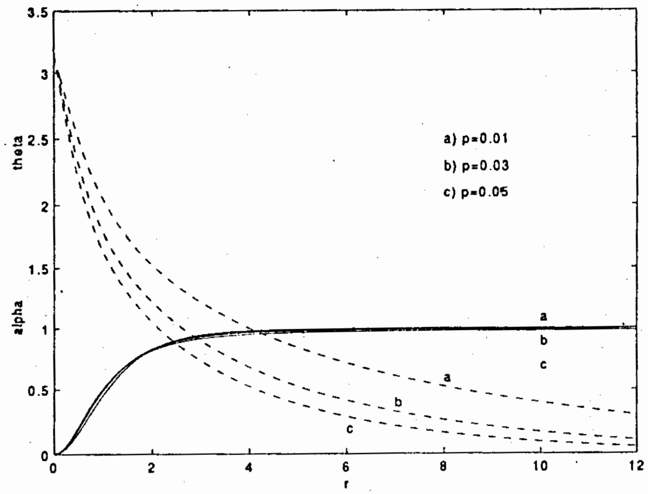

Fig.1 Soliton profile functions $\alpha(r)$ and $\theta(r)$ for $p = 0.01, 0.03$ and 0.05.

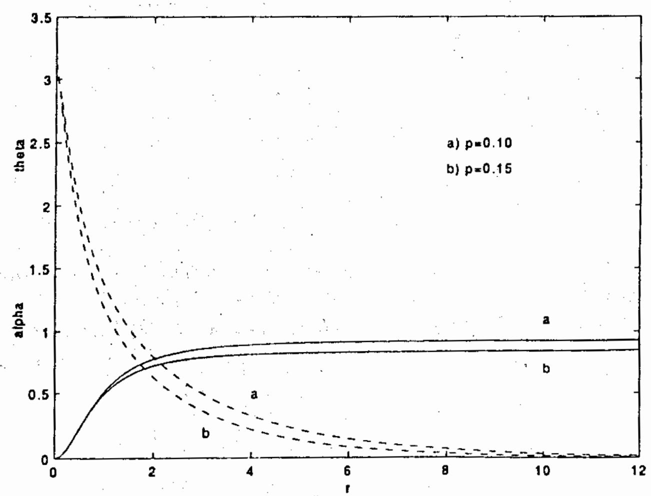

Fig.2 Soliton profile functions $\alpha(r)$ and $\theta(r)$ for $p = 0.10$ and 0.15.

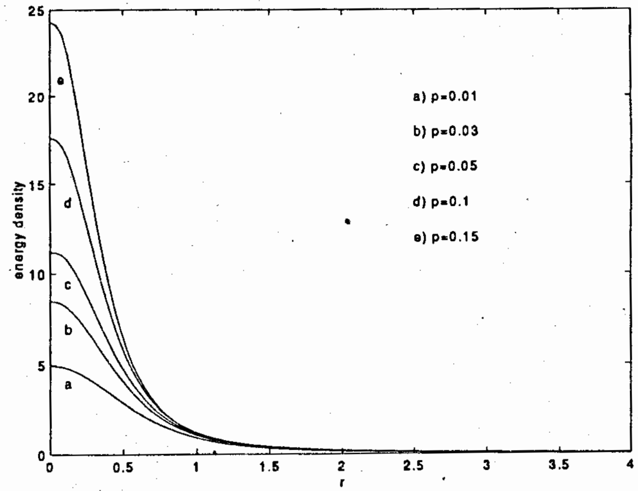

Fig.3 Energy density of the 2D A3M solitons vs radius for various p.

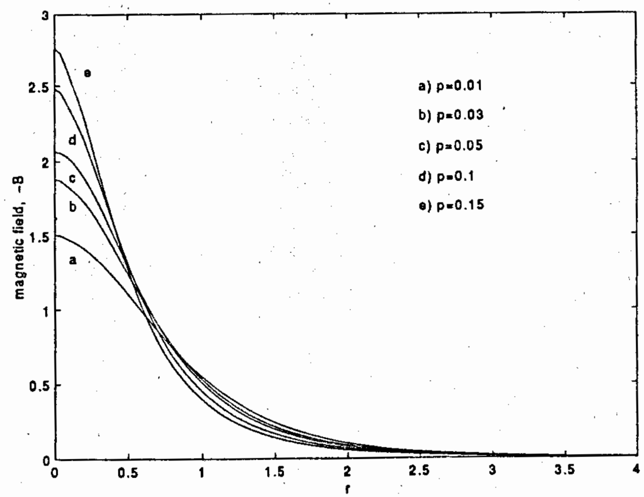

Fig.4 Magnetic field of the 2D A3M solitons vs radius for various p.

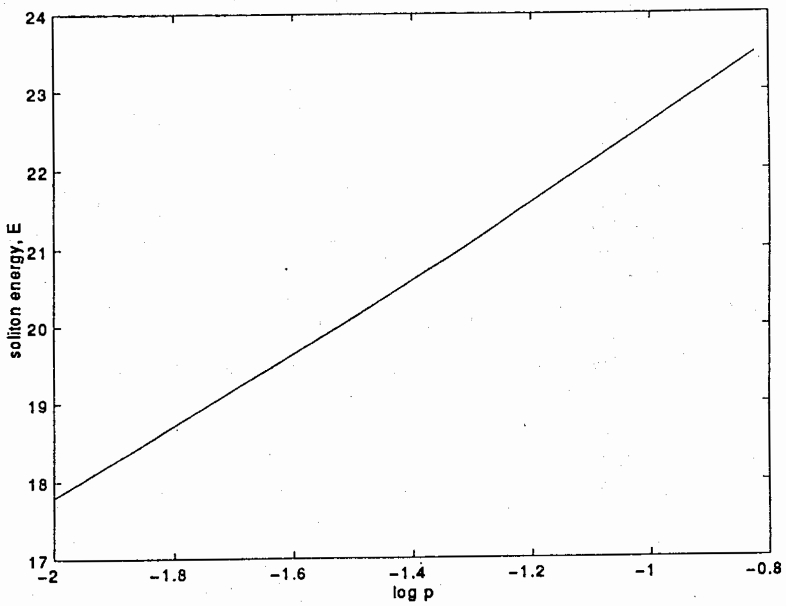

Fig.5 The dependence of the 2D A3M soliton energy E on $log(p)$.

localized within much smaller regions (see Figs. 3,4). One can easily find that the magnetic flux of the 2D topological solitons in the A3M model is quantized,

$$\Phi = \int \mathbf{B} d\mathbf{S} = \int A_k dx_k = -\int_0^{2\pi} \alpha(\infty) d\varphi = -2\pi m = -2\pi Q_t \qquad (29)$$

The dependence of the soliton energy $E = 2\pi \int \mathcal{H}(r) r dr$ on the p value is depicted in Fig.5; it is natural to conjecture that the dependence $E(\log p)$ is exactly linear. It is important to note that $E(p) < 8\pi$ for $p < p_0 \approx 0.3$ (recall that 8π is the energy value of the Belavin-Polyakov localized solutions in $D = 2$ isotropic Heisenberg ferromagnet [11]). It means that at least for $p < p_0$ the particle-like solutions (17)-(21) of the A3M model describe spatially localized *bound* states of the A3- and the Maxwell fields, and hence it is natural to conjecture these 2D solitons to be stable for $p < p_0$ (indeed, to destroy them it is necessary to supply an amount of energy exceeding the appropriate "mass defect").

4. One can consider the 2D A3M solitons as strings with finite radia embedded into 3D space. Such strings-solitons can be used when elaborating cosmological models of evolution of the early Universe and its present large-scale structure (for review see, e.g., [12]). From mathematical point of view these solitons are well suited for computer simulation of interaction of extended cosmic strings, indeed, to investigate the interaction of M strings within the A3M model initial data for the evolutionary PDE problem could be set as a *sum* of M solitons well separated from each other. Recall that such the physically natural prescription cannot be used for computer simulation of interactions of the Abrikosov-Nielsen-Olesen strings-vortices[13] within the abelian Higgs model[14], which are widely discussed (along with their non-abelian generalizations) in cosmological models[12].

Further computer studies of the 2D solitons within the A3M model are in progress, in particular, investigation of the boundary value problem (19)-(22) for various m and p. Next, we are starting computer studies of dynamical processes in which the 2D A3M topological solitons take part, using evolutionary PDEs of the model presented above.

From the viewpoint of particle physics the most important line of soliton studies within the A3M model is certainly the search for stable 3D solitons. Our preliminary analysis shows that existence of stationary 3D A3M solitons with nonzero integer values of the topological charge of the $A3$-field (the Hopf index, see, e.g., [1a,15]) is not forbidden and, moreover, it is quite likely; however, serious mathematical difficulties should be overcome first in order to make appropriate computer investigations reliable and successful.

References

[1] a) L.D.Faddeev, in: Proceedings of the IV International Conference on Nonlocal and Nonlinear field theories, Alushta, 1976 (Joint Institute for Nuclear Research, Dubna, D-29788, 1976) p.207; b) L.D.Faddeev, Pis'ma v Zh.E.T.F., 21 (1975) 141; c) L.D.Faddeev, Lett. Math. Phys., 1 (1976) 289.

[2] Proceedings of the Conference on Extended systems in field theory, Phys. Rep., 23 (1976) 237.

[3] S.Coleman. Classical lumps and their quantum descendants, in: New Phenomena in Subnuclear Physics, Ed. A.Zichichi (Plenum Press, New York, 1977).

[4] R.Friedberg, T.D.Lee and A.Sirlin, Phys.Rev., D13 (1976) 2739.

[5] R.Jackiw, Rev. Mod. Phys., 49 (1977) 681.

[6] R.Rajaraman, Solitons and instantons (North-Holland, Amsterdam, 1982).

[7] G.H.Derrick, J. Math. Phys., 5 (1964) 1252.

[8] I.L.Bogolubsky and A.A.Bogolubskaya, JINR, E5-96-73, Dubna, 1996.

[9] I.L.Bogolubsky and A.A.Bogolubskaya, Soliton analogs of Abrikosov-Nielsen-Olesen vortices, in: Proc. of the 1st Workshop "Nonlinear Physics. Theory and Applications", Gallipoli (Lecce), 1995, Eds. E.Alfinito, M.Boiti, L.Martina and F.Pempinelli (World Scientific, Singapore, 1996).

[10] A.S.Schwartz, Quantum field theory and topology (Springer-Verlag, 1993).

[11] A.A.Belavin and A.M.Polyakov, Pis'ma v Zh.E.T.F., 22 (1975) 503 [JETP Lett., 22 (1975) 245].

[12] A.Vilenkin and E.P.S.Shellard, Cosmic strings and other topological defects (Cambridge University Press, Cambridge, 1994).

[13] A.A.Abrikosov, Zh.E.T.F., 32 (1957) 1442; H.B.Nielsen and P.Olesen. Nucl. Phys., B61 (1973) 45 .

[14] P.Higgs, Phys. Lett., 12 (1964) 132.

[15] D.A.Nicole, J.Phys., G4 (1978) 1363; H.J.de Vega, Phys.Rev., D18 (1978) 2945; I.L.Bogolubsky, Phys. Lett., A126 (1988) 511.

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