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GROUP ANALYSIS AND RENORMGROUP SYMMETRIES

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1 Introduction

The paper is devoted to the problem of constructing a special class of symmetries for boundary value problems (BVP) in mathematical physics, namely **renormalization group symmetries** (hereinafter referred to as RG-symmetries).

> By RG-symmetry we mean a symmetry that characterizes a solution of a BVP and corresponds to *transformations* involving both "dynamical" (i.e., equation) *variables* and *parameters* entering solution via equations and boundary conditions.

Symmetries of this type appeared about forty years ago in the context of the renormalization group (RG) concept. This concept originally arose [1, 2, 3, 4] in the "depth" of quantum field theory (QFT) and was connected with a complicated procedure of renormalization, that is 'removing of ultraviolet infinities'. In QFT the RG was successively used for improving approximate perturbation solution to restore a correct structure of solution singularity.

Later on, in the seventies, it was found that the RG concept is fruitful in some other fields of microscopic physics: phase transitions in large statistical systems, polymers, turbulence, and so on. However, in some cases, following Wilson approach [5] to spin lattice, the original *exact symmetry* underlying the renormalization group notion in QFT was changed to an approximate one with the corresponding transformations forming a semigroup (not a group as in the QFT case). Here, in this paper, by RG-symmetry we mean the original exact property of solution – as it was formulated in Refs.[3, 4] (see also [6]) by N. Bogoliubov and one of the present authors.

Later on symmetry underlying the renormgroup invariance was also found in a number of problems of macroscopic physics like, e.g., mechanics, transfer theory, hydrodynamics and a close relation of renormgroup symmetry to the notion of self-similarity has been established [7, 8].

The simplest variant of RG transformation is given by a simultaneous one-parameter point transformation

$$T_a : \{x' = x/a \ , \ g' = \bar{g}(a,g) \} \ , \ \bar{g}(1,g) = g \tag{1.1}$$

of a dimensionless "coordinate" x and one solution parameter (e.g., initial value) g, which characterise some partial solution of a problem. The transformation function $\bar{g}(x,g)$, that depends upon two arguments[2], should satisfy the functional equation

$$\bar{g}(x,g) = \bar{g}(x/a, \bar{g}(a,g)), \tag{1.2}$$

that guarantees the group property $T_a \cdot T_b = T_{ab}$ fulfillment.

The functional equation (1.2) and transformation (1.1) arise, for example, in the massless QFT with one coupling. In that case $x = Q^2/\mu^2$ is the ratio of a 4-momentum Q squared to a "reference momentum" μ squared, g is the coupling constant and $\bar{g}$ is the so-called *effective coupling.*

The infinitesimal form of transformation (1.1) can be be written down in a differential form

$$R\,\bar{g} = 0\,, \tag{1.3}$$

where R is the *infinitesimal operator* of RG-symmetry (or simply RG-operator) with the coordinate β defined by the first derivative of the function $\bar{g}$ at the point $a = 1$

$$R = x\partial_x - \beta(g)\partial_g\,, \quad \beta(g) = \left.\frac{\partial\bar{g}(a;g)}{\partial a}\right|_{a=1}. \tag{1.4}$$

Therefore, instead of equations (1.2) and , RG transformation can be introduced by means of an RG-operator, as well. And vice versa, being given an RG-operator, one can easily reconstruct the functional equation for tranformation function $\bar{g}$ with the help of the corresponding Lie equations

$$\begin{gathered} \frac{dx'}{x'} = -\frac{da}{a}, \quad \frac{dg'}{\beta(g')} = \frac{da}{a}, \quad d\bar{g}' = 0; \\ x'|_{a=1} = x, \; g'|_{a=1} = g, \; \bar{g}'|_{a=1} = \bar{g}. \end{gathered} \tag{1.5}$$

Thus, for given form of a β-function defined by the behavior of $\bar{g}$ in the vicinity of $a = 1$ or, in other words, for given RG-symmetry, one can get the explicit solution for $\bar{g}(x,g)$. However, the procedure of finding RG-symmetry in any particular case usually is based upon atypical manipulations [10, 11] and the constucting of a regular approach to revealing RG-symmetries is of principal interest.

In a particular case, linear in the last argument, $\beta = kg$, equation (1.3) yields a solution that has a power x dependence, i.e., $\bar{g}(x,g) = gx^k$ with k

[2]The generalization to the case of several parameters is straightforward – see, e.g.,[9].

being an arbitrary number. Then, Eq.(1.1), takes a form of a power scaling (or *power self-similarity*) transformation

$$x' = x/a\,, \quad g' = ga^k\,,$$

that is well-known in mathematical physics and widely used in the problems of hydrodynamics of liquids and gases.

Therefore, transformation (1.1) can be considered as a functional generalization $gx^k \to \bar{g}(x,g)$ of a usual (i.e., power) self-similarity transformation. One can refer to it as to *functional self-similarity transformation.*

Relationship (1.3), that reflects the invariance of $\bar{g}$ under the RG transformation (i.e. functional self-similarity property) in a general case appears as an equation for finding all invariants of the group with the RG-operator R. In this sense Eq.(1.3) should be written for the function G of three arguments[3] x, g and $\bar{g}$

$$xG_x - \beta(g)G_g = 0\,. \tag{1.6}$$

General solutions of RG equations of (1.6)-type (both in massive and in massless cases, with one and two couplings) were presented in [12]; one particular solution was also found in [2]. These solutions depend upon arbitrary functions. Generally, after substituting G by $\bar{g}$, Eq.(1.6) can be treated as the vanishing condition for the coordinate æ of the RG-operator (1.4) in the canonical form [13]

$$R = \text{æ}\,\partial_{\bar{g}}\,, \quad \text{æ} \equiv x\bar{g}_x - \beta(g)\bar{g}_g = 0\,, \tag{1.7}$$

identically valid on a particular BVP solution $\bar{g} = \bar{g}(x,g)$; the latter property can be treated as a formulation of the functional self-similarity.

This rises the question:

– does there exists any RG-symmetries defined by RG-operators of the form that differs from (1.4) but with canonical coordinates that identically turn to zero on the solution of the BVP and (if "yes")

– does there exist a regular way of finding these symmetries?

2 Approach to constructing RG-symmetries

First of all, the desired regular approach to constructing RG-symmetries turns out to be possible for those mathematical models of physical systems

[3]The number of arguments depends upon a particular physical system under consideration: in the inhomogeneous (massive – for QFT) case this equation contains one additional term yG_y .

that are based on differential equations. The key idea uses[14] the fact that such models can be analyzed by algorithms of modern group analysis.

The proposed scheme comprises a sequence of the four steps[4].

I. A specific manifold (differential, integro-differential, etc.) should be primarily constructed which is then used to calculate RG-symmetries. This manifold that will be referred to as *renormgroup manifold* (RG-manifold) generally differs (see below) from the manifold given by the original system of differential equations (DEs).

II. The second step deals with calculating the most general group $\mathcal{G}$ admitted by RG-manifold.

III. The restriction of the group $\mathcal{G}$ on the BVP solution (exact or approximate) constitutes the next step. The group of transformations thus obtained (renormgroup) is characterized by a set of infinitesimal operators (RG-operators), each containing the solution of a BVP in its invariant manifold.

IV. The last, fourth step implies utilization of RG-operators to find analytical expressions for solutions of the BVP.

Comment to I. In the scheme described above the first step, namely constructing the RG-manifold, is of fundamental importance. The form of its realization depends both on a mathematical model and on a form of a boundary condition.

Different approaches to RG-manifold constructing lead to the various methods of finding RG-symmetries:

Ia). In the first, more simple case the RG-manifold, as usual in classical group analysis, is presented by a system of basic DEs with the only substantial difference: parameters, entering into a solution via the equation and boundary data, are included in the list of independent variables.

Ib). The second approach to constructing the RG-manifold implies an extension of a space of variables involved in group transformations, for example, by taking into account Lie-Bäcklund transformation groups and nonlocal transformation groups.

Ic). The third approach is based on a combining the group analysis technique with the invariant embedding method [15]. Here, the group analysis is performed for the system of equations that consists of original DE and/or embedding equations which correspond to the BVP under

[4]It should noticed that the formulated approach is original and differs from the traditionally used in theoretical physics. As an illustration of this difference we point on an extended interpretation of the notion of the renormalization group.

consideration.

Id). The fourth approach to some extent is similar to the previous one. In this event boundary conditions are reformulated in terms of a differential constraint combined with original equations.

Ie). The last, fifth approach utilizes approximate transformation groups. Here, the RG-manifold is given by a system of DEs with small parameters and can be analyzed by perturbation methods [16].

Comment to III. Whereas the second step is a standard procedure in the group analysis, one further comment should be made concerning the step III, namely the procedure of the group restriction. The goal of such a restriction is the construction of a transformation group with a tangent vector field (point, Lie-Bäcklund, etc.) infinitesimal operators of which contain the desired BVP solution in an invariant manifold. This means, that the coordinate of the canonical operator of RG-symmetry becomes equal to zero on the BVP solution and on its differential consequences.

Mathematically, the procedure of a group restriction appears as a "combining" different coordinates of group generators admitted by the RG-manifold. The vanishing condition for the combination of these coordinates on a solution of the BVP leads to algebraic equalities that couple different coordinates and give rise to desired RG-symmetries. In a particular case, when RG is constructed from a Lie group admitted by the original system of DEs, it turns out to be a subgroup of this group and a solution of the BVP appears as an invariant solution with respect to the point RG obtained (compare with [17]). In the general case, not only Lie point group, but Lie-Bäcklund groups, approximate groups, nonlocal transformation groups, etc., are also employed as basic groups which are then to be restricted on the solution of a BVP.

The merits of the described approach were first realized while solving the problem of nonlinear interaction of a powerful laser radiation with inhomogeneous plasma [18, 19]. The mathematical model is given by a system of nonlinear DEs for components of electron velocity, electron density and the electric and magnetic fields. The presence of small parameters in the original system of equations (such as weak inhomogeneity of the ion density, low electron thermal pressure and small angles of incidence of a laser beam on plasma surface) enables us to construct RG-symmetries using approximate group methods. In a particular case that is of interest from the physical standpoint the desired RG-symmetry appears as a Lie point symmetry that takes account of transformations of the nonlinearity parameter; this parameter is proportional to the amplitude of the magnetic field at the laser frequency at a critical density point and is assumed to be small in the perturbation theory approximation. RG-symmetries obtained enable us to reconstruct the exact solution of original equations from the perturbation theory results. The solutions found were used to calculate the nonlinear structure of the electromagnetic field in plasma and to evaluate the efficiency of harmonics generation.

Further development of the approach was concerned with the initial value problem for the modified Burgers equation with parameters of nonlinearity and dissipation included explicitly. This example [20, 21, 22] yielded a detailed illustration of the method of constructing RG-symmetries when a basic RG-manifold is given by an original differential equation with parameters included in the list of independent variables.

To demonstrate the method of constructing Lie-Bäcklund RG-symmetries, the initial value problem for a linear parabolic equation was considered in [14]. The same mathematical model was also employed to find RG-symmetries when the boundary condition is described by a differential constraint [23].

An idea of a specific method of constructing RG-symmetries based on embedding equations was first realized in [24] for ordinary differential equations. Some possible applications of this method were discussed in [25].

Worthy of mention is an example that demonstrates the utilization of RG-symmetries to constructing solutions of the BVP for a system of two first-order partial differential equations that describes the propagation of a laser beam in a nonlinear focusing medium [26, 27, 35, 14, 29]. Both Lie point and Lie-Bäcklund symmetries (exact and approximate) were used to find RG-symmetries that were then used to find an analytical solution of the problem.

To clarify the idea of constructing RG-symmetries several examples are given below which demonstrate different approaches to the problem. To gain better understanding of these approaches, a single simple mathematical model is used[5] that corresponds to BVP for a system of two first-order partial differential equations that were studied by Chaplygin [30] in gas

[5] A disadvantage of this is that some examples have a methodological, rather than a physical significance.

dynamics

$$v_t + vv_x - a\varphi(n)n_x = 0\,, \quad n_t + vn_x + nv_x = 0\,; \qquad v(0,x) = V(x)\,, \quad n(0,x) = N(x)\,, \tag{2.1}$$

where $\varphi(n)$ is an arbitrary function of n and a is a nonlinearity parameter. Despite its simplicity, this mathematical model has a wide field of application and was used to describe various physical phenomena[6]. In such a case the physical meaning of variables t, x, v and n may differ from that in gas dynamics. For example, in nonlinear geometrical optics, t and x are coordinates, respectively, along and transverse to the direction of laser beam propagation, v is the derivative of eikonal with respect to x, and n is a laser beam intensity. In this case, functions V and N characterize the curvature of the wave front and the beam intensity distribution upon the coordinate x and the entrance of a medium $t = 0$.

3 RG as Lie point subgroup

This section presents an illustration of the method of constructing RG-symmetries when a basic RG-manifold is given by the original DEs with parameters included in the list of independent variables. Boundary conditions are taken into account while restricting the group admitted by RG-manifolds up to the desired RG on the exact or approximate solution of a BVP which thus appears as an invariant solution with respect to any of RG operators obtained.

3.1. First we shall consider a trivial particular case of equations (2.1) when the nonlinearity parameter a is equal to zero: in application to optical equations discussed above this means that nonlinear effects are neglected. Then by introducing a normalized variable $v = \varepsilon u$, the system of equations (2.1) is rewritten in the following form

$$u_t + \varepsilon uu_x = 0\,, \quad n_t + \varepsilon un_x + \varepsilon nu_x = 0\,; \tag{3.1}$$

$$u(0,x) = U(x)\,, \quad n(0,x) = N(x)\,. \tag{3.2}$$

The continuous point Lie group admitted by the differential manifold (3.1) (RG-manifold) is given by the infinitesimal operator (a general element of

[6] In the so-called quasi-gaseous media [31].

Lie algebra) with six independent terms

$$X = \xi^1\partial_t + \xi^2\partial_x + \xi^3\partial_\varepsilon + \eta^1\partial_u + \eta^2\partial_n \equiv \sum_{i=1}^{6} X_i\,, \qquad (3.3)$$

$$X_1 = (1/\varepsilon)\left(\varepsilon t J^1_\chi - J^1_u\right)\partial_t + \left(J^1 + u\left(\varepsilon t J^1_\chi - J^1_u\right)\right)\partial_x - nJ^1_\chi\partial_n\,,$$

$$X_2 = \frac{1}{n}J^2\left(\partial_t + \varepsilon u\partial_x\right),\ X_3 = nJ^3\partial_n\,,\ X_4 = \frac{1}{n}J^4\left(-t\partial_t - \varepsilon ut\partial_x + n\partial_n\right),$$

$$X_5 = t\left(\varepsilon t J^5_\chi - J^5_u\right)\partial_t + \varepsilon t\left(J^5 + u\left(\varepsilon t J^5_\chi - J^5_u\right)\right)\partial_x$$

$$+ J^5\partial_u - n\left(\varepsilon t J^5_\chi - J^5_u\right)\partial_n\,,$$

$$X_6 = \left(J^6/\varepsilon\right)\left(-t\partial_t + n\partial_n + \varepsilon\partial_\varepsilon\right).$$

This infinite-dimensional group depends upon five functions $J^i(\chi, u, \varepsilon)$, $i = 1, \ldots, 5$ that appear as arbitrary functions of their arguments $\chi = x - vt$, u and ε. The last function J^6 entering into the operator that describes the group transformation of parameter ε is an arbitrary function only of this parameter. The restriction of the group admitted by the RG-manifold (3.1) on the BVP solution $u = \bar{u}(t, x, \varepsilon)$, $n = \bar{n}(t, x, \varepsilon)$ leads to zero equalities for two coordinates of the operator (3.3) in the canonical form, the conditions of functional self-similarity:

$$\eta^1 + \xi^1\varepsilon\bar{u}\bar{u}_x - \xi^2\bar{u}_x - \xi^3\bar{u}_\varepsilon = 0\,, \quad \eta^2 + \xi^1\varepsilon(\bar{n}\bar{u})_x - \xi^2\bar{n}_x - \xi^3\bar{n}_\varepsilon = 0\,. \quad (3.4)$$

Equalities (3.4) should be valid for any values of t, and certainly for $t = 0$, when dependences $\bar{u}$ and $\bar{n}$ upon x are given by boundary conditions (3.2). This yields two relations between J^i:

$$J^4 = N_xJ^1 + NJ^1_\chi - N(U_x)_uJ^1 - \varepsilon U_xJ^2 - NJ^3 - NJ^6/\varepsilon\,, \quad J^5 = U_xJ^1\,. \quad (3.5)$$

In formulas (3.5) and in what follows, the functions U and N and their derivatives with respect to x should be expressed either in terms of u or in terms of χ. Substituting (3.5) into (3.3) gives the desired RG-symmetries described by the following RG-operator

$$R = \sum_{i=1}^{4} R_i\,, \qquad (3.6)$$

$$R_1 = X_1 + U_xJ^1\left(\varepsilon t\partial_x + \partial_u\right) + \left[\left(\varepsilon t(U_x)_\chi - \left(1 - \frac{N}{n}\right)(U_x)_u - \frac{N_x}{n}\right)J^1\right.$$

$$\left. + \left(\varepsilon tU_x - \frac{N}{n}\right)J^1_\chi - U_xJ^1_u\right]\left(t\partial_t + \varepsilon ut\partial_x - n\partial_n\right),$$

$$R_2 = X_2 + \frac{\varepsilon U_x}{n}J^2\left(t\partial_t + \varepsilon ut\partial_x - n\partial_n\right),$$

$$R_3 = X_3 + \frac{N}{n}J^3\left(t\partial_t + \varepsilon ut\partial_x - n\partial_n\right),$$

$$R_4 = X_6 + \frac{N}{\varepsilon n}J^6\left(t\partial_t + \varepsilon ut\partial_x - n\partial_n\right).$$

It follows from these formulas that RG-symmetries for the BVP (3.1)-(3.2) are obtained as a combination of symmetries of infinite-dimensional algebra with the infinitesimal operator (3.3). Any of the four RG operators obtained (and their linear combinations with coefficients that are arbitrary functions of ε) contains the solution of the BVP $u = \bar{u}(t, x, \varepsilon)$ and $n = \bar{n}(t, x, \varepsilon)$ in the invariant manifold and enables us to obtain the group transformation of both group variables and different functionals of the solution (a method of calculating the transformation of functionals was described in [35]).

As is well known the renormalization group should be capable of improving the perturbation theory solution. As an example, we consider the perturbative solution of the BVP (3.1)-(3.2) for small $\varepsilon t \ll 1$

$$u = U(x) - UU_x(\varepsilon t) + O\left((\varepsilon t)^2\right),\ n = N(x) - \left(UN_x + NU_x\right)(\varepsilon t) + O\left((\varepsilon t)^2\right). \quad (3.7)$$

The approximate solution (3.7) in the limit $(\varepsilon t) \to 0$ is invariant under the RG transformation defined by the operator R_2 with arbitrary $\varepsilon J^2 \neq 0$. Assuming $J^2 = 1/\varepsilon$, we obtain the explicit expression for the RG-operator

$$R = \frac{1}{n}\left[\left(1 + \varepsilon tU_x\right)\left(\partial_t + \varepsilon u\partial_x\right) - \varepsilon U_xn\partial_n\right], \qquad (3.8)$$

and the following invariance conditions written in the form of two first-order differential equations:

$$u_t + \varepsilon uu_x = 0\,, \qquad (1 + \varepsilon tU_x)(n_t + \varepsilon un_x) + \varepsilon nU_x = 0\,. \qquad (3.9)$$

Solving Lie equations that correspond to RG-operators (3.8) (and coincide with characteristics equations for (3.9)) allows us to reconstruct the desired exact solution of the BVP (3.1)-(3.2) from the perturbation theory solution (3.7)

$$u = U(x - \varepsilon ut), \quad n = \frac{1}{1 + \varepsilon t U_x} N(x - \varepsilon tu), \tag{3.10}$$

where the derivative U_x should be expressed in terms of u. For example, in particular case of $N(x) = N_0 \exp(-x^2)$, $U(x) = -x$ and $\varepsilon = 1/T$, the latter formulas describe the focusing of the gaussian laser beam in geometrical optics

$$n = \frac{T}{T-t} N_0 \exp\left(-x^2 \left(\frac{T}{t-T}\right)^2\right), \quad u = x\frac{T}{t-T}, \quad t \leq T.$$

3.2. Now let us turn to a more general case of $a \neq 0$. By means of hodograph transformations the basic equations (2.1) are transformed to a system of linear partial differential equations for functions $\tau = nt$ and $\chi = x - vt$, which is more convenient for a group analysis

$$\tau_v - \psi(n)\chi_n = 0, \quad \chi_v + \tau_n = 0, \quad a\varphi\psi = n. \tag{3.11}$$

The Lie point symmetry group admitted by the RG-manifold (3.11) is characterized by a canonical infinitesimal operator [13] with six independent terms X_i, $i = 1, \ldots, 5$ and X_∞

$$X = X_\infty + \sum_{i=1}^{5} c_i X_i \equiv \left(\bar{f} + \sum_{i=1}^{5} c_i f_i\right)\partial_\tau + \left(\bar{g} + \sum_{i=1}^{5} c_i g_i\right)\partial_\chi, \tag{3.12}$$

where coordinates f_i and g_i are linear combinations of τ and χ and their first derivatives $\tau_1 = (\partial\tau/\partial n)$ and $\chi_1 = (\partial\chi/\partial n)$ with coefficients depending only on v and n [35, 29]. For a particular case $\varphi = 1$, they are presented as follows

$$f_1 = \tau; \quad g_1 = \chi; \quad f_4 = -(1/a)n\chi_1, \quad g_4 = \tau_1;$$

$$f_2 = -\tau/2 + n\tau_1 + (1/2a)nv\chi_1, \quad g_2 = -(v/2)\tau_1 + n\chi_1;$$

$$f_3 = -(1/2)n\chi + vn\tau_1 + \left[(1/4a)v^2 - n\right] n\chi_1, \tag{3.13}$$

$$g_3 = (a/2)\tau + (1/2)v\chi + vn\chi_1 + \left[an - (1/4)v^2\right]\tau_1;$$

$$f_5 = (n\tau_1 - \tau) - a\tau_a, \quad g_5 = n\chi_1 - a\chi_a.$$

"Evident" symmetries f_1, g_1 and f_4, g_4 describe dilations of τ and χ and translations along the v-axis, respectively, for arbitrary nonlinearities $\varphi(n)$. Two more symmetries f_2, g_2 and f_3, g_3 appear on account of a special form of the function $\varphi = 1$ under consideration. The symmetry f_5, g_5 takes account of the transformation of a parameter a along with the transformations of dynamic variables.

The operator X_∞ with coordinates $\bar{f} = \xi^1(v,n)$ and $\bar{g} = \xi^2(v,n)$ that are arbitrary solutions of partial differential equations

$$\xi^1_v - (n/a)\xi^2_n = 0, \quad \xi^2_v + \xi^1_n = 0, \tag{3.14}$$

results from the linearity of basic equations (3.11); it is an ideal of an infinite-dimensional Lie algebra L_∞ formed by operators $X_1, \ldots, X_5$ and X_∞.

The restriction of the group obtained on a solution of the BVP means that coordinates f and g of the canonical operator (3.12) turn to zero on this solution

$$\bar{f} = -\sum_{i=1}^{5} c_i f_i, \quad \bar{g} = -\sum_{i=1}^{5} c_i g_i. \tag{3.15}$$

Formulas (3.15) should be treated as relationships that yield expressions for functions $\bar{f}$ and $\bar{g}$ in terms of f_i, g_i, $i = 1, \ldots, 5$ taken on a solution $\tau = \bar{\tau}(v,n)$, $\chi = \bar{\chi}(v,n)$ of a BVP (exact or approximate). Substitution of (3.15) into (3.12) gives five RG-operators

$$R = \sum_{i=1}^{5} c_i(a) R_i, \tag{3.16}$$

each being determined by the corresponding coordinates f_i, g_i and by a pair of functions A^i, B^i explicitly defined by the solution $\bar{\tau}(v,n)$, $\bar{\chi}(v,n)$

$$R_1 = (\tau - A^1)\partial_\tau + (\chi - B^1)\partial_\chi,$$

$$R_2 = (-(\tau/2) - A^2)\partial_\tau - B^2\partial_\chi - (v/2)\partial_v - n\partial_n,$$

$$\begin{aligned} R_3 = {} & \left(-(n/2)\chi - A^3\right)\partial_\tau + \left((a/2)\tau + (v/2)\chi - B^3\right)\partial_\chi \\ & + \left(-(1/4)v^2 + an\right)\partial_v + vn\partial_n, \end{aligned} \tag{3.17}$$

$$R_4 = -A^4\partial_\tau - B^4\partial_\chi + \partial_v,$$

$$R_5 = \left(-\tau - A^5\right)\partial_\tau - B^5\partial_\chi - n\partial_n + a\partial_a.$$

Here functions A^i and B^i are defined by the corresponding formulas (3.13) for f^i and g^i where one should replace τ, χ by $\bar{\tau}(n,v)$, $\bar{\chi}(n,v)$. Explicit formulas for RG-operators depend upon the specific solution of the BVP. For example, for the particular solution of the BVP (2.1) with $V=0$ and $N(x)=\cosh^{-2}(x)$ described by formulas [32]

$$\tau=\frac{(v/2)^{1/2}}{a^{3/4}}\left(\sqrt{\kappa^2+1}-\kappa\right)^{1/2}, \qquad \kappa=\frac{\sqrt{a}}{v}\left(1-n-\frac{v^2}{4a}\right),$$

$$\chi=-\frac{1}{2}\ln\frac{(v/2\sqrt{a})^{1/2}+\left(\sqrt{\kappa^2+1}-\kappa\right)^{1/2}}{-(v/2\sqrt{a})^{1/2}+\left(\sqrt{\kappa^2+1}-\kappa\right)^{1/2}}, \tag{3.18}$$

the functions A^5 and B^5 in (3.17) are expressed as follows [28]:

$$A^5=-\frac{(v/2)^{1/2}}{4a^{3/4}\sqrt{1+\kappa^2}}\left(\sqrt{\kappa^2+1}-\kappa\right)^{1/2}\left(\kappa+\sqrt{1+\kappa^2}-2\frac{\sqrt{a}}{v}\right),$$

$$B^5=\frac{(v/2)^{1/2}}{4a^{1/4}\sqrt{1+\kappa^2}}\frac{\left(\sqrt{\kappa^2+1}-\kappa\right)^{1/2}}{\left(\sqrt{\kappa^2+1}-\kappa-(v/2\sqrt{a})\right)}$$

$$\times\left(\sqrt{1+\kappa^2}-3\kappa+2\frac{\sqrt{a}}{v}-\frac{v}{\sqrt{a}}\right).$$

It should be noticed, that the solution of the BVP presented above is unique, but the number of RG-operators that give rise to this solution is not equal to one (in the first example, we have four RG-operators with arbitrary functions of $(n,\ \chi)$, and in the second example, five RG-operators with arbitrary functions of a). In the next section, we shall see that the number of RG operators may be enlarged arbitrarily provided not only point but Lie-Bäcklund groups are taken into account.

4 RG as Lie-Bäcklund subgroup

The method of constructing RG-symmetries from Lie point symmetries admitted by the original DE is naturally generalized to include Lie-Bäcklund symmetries. The extension of the space of differential variables increases the amount of BVPs that allow restriction of a group on their solution. A complete set of RG-symmetries is obtained by appending Lie-Bäcklund

RG-symmetries to point RG-symmetries. This section presents an example of constructing Lie-Bäcklund RG-symmetries of the second order for the BVP (2.1). As in the previous section we use a transformed form of the basic equations (3.11).

Lie-Bäcklund symmetries admitted by the RG-manifold (3.11) are characterized by the same canonical infinitesimal operator (3.12) where additional terms in coordinates f and g proportional to higher-order derivatives of τ and χ should be taken into account. Similarly to first-order symmetries, these terms are linear combinations of τ and χ and their derivatives $\tau_i=(\partial^i\tau/\partial n^i)$ and $\chi_i=(\partial^i\chi/\partial n^i)$ with coefficients that depend only on v and n [27, 28, 29]. For the second-order Lie-Bäcklund symmetries in a particular case $\varphi(n)=1$, we have five additional operators with $i=7,\ldots,12$ (the term with $i=6$ corresponds to X_∞ and is omitted in the sum, i.e. $c_6=0$)

$$X=X_\infty+\sum_{i=1}^{12}c_iX_i\equiv\left(\bar{f}+\sum_{i=1}^{12}c_if_i\right)\partial_\tau+\left(\bar{g}+\sum_{i=1}^{12}c_ig_i\right)\partial_\chi. \tag{4.1}$$

It should be noted that expressions for all coordinates here can be obtained by the action of the following three recursive operators [29] L_i, $i=1,2,3$

$$L_1=\begin{pmatrix}0 & -(n/a)D_n\\ D_n & 0\end{pmatrix},\quad L_2=\begin{pmatrix}2nD_n-1 & (n/a)vD_n\\ -vD_n & 2nD_n\end{pmatrix},$$

$$L_3=\begin{pmatrix}2nvD_n & n(v^2/2a-2n)D_n-n\\ (-v^2/2+2an)D_n+a & 2nvD_n+v\end{pmatrix}, \tag{4.2}$$

on the "trivial" operator with $f=\tau$ and $g=\chi$ (here D_n is the operator of total differentiation with respect to n). Below we present only three of these five second-order Lie-Bäcklund operators

$$f_7=n\tau_2,\qquad g_7=\chi_1+n\chi_2;$$

$$f_8=(1/2a)n\left[-\chi_1+v\tau_2-2n\chi_2\right],\qquad g_8=(1/2a)v\chi_1+n\tau_2+\frac{1}{2a}nv\chi_2;$$

$$f_9=(1/4)\tau-n\tau_1-(5/4a)vn\chi_1+\left(-n+(1/4a)v^2\right)n\tau_2-(1/a)vn^2\chi_2,$$

$$g_9=(3/4)v\tau_1-\left(2n-(1/4a)v^2\right)\chi_1+vn\tau_2+\left(-n+(1/4a)v^2\right)n\chi_2. \tag{4.3}$$

The procedure of restriction of the Lie-Bäcklund group obtained on the solution of the BVP leads to expressions for $\bar{f}$ and $\bar{g}$ akin to (refeq3.15)

$$\bar{f} = -\sum_{i=1}^{12} c_i f_i, \quad \bar{g} = -\sum_{i=1}^{12} c_i g_i . \tag{4.4}$$

Substitution of (refeq4.4) in (refeq4.1) yields additional terms in the expression (3.16) for the RG-operator R that depend on higher-order derivatives of τ and χ

$$R = \sum_{i=1}^{12} c_i(a) R_i \equiv \sum_{i=1}^{12} c_i(a) \left((f_i - A^i)\partial_{\mathcal{T}} + (g_i - B^i)\partial_{\mathcal{X}} \right) . \tag{4.5}$$

Here functions A^i and B^i are given by the corresponding formulas for coordinates f_i and g_i to be evaluated on the solution $\bar{\tau}(n,v)$ and $\bar{\chi}(n,v)$. It appears that coordinates of Lie-Bäcklund RG-operators are obtained from point RG-operators with the help of the above-mentioned recursive operators, hence, one can obtain Lie-Bäcklund RG-operators of an arbitrary high order. Despite an unusual form, we still call them RG-operators since they possess the main property of RG-operators, namely, they contain a solution of the BVP in their invariant manifold.

The problem of utilization of Lie-Bäcklund RG-operators is not as trivial as for point RG-operators. Yet we can indicate two ways in which Lie-Bäcklund RG-symmetries can be employed.

Firstly, coordinates of canonical Lie-Bäcklund RG-operators can be used to construct a set of relationships, differential constraints, that are compatible with the original DEs and satisfy specific boundary conditions. The use of such constraints is described in section 6. In the general case, for an arbitrary Lie-Bäcklund group of a given order, coordinates of the corresponding canonical operator can be treated as a set of differential expressions, zero equalities for which impose appropriate restrictions on the basic DEs, consistent either with physical or with symmetrical reasons. These equalities can also be treated as embedding equations (see also [14]).

Secondly, Lie-Bäcklund RG-operators can be used to construct invariant solutions that automatically fit boundary conditions. It should be noticed that in some particular cases, Lie-Bäcklund RG-symmetries can be constructed from a Lie-Bäcklund group with a finite number of operators. For example, RG-symmetry for the BVP (2.1) with boundary conditions $V = 0$ and $N = \cosh^{-2}(x)$ appears as a linear combination of

three Lie-Bäcklund symmetries

$$R = (f_2 + 2(f_7 + f_9))\partial_\tau + (g_2 + 2(g_7 + g_9))\partial_\chi . \tag{4.6}$$

The desired solution of the BVP is found as the invariant solution with respect to RG-operator (4.6) and is presented by formulas (3.18).

The recipe of constructing the Lie-Bäcklund renormgroup formulated in this section goes far beyond a simple illustrative example for the BVP (2.1). In a similar way, Lie-Bäcklund renormgroups are constructed for different BVPs of mathematical physics that admit Lie-Bäcklund symmetries; other examples are presented in [14] for the linear parabolic and modified Burgers equation. It is essential that when parameters entering into the equation and boundary conditions are involved in group transformations, coordinates of canonical Lie-Bäcklund RG-operators contain not only first but higher-order derivatives with respect to these parameters. This means that in addition to recursive operators that contain operators of total differentiation with respect to n (for BVP (2.1)), new recursive operators appear with terms that contain operators of differentiation with respect to parameters, as well ($\propto D_\epsilon$ and D_a in the case of BVPs (3.1) and (3.11)).

5 RG devising based on embedding equations

A specific method of constructing RG-symmetries particularly promising for physical systems described by ordinary differential equations (ODEs) is based on embedding equations [24, 14]. In the context of the discussed model of quasi-Chaplygin media such equations arise, for example, while studying invariant solutions with respect to symmetries obtained. Below, for simplicity the idea of this method is demonstrated for the BVP for the first-order ODE[7]

$$u_t = f(t, u, a); \qquad t = \tau, \quad u = x . \tag{5.1}$$

Extension of the original differential manifold by adding, to the original equation, the embedding equation [15] that appears as a linear first-order

[7] In the context of the discussed mathematical model (2.1) of quasi-gaseous media such equations arise, for example, while studying invariant solutions with respect to symmetries obtained.

partial DE

$$u_\tau + f(\tau, x, a)u_x = 0\,, \tag{5.2}$$

gives the desired RG-manifold. Performing the group analysis for this RG-manifold involves boundary data and parameter a in group transformations, while the subsequent restriction of the group obtained on any solution of the BVP yields the desired RG-symmetries. Below two examples of such calculations for $f = au^2$ and $f = u^2 + au^3$ are presented.

5.1. In the event of $f = au^2$ the RG-manifold (5.1)-(5.2) is given by two equations

$$u_t = au^2\,, \quad u_\tau + ax^2u_x = 0 \tag{5.3}$$

that admit an infinite-dimensional Lie point algebra characterized by five independent terms

$$X = \sum_{i=1}^{5} \alpha_i X_i\,,$$

$$X_1 = \partial_t + au^2\partial_u\,, \quad X_2 = \partial_\tau + ax^2\partial_x\,, \tag{5.4}$$

$$X_3 = u^2\partial_u\,, \quad X_4 = x^2\partial_x\,, \quad X_5 = x^2\tau\partial_x + u^2t\partial_u + \partial_a\,.$$

Here functions α_1 and α_2 depend upon five variables $\{t,\, \tau,\, x,\, a,\, u\}$ whereas α_3, α_4, α_5 are arbitrary functions of the following combinations

$$at + \frac{1}{u}, \quad a\tau + \frac{1}{x}, \quad a\,.$$

The procedure of restriction of the group obtained leads to the following condition

$$U^2(\alpha_3 + a\alpha_1 + \alpha_5 t) - \alpha_1 U_t - \alpha_2 U_\tau - x^2(\alpha_4 + a\alpha_2 + \alpha_5\tau)U_x - \alpha_5 U_a = 0\,. \tag{5.5}$$

to be fulfilled on the exact or approximate solution $u = U(t, x, \tau, a)$ of the BVP (5.1)-(5.2); for example, one can take the perturbation theory solution of the problem (5.1)-(5.2) with respect to powers of the parameter a

$$u = U(t, x, \tau, a) \equiv x + ax^2(t - \tau) + O(a^2)\,, \qquad a \ll 1\,. \tag{5.6}$$

Substituting (5.6) into (5.5) shows that the invariance condition (5.5) is fulfilled for $\alpha_3 = \alpha_4 \equiv \alpha$ and arbitrary α_1, α_2 and α_5. Assuming $\alpha_1 = \alpha_2 = \alpha = 0$ and $\alpha_5 = 1$ yields one of the RG-operators

$$R = x^2\tau\partial_x + \partial_a + u^2t\partial_u\,, \tag{5.7}$$

which enables us to transform the perturbation theory solution of the BVP (5.1) for small $a \ll 1$ to the exact solution in the form of a sum of a geometric progression

$$u = \frac{x}{1 - ax(t - \tau)}\,.$$

This result is found by solving the Lie equations, that correspond to the RG-operator (5.7).

5.2. For another value of the function $f = u^2 + au^3$, the RG-operator that is similar to (5.7) is given as follows

$$R = \left(x^2(1 + ax)\tau + x\right)\partial_x + \left(u^2(1 + au)t + u\right)\partial_u - a\partial_a\,. \tag{5.8}$$

The invarince condition for the solution of the BVP with respect to the RG-operator (5.8) has the form of the first-order partial differential equation

$$-\left(x^2(1 + ax)\tau + x\right)u_x + au_a + u^2(1 + au)t + u = 0\,. \tag{5.9}$$

Solving the characteristic equations for (5.9) (Lie equations) yields the following exact solution of the BVP (5.1) with $f = u^2 + au^3$

$$t - \tau = \frac{1}{x} - \frac{1}{u} + a\ln\left|\frac{x\,(1 + au)}{u\,(1 + ax)}\right|\,.$$

What all renormgroups obtained for the BVP for the first-order DE in the above examples have in common is that their operators depend upon arbitrary functions α^i, which means that RG can be expressed in terms of different RG-operators with various particular expressions for their coordinates. This situation is the same as that one obtains for the BVP in the case of partial differential equations: different RG-operators yield the unique specific solution of the given BVP contained in the invariant manifold of RG-operators. The previous procedure of RG constructing for the BVP for the ODE was based on the utilization of point groups. However, Lie-Bäcklund groups can also be employed for constructing RG-symmetries for the first-order ODE, especially, in view of embedding equations [25].

One should bear in mind that the structure of embedding equations depends not only on the form of the original equation, but also on the boundary conditions. It means that for given basic equations we may obtain different embedding equations defined by a specific form of boundary data. For example, if the function f in the right-hand side of the original equation (5.1) depends upon the variable x,

$$u_t = f(t, x, a, u)\,; \quad t = \tau\,,\ u = x \tag{5.10}$$

an integro-differential embedding equation is obtained

$$u_\tau + f(\tau, x, a, x)u_x = f(\tau, x, a, x)$$

$$+ \int_\tau^t dt' f_x(t', x, a, u(t')) \exp\left[- \int_t^{t'} dt'' f_u(t'', x, a, u(t'')) \right] . \tag{5.11}$$

Hence, the RG manifold in this case is defined by a system of integro-differential equations (5.10) and (5.11) and one should employ the modern group analysis techniques which give a possibility of analyzing integro-differential equations, as well [33, 34].

6 RG and differential constraint

To take boundary data into account, instead of an embedding equation, an additional differential constraint can be used that should satisfy two conditions: firstly, it should be compatible with the original DE and, secondly, it should explicitly take boundary conditions into account. This constraint naturally emerges when a coordinate of a canonical operator of the Lie-Bäcklund RG admitted by BVP is assumed equal to zero. Adding this constraint to original equations we obtain the RG-manifold.

6.1. To illustrate this approach, we consider first a trivial case of a BVP (2.1) with $a = 0$ which we rewrite using hodograph transformations in a simple form (compare with (3.11))

$$\chi_n = 0 , \quad \chi_v + \tau_n = 0 . \tag{6.1}$$

Lie-Bäcklund symmetries of the system of DE (6.1) are given by a canonical operator

$$X = f\partial_\tau + g\partial_\chi , \tag{6.2}$$

characterized by an arbitrary dependence of its coordinates f and g upon v and derivatives $\tau_s + n\chi_{s+1}$, χ_s of an arbitrary order $s \geq 0$

$$f = F(v, \chi_s, \tilde{\tau}_s) - n\left[\partial_v + \sum_{k=0}^{\infty} (\tilde{\tau}_{k+1}\partial_{\tilde{\tau}_s} + \chi_{k+1}\partial_{\chi_k}) \right] G, \quad g = G(v, \chi_s, \tilde{\tau}_s) ,$$

$$\tilde{\tau}_s = \tau_s + n\chi_{s+1} , \quad \tau_s = (\partial^s\tau/\partial v^s) , \quad \chi_s = (\partial^s\chi/\partial v^s) . \tag{6.3}$$

Let us consider a particular case of a BVP (2.1) with a linear function $V(x) = -\varepsilon x$ and arbitrary $N(x)$. In terms of the variables τ and χ, these conditions are described, for example, by a pair of differential constraints

$$\chi_{vv} = 0 , \quad \tau_{vv} - N_{vv}\chi_v - N_v\chi_{vv} = 0 . \tag{6.4}$$

Here the dependence of N upon x is expressed in terms of v by using the boundary condition.

It is easily checked by direct substituting into (6.3) that left-hand sides of these equalities are the corresponding coordinates g and f of the second-order Lie-Bäcklund symmetry operator (6.2). Adding differential constraints (6.4) to the original equation (6.1), we obtain the desired RG-manifold

$$\chi_n = 0 , \quad \chi_v + \tau_n = 0 , \quad \chi_{vv} = 0 , \quad \tau_{vv} - N_{vv}\chi_v = 0 . \tag{6.5}$$

The latter admits a 17-parameter group of point transformations given by the following infinitesimal operators

$$X = \sum_{i=1}^{m} c_i X_i , \quad m = 17 , \tag{6.6}$$

$$X_1 = v^2\partial_v + v(2(n - N) + vN_v)\partial_n + (\chi(N - n) + \tau v)\partial_\tau + v\chi\partial_\chi ,$$

$$X_2 = v\chi\partial_v + (\chi(n - N) + v(\chi N_v - \tau))\partial_n + 2\tau\chi\partial_\tau + \chi^2\partial_\chi ,$$

$$X_3 = -v\partial_v + (N - n - vN_v)\partial_n , \quad X_4 = v\chi\partial_n - \chi^2\partial_\tau ,$$

$$X_5 = v\partial_n , \quad X_6 = (N - n)\partial_n + \chi\partial_\chi , \quad X_7 = (n - N)\partial_n + \tau\partial_\tau ,$$

$$X_8 = \partial_\tau , \quad X_9 = v\partial_\tau , \quad X_{10} = (N - n)\partial_\tau + v\partial_\chi \quad X_{11} = \partial_\chi ,$$

$$X_{12} = \chi\partial_\tau , \quad X_{13} = -v^2\partial_n + v\chi\partial_\tau , \quad X_{14} = -\partial_v - N_v\partial_n ,$$

$$X_{15} = \partial_n , \quad X_{16} = \chi\partial_n , \quad X_{17} = -\chi\partial_v + (\tau - \chi N_v)\partial_n ,$$

The usual procedure of restriction of the group obtained on a solution of the BVP (6.1), relates different coefficients in the sum (6.6) and gives the desired RG operators

$$R = \sum_{i=1}^{13} c_i R_i , \tag{6.7}$$

$$R_1 = X_1\,, \quad R_2 = X_2\,, \quad R_3 = X_3 + \varepsilon X_{17}\,, \quad R_4 = X_4\,,$$

$$R_5 = X_5 + \epsilon X_{16}\,, \quad R_6 = X_6 + \varepsilon X_{17}\,, \quad R_7 = X_7 + \varepsilon X_{16}\,,$$

$$R_8 = X_8 + \varepsilon X_{15}, \quad R_9 = X_9 - \varepsilon^2 X_{16}, \quad R_{10} = X_{10} - \varepsilon^2 X_{17},$$

$$R_{11} = X_{11} + \varepsilon X_{14}\,, \quad R_{12} = X_{12} + \varepsilon X_{16}\,, \quad R_{13} = X_{13}\,.$$

The exact solution of the BVP $\chi = -v/\varepsilon$, $\tau = (1/\varepsilon)(n - N)$ is found either by solving Lie equations corresponding to any of these RG-operators, or as the intersection of all invariant manifolds.

6.2. Now let us turn to a general case of a BVP (2.1) with $a \neq 0$. We shall consider the problem of constructing RG-symmetries using the RG-manifold given by basic equations in the form (3.11) and the most simple differential constraint yielded by the linear combination of the second-order Lie-Bäcklund symmetry (4.3) f_7, g_7 and trivial infinite-dimensional symmetry $f_\infty = 0$, $g_\infty = -1$

$$a\tau_v - n\chi_n = 0\,, \quad \chi_v + \tau_n = 0; \quad n\tau_{nn} = 0\,, \quad \chi_n + n\chi_{nn} - 1 = 0\,. \quad (6.8)$$

This differential constraint describes, in particular, a linear dependence of N upon x and $V(x) = 0$. The Lie point group admitted by the RG-manifold (6.8) is characterized by seven infinitesimal operators (see above the formula (6.6) for $m = 7$)

$$X_1 = -2v\partial_v - 4n\partial_n - 6n(v/a)\partial_\tau + (2\chi - 6n + 3v^2/a)\partial_\chi\,,$$

$$X_2 = v\partial_v + 2n\partial_n + (\tau + 2nv/a)\partial_\tau + (2n - v^2/a)\partial_\chi\,, \quad X_3 = \partial_v\,,$$

$$X_4 = n\partial_\tau - v\partial_\chi\,, \quad X_5 = (v/a)\partial_\tau + \ln n\partial_\chi\,, \quad X_6 = \partial_\tau\,, \quad X_7 = \partial_\chi.$$

The restriction of this group on the solution of the BVP with the above-mentioned boundary conditions leads to the three-parameter renormgroup with operators

$$R_1 = X_1, \quad R_2 = X_2, \quad R_3 = aX_3 + X_4\,.$$

As in the previous case, the exact solution of the BVP $\tau = nw$, $\chi = n - aw^2/2$ appears as an intersection of all invariant manifolds that correspond to these RG-operators.

The characteristic feature of the described approach is the formulation of boundary data in the form of a differential constraint and the subsequent search of the group admitted by this constraint and basic equations. It is evident that there exists an infinite number of other differential constraints that adequately describe boundary data and the use of which leads to different RG algebras. As an example, we can point to differential constraints that arise from the zero equality of appropriate coordinates of the infinite Lie-Bäcklund algebra.

The example of constructing RG-symmetries on the basis of Lie-Bäcklund symmetries reveals the practical importance of the latter and, on the other hand, demonstrates point RG-symmetries that are not admitted by the original equation. The procedure of constructing RG-symmetries with the help of a differential constraint was also carried out in [23] for the linear parabolic equation.

7 RG as a subgroup of an approximate symmetry group

Probably, one of the most attractive methods of RG constructing is that based on approximate symmetries [16]. This method can be applied to physical systems described in terms of mathematical models based on DEs with small parameters. These small parameters allows us to consider a simple subsystem of the original DEs that usually admits an extended symmetry group inherited by the original DEs (in the small-parameter approximation). Restricting this approximate group on the solution of the BVP yields the desired RG-symmetries. The merits of the described method is illustrated below for the BVP (2.1) with a small nonlinearity parameter $a \ll 1$. Using hodograph transformations and the change of variables $\tau = nt$, $\chi = x - vt$ and $w = v/a$, instead of (3.11), the following system of linear equations is obtained:

$$\tau_w - (n/\varphi(n))\chi_n = 0\,, \quad \chi_w + a\tau_n = 0\,. \quad (7.1)$$

In the event of $a = 0$, these equations admit an infinite Lie-Bäcklund group with the operator

$$X = f\partial_\tau + g\partial_\chi\,, \quad (7.2)$$

characterized by an arbitrary dependence of the zero-order coordinates $f = f^0$ and $g = g^0$ upon n, τ, χ and the derivatives $\tilde{\tau}_s$, χ_s of an arbitrary order

$$f^0 = F^0(n, \chi_s, \tilde{\tau}_s) + \int dw \left\{ \frac{n}{\varphi} \left[\partial_n + \sum_{s=0}^{\infty} (\tau_{s+1}\partial_{\tau_s} + \chi_{s+1}\partial_{\chi_s}) \right] g^0 \right\},$$

$$g^0 = G^0(n, \chi_s, \tilde{\tau}_s), \qquad \tilde{\tau}_s = \tau_s - w \sum_{p=0}^{s} \binom{s}{p} (n/\varphi)_p \, \chi_{s-p+1},$$

$$\tau_s = (\partial^s \tau / \partial n^s), \quad \chi_s = (\partial^s \chi / \partial n^s). \tag{7.3}$$

Here and below in (7.5) F^0 and G^0 are arbitrary functions of their arguments, and expressions in brackets before integrating over w should be given in terms of $\tilde{\tau}_s$, χ_s, n, w.

For small nonzero values of the parameter $a \ll 1$, this symmetry is inherited as an approximate one by equations (7.1) which thus represent an approximate RG-manifold. For example, for $\varphi(n) = 1$ the following result is obtained:

$$f = \sum_{i=0}^{\infty} a^i f^i; \quad g = \sum_{i=0}^{\infty} a^i g^i, \tag{7.4}$$

$$\begin{aligned} f^i &= F^i(n, \chi_s, \tilde{\tau}_s) \\ &+ \int dw \left\{ \sum_{s=0}^{\infty} \tau_{s+1}\partial_{\chi_s} f^{i-1} + n \left[\partial_n + \sum_{s=0}^{\infty} (\tau_{s+1}\partial_{\tau_s} + \chi_{s+1}\partial_{\chi_s}) \right] g^i \right\}, \\ g^i &= G^i(n, \chi_s, \tilde{\tau}_s) \\ &+ \int dw \left\{ \sum_{s=0}^{\infty} \tau_{s+1}\partial_{\chi_s} g^{i-1} - \left[\partial_n + \sum_{s=0}^{\infty} (\tau_{s+1}\partial_{\tau_s} + \chi_{s+1}\partial_{\chi_s}) \right] f^{i-1} \right\}, \\ & i \geq 1, \quad \tilde{\tau}_s = \tau_s - w(n\chi_{s+1} + s\chi_s). \end{aligned} \tag{7.5}$$

It follows from these formulas that the symmetry of equations for $a \neq 0$ is inherited by the system of equations (7.1) up to an arbitrary order of the parameter a. It should be noticed that both zero-order and higher-order (in the nonlinearity parameter a) approximate symmetries may appear as

Lie point symmetries or Lie-Bäcklund symmetries, and this parameter can be involved in group transformations, as well.

The restriction of the approximate group obtained on a particular solution of the BVP defines the specific form of the zero-order symmetries. It means that while constructing RG-symmetries of the BVP (2.1), (7.1) coordinates f^0, g^0 and "integration constants" F^i, G^i, $i \geq 1$ are not arbitrary functions, but should be chosen so that relationships $f = 0$, $g = 0$ satisfy desired boundary conditions $\tau_s = 0$, $\chi = H(n)$ at $w = 0$. Provided that the functions F^i and G^i are also equal to zero in this case, boundary conditions are fully correlated with the form of functions f^0 and g^0. In fact, invariance conditions $f = 0$ and $g = 0$ appear as differential constraints (or simply algebraic relationships) to be fitted by boundary data.

Of special interest are such zero-order functions f^0 and g^0 for which infinite series (7.5) are truncated for some finite value of $i = i_{max}$, and we obtain finite sums. In this case, instead of an approximate group with respect to a small parameter a we obtain the exact symmetry group (compare with [16, §11]). A simple example of this is given by the RG-operator (4.6). It is easily checked that in terms of n and w, the combinations of coordinates $f_2 + 2(f_7 + f_9)$ and $g_2 + 2(g_7 + g_9)$ are expressed as binomial in a, i.e. expressions for f and g are represented as zero-order and first-order terms $f = f^0 + af^1$ and $g = g^0 + ag^1$, where f^0, g^0 and f^1, g^1 according to (7.3), (7.5) are defined by the formulas

$$\begin{aligned} f^0 &= 2n(1-n)\tau_2 - n\tau_1 - 2nw(\chi_1 + n\chi_2), \\ g^0 &= 2n(1-n)\chi_2 + (2 - 3n)\chi_1, \\ f^1 &= \frac{1}{2} n w^2 \tau_2, \quad g^1 = 2nw\tau_2 + w\tau_1 + \frac{1}{2}(nw^2\chi_2 + w^2\chi_1). \end{aligned} \tag{7.6}$$

From here, in view of (7.5), it follows that higher-order corrections vanish, and we obtain an exact second-order Lie-Bäcklund symmetry of DEs (7.1) at $\varphi = 1$ for arbitrary $a \neq 0$; this symmetry gives rise to the rigorous solution [26, 27] satisfying the boundary condition $N = \cosh^{-2}(x)$ defined by the zero order term g^0.

The arbitrariness in functions f^0, g^0 enables us to construct RG-symmetries for any boundary conditions. As an illustration, we present RG-symmetries for the BVP with

$$H(n) = (\ln(1/n))^{1/2}, \tag{7.7}$$

describing space evolution (self-focusing) of the gaussian beam with the originally plane phase front at $\tau = 0$. To satisfy the initial distribution (7.7), one can choose the following functions f^0 and g^0:

$$f^0 = 1 + 2n\chi\chi_1, \quad g^0 = 0. \tag{7.8}$$

For this value of f^0 the inherited point group of the BVP is constructed with the help of formulas (7.4), (7.5) and is given by the operator

$$R = -2\chi\partial_w + 2a\tau\partial_n + \left(1 + \frac{a\tau^2}{n}\right)\partial_\tau. \tag{7.9}$$

The invariance condition for the solution of the BVP with respect to RG with this operator is presented in the form of two partial differential equations

$$\chi\chi_w - a\tau\chi_n = 0, \quad 2\chi\tau_w - 2a\tau\tau_n + 1 + \frac{a\tau^2}{n} = 0,$$

the solution of which yields the desired approximate analytical solution of the problem

$$x^2 = (ant^2 - \ln n)\left[1 - 2Q(\sqrt{ant^2})\right]^2, \quad v = -2\frac{x}{t}\frac{Q(\sqrt{ant^2})}{\left[1 - 2Q(\sqrt{ant^2})\right]}. \tag{7.10}$$

Here the function $Q(z)$ is expressed as follows

$$Q(z) = ze^{-z^2/2}\int\limits_0^z dt e^{t^2/2}.$$

The first-order approximate symmetry obtained can be used to calculate a higher-order approximation of the RG-operator (7.9) and, thus, to improve the analytical solution (7.10). One can also obtain new-type RG-operators just by substituting the approximate solution (7.10) into formulas (3.17).

8 Conclusion

This paper presents a new approach to constructing RG-symmetries based on the mathematical apparatus of classical and modern group analysis. It

differs from the traditionally used methods of constructing RGs in theoretical physics [11] and is formulated as a sequence of the following steps:

I) *constructing the RG-manifold*, that takes into account both basic equations and the corresponding boundary conditions;

II) *calculating the symmetry group*, admitted by RG-manifold;

III) *restricting the group* obtained *on the solution of the BVP*;

IV) utilizing of RG-operators to *find analytical expressions for solutions.*

As it was shown there exists a set of different algorithms for finding RG-symmetries. The choice of a particular one for a given physical problem depends on a mathematical model used for the problem description.

It should be noted, that different methods of constructing RG-symmetries described above do not exhaust the suggested approach (see, e.g., above-mentioned [18]-[28]). Moreover, procedure of constructing RG–symmetries may combine different algorithms; for example, of interest is a simultaneous use of the method based on approximate symmetries and the invariant embedding method, and so on.

The new approach reveals a close relation of *functional self-similarity* property (i.e., "classical" RG–symmetry as an exact property of a solution) to an invariance condition of a BVP solution with respect to RG-operator. Mathematically, the latter is formulated as the zero equality of the coordinate of a canonical RG-operator on a solution of BVP.

One can readily see that RG-operators may appear in the form, that is different from QFT case[6], e.g., operators of Lie-Bäcklund RG-symmetries. However, in some cases our RG-operators can look like that ones in QFT renormalization group. For example, linear combination of operators $\alpha_1 X_1$ and $\alpha_3(X_3 + X_4)$ for the BVP (5.1) with $\alpha_1 = 1$, $\alpha_3 = -a$ gives

$$R = \partial_t - ax^2\partial_x, \tag{8.1}$$

which is formally equivalent (with appropriate change of variables $t = \ln x$, $x = g$ and $\beta(g) = ag^2$) to differential Ovsiannikov – Callan – Symanzik operator for one-coupling massless quantum field model in one-loop approximation.

Up to now this approach is feasible for systems that can be described by differential equations and is based on the formalism of modern group analysis.

It seems also possible to extend our approach to physical systems that are not described just by differential equations. A chance of such extension is based on recent advances in group analysis of systems of integro-differential equations [33, 34] that allow transformations of both dynamical variables and functionals of a solution to be formulated [35]. More intriguing is the issue of a possibility of constructing a regular approach for more complicated systems, in particular to that ones having an infinite number of degrees of freedom. The formers can be represented in a compact form by functional integrals (or *path integrals*).

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Групповой анализ и ренормгрупповые симметрии

Предложен новый подход к построению специального класса симметрий краевых задач — ренормгрупповых симметрий. Описаны различные методы вычисления РГ-симметрий на основе современного группового анализа. Приложения этого подхода к краевым задачам продемонстрированы с помощью простой математической модели.

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Group Analysis and Renormgroup Symmetries

An original regular approach to constructing special type symmetries for boundary value problems, namely renormgroup symmetries, is presented. Different methods of calculating these symmetries based on modern group analysis are described. An application of the approach to boundary value problems is demonstrated with the help of a simple mathematical model.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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