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GEOMETRY OF ADIABATIC CHANGES

General Analysis

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Thus $\gamma_n(C)$ is given by a circuit integral in parametric space and is independent of how the circuit is traversed (provided of course that this is slow enough for the adiabatic approximation to hold).

M.V. Berry

I. Introduction

The geometric phase can be qualified as one of the most actively developed modern physical conceptions. The number of publications analyzing different mathematical aspects and experimental consequences of its existence, beginning with Berry's pioneer paper, is impressive [1]. Moreover, recent investigations [2] further improve the general picture drawn by Berry, Simon, Hannay, Aharonov and Anandan et al. Nevertheless, it seems necessary to return and consider in greater detail the most typical case that represents the evolution of neutron spin accompanying the excursion of the magnetic field. This paper originated with the unsuccessful attempt to apply existing statements of the geometric phase calculation in the case of essentially non-adiabatic spin (characteristic vector) evolution under an arbitrary and finite excursion rate of the magnetic field (parametric vector).

The two references directly connecting with this question also belong to Berry [3], [4]. Unfortunately, the scheme of adiabatic iteration developed in [3] can not be accepted as consistent and applicable. Berry's second consideration [4] of some years ago, seems to have been done because of dissatisfaction with the previous analysis. The argument which could initiate a new consideration arises naturally at the level of intuition: for an arbitrary excursion, a complex generalization of the geometric phase seems inevitable.

It must be stressed that the notion of a complex geometric phase is not new and was well developed by Garrison and Wright for dissipative systems in [5]. We will not discuss this first consideration because it represents an example of the standard way of generalizing on the basis of non-Hermitian Hamiltonians which, as a consequence, usually leads from real parameters to their complex counterparts. Another consideration leading to the notion of the complex geometric phase was developed by Alber and Marsden [6] in connection with solitons.

Berry's second consideration appeals to Dykhne's calculation connected with integration of the quantum amplitude through the introduction of the complex time plane [4]. Is the picture of a geometric amplitude drawn in this paper complete? Corresponding remarks can be found in Sec.VI.

From the other side, an arbitrary excursion implies the possibility of a non-adiabatic evolution of the system. Is Aharonov and Ananadan's consideration of the non-adiabatic case [7], founded on the assumption of unitarity, the most general one? As it will be shown below in Sec.IV, the latter, particularly, does not cover the case presented in Sec.V.

II. Adiabatic changes

The conclusions leading to the notion of adiabatic changes could be initiated with the following simple illustration. Let us take a watch arrow. The rod could be used as an axis in a manner which permits comparatively free turning of the arrow.

Now, let us transport this construction along a plane, always keeping the rod vertical. We would see that the initial and final positions of the arrow, after an excursion along the closed lines, would have coincided if the speed of the traverse were slow enough. From the geometric point of view, this result points out that the parallel transport law was fulfilled locally by the arrow.

Definition 1. The changes in a physical system are adiabatic if they are caused by the parametric vector traversing slowly enough that the characteristic vectors of the physical system move in accordance with the parallel transport law.

Now, if we place our construction on the surface of a cone, we observe the angle between the initial and final positions of the arrow even under the excursion with very slow speed. This is the so-called (an)holonomy effect, caused by the non-local difference between the cone surface and the plane.

The value of this angle is equal to the cone angle and is obtained as a result of local conservation of the parallel transport law. One could easily generalize this result for a particular excursion on the sphere figured here as a sequence of parallel excursions along the corresponding tangential cones, and arrive at the famous solid angle law

$$\alpha_N = \Omega_N = (1 - \cos\theta)\Delta\phi$$

for the excursion crossing the North pole, and

$$\alpha_S = \Omega_S = (1 + \cos\theta)\Delta\phi.$$

for the excursion crossing the South pole.

Furthermore, it can be easily shown that the solid angle law remains valid for the excursion of the arrow having an arbitrary configuration on the sphere and under an arbitrary radial deformation. Consequently, we can confirm that the α_N and α_S are the t o p o l o g i c a l i n v a r i a n t s. In a quantum setting, consideration passes to the Hilbert space with the corresponding notions of horizontal and vertical l i f t s for the wave functions [8].

Is this picture of the adiabatic changes conserved in the case of quantum evolutions of the neutron spin in a magnetic field? Is there a real difference between the terms "topological" and "geometrical" (phase), which are usually identified in the frame of holonomy analysis?

The Cartesian representation of the Hamiltonian of a neutron spin in a magnetic field is

$$\begin{pmatrix} Z & X - iY \\ X + iY & -Z \end{pmatrix}.$$

But this representation is not convenient for the consideration below. The natural map for investigating of the holonomy effects is a spherical one. Particularly, it is evident from the natural *ad hoc* separation of the variables for a slow excursion: H, responsible for accounting for the dynamic phase, and the angular variables ϕ and θ, which, in principle, can generate holonomic phenomena. So, let us rewrite and hereafter use the Hamiltonian in spherical representation

$$H \cdot \begin{pmatrix} \cos\theta & \sin\theta e^{-i\Phi} \\ \sin\theta e^{i\Phi} & -\cos\theta \end{pmatrix}.$$

Two corresponding two solutions of Pauli's equation

$$i\hbar\dot{\Psi} = -2\mu\vec{H}\hat{\vec{s}}\Psi$$

for opposite projections of the spin can be obtained easily and look as follows

$$\Psi^o_+(\theta,\phi) = e^{-i\frac{\omega_L t}{2}} \begin{pmatrix} \cos\frac{\theta}{2}\cdot e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}, \tag{1}$$

$$\Psi^o_-(\theta,\phi) = e^{i\frac{\omega_L t}{2}} \begin{pmatrix} -\sin\frac{\theta}{2}\cdot e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}, \tag{2}$$

where θ and ϕ are the respective polar and azimutal angles of the quantization axis ($\vec{H}$) in the chosen coordinate frame, $\omega_L = 2|\mu|H/\hbar$ is the frequency of the Larmor precession, and μ is the magnetic momentum of the neutron, $H = |\vec{H}|$. The following must additionaly be stressed: these often cited and dedicated solutions describe the neutron spin evolution in an arbitrarily oriented and homogeneous magnetic field. However, the problem of neutron spin evolution in a magnetic field is unique. After passing from the stationary problem (1), (2), one also has an exactly-solvable problem for the case of a precessing magnetic field with the corresponding (time-dependent) Schrödinger equation [14]. Moreover, as will be shown in Sec.V, we can point to an alternative to the precession mode that completes the picture of neutron spin evolution, naturally and which also has an exact solution.

For the conclusions below, in addition to the definition of adiabatic changes, the following two notions need to be determined:

Definition 2. The weak non-adiabatic changes in a physical system are those caused by the vector-parameter's traversing with finite rate, but which conserve the parallel transport law for the characteristic vectors of the physical system.

Definition 3. The strong non-adiabatic changes in the physical system are those which are accompanied by with violations of the characteristic vector parallel transport law and can be caused by infinitely slow traverses of the external vector-parameter.

Let us go to a more detailed consideration.

3

III. Precession

By a precessing (or rotating) field configuration, we mean the following specific time-dependence of the magnetic field components:

$$
\begin{aligned}
H_x &= H \sin\theta \cos(\omega t + \phi), \\
H_y &= H \sin\theta \sin(\omega t + \phi), \qquad (3) \\
H_z &= H \cos\theta,
\end{aligned}
$$

where ω is the angular rate corresponding to a rotation around the z axis carrying the strength H and polar angle θ constants.

The solutions of Pauli's equation for the precessing field case are well known [14] and can be written as follows

$$\Psi(t) = C_+\Psi_+(t) + C_-\Psi_-(t)\,,\ |C_+|^2 + |C_-|^2 = 1\,, \qquad (4)$$

$$\Psi_+(t) = e^{-i(\Lambda-\omega)t/2} \begin{pmatrix} \sqrt{\frac{\Lambda+\omega_L\cos\theta-\omega}{2\Lambda}}\, e^{-i(\omega t+\phi)} \\ \sqrt{\frac{\Lambda-\omega_L\cos\theta+\omega}{2\Lambda}} \end{pmatrix}, \qquad (5)$$

$$\Psi_-(t) = e^{i(\Lambda+\omega)t/2} \begin{pmatrix} -\sqrt{\frac{\Lambda-\omega_L\cos\theta+\omega}{2\Lambda}}\, e^{-i(\omega t+\phi)} \\ \sqrt{\frac{\Lambda+\omega_L\cos\theta-\omega}{2\Lambda}} \end{pmatrix}, \qquad (6)$$

where

$$\Lambda = \sqrt{(\omega - \omega_L\cos\theta)^2 + \omega_L^2 \sin^2\theta}\,.$$

(As was mentioned by D.A. Korneev, Rabi's problem of spin resonance turning corresponds to the replacement $\omega \to -\omega$, i.e., rotation in a back direction.)

It must be stressed that $\Psi_\pm$ are orthogonal:

$$(\Psi_+, \Psi_-) = 0.$$

The spinors in the expressions above, under the substitutions

$$\cos\frac{\Theta}{2} = \sqrt{\frac{\Lambda + \omega_L\cos\theta - \omega}{2\Lambda}}, \qquad (7)$$

$$\Phi = \omega t + \phi, \qquad (8)$$

can be rewritten in a form similar to (1), (2):

$$\Psi_+(t) = e^{i\alpha_+}\tilde{\Psi}_+(t) = e^{-i(\Lambda-\omega)t/2} \begin{pmatrix} \cos\frac{\Theta}{2} e^{-i\Phi} \\ \sin\frac{\Theta}{2} \end{pmatrix}, \qquad (9)$$

$$\Psi_-(t) = e^{i\alpha_-}\tilde{\Psi}_-(t) = e^{i(\Lambda+\omega)t/2} \begin{pmatrix} -\sin\frac{\Theta}{2} e^{-i\Phi} \\ \cos\frac{\Theta}{2} \end{pmatrix}. \qquad (10)$$

These are the states with the following definite projection on the z axis:

$$(\Psi_\pm, s_z\Psi_\pm) = \pm\frac{1}{2}\cos\Theta. \qquad (11)$$

The Aharonov–Anandan approach, as was shown particularly in [11], points out that the phases of the exponents before the spinors in (9), (10) can be expressed in a suprising manner through Θ and Φ, too:

$$\alpha_+ = -\frac{\Lambda-\omega}{2}t = -\frac{\omega_L\cos(\theta-\Theta)}{2}t + \frac{(1+\cos\Theta)\Delta\Phi}{2}, \qquad (12)$$

$$\alpha_- = \frac{\Lambda+\omega}{2}t = \frac{\omega_L\cos(\theta-\Theta)}{2}t + \frac{(1-\cos\Theta)\Delta\Phi}{2}, \qquad (13)$$

where $\Delta\Phi = \Phi - \phi (= \omega t)$. As a result, we arrive at the forms showing that the values of the dynamic phases

$$\beta_\pm = \mp\frac{\omega_L\cos(\theta-\Theta)}{2}t \qquad (14)$$

deviate by the Aharonov–Anandan phases:

$$\frac{(1\pm\cos\Theta)\Delta\Phi}{2}. \qquad (15)$$

The Aharonov–Anandan phases are equal and can be obtained in the considered case directly from the known spinor parts, also, through integration over the corresponding holonomy connections

$$\gamma_\pm = \alpha_\pm - \beta_\pm = i\int_0^t (\tilde{\Psi}_\pm, \frac{\partial}{\partial\tau}\tilde{\Psi}_\pm)d\tau\,. \qquad (16)$$

These changes vanish when there is no precession ($\omega = 0$) and equal the Berry phases in the adiabatic mode ($\omega \to 0$)

$$\frac{(1 \pm \cos\theta)\Delta\Phi}{2}. \tag{17}$$

For the components of the observable polarization vector, in the precession case after the cyclic evolution ($t = 2\pi/\omega$), and with the non-essential simplified assumptions

$$(C_+, C_-) = (1/\sqrt{2}, i/\sqrt{2}),\ \phi = 0,$$

the following expressions can be obtained

$$P_x = \sin\Theta \sin(\alpha_- - \alpha_+), \tag{18}$$

$$P_y = \cos(\alpha_- - \alpha_+), \tag{19}$$

$$P_z = -\cos\Theta \sin(\alpha_- - \alpha_+), \tag{20}$$

where

$$\alpha_- - \alpha_+ = \frac{2\pi\omega_L \cos(\theta - \Theta)}{\omega} + \Omega(\Theta)$$

and

$$\Omega(\Theta) = 2\pi(1 - \cos\Theta)$$

is the solid angle traversed by the magnetic field.

These expressions show that if we subtract the Larmor (local) precession, the polarization of the neutron evaluates as a classic vector parallel transport under the field precession with a finite angular rate ω: the solid angle law is conserved. So, in agreement with Definition 2, the neutron spin evolutions under magnetic field precession can be qualified as weak non-adiabatic changes.

IV. General setting

The consideration of spin evolution in a precessing field (in Sec.III) was to illustrate the holonomic phenomena of Berry and Aharonov–Anandan on the level of an exactly solvable problem. But, is it enough to know the precessing field case for the solving the problem announced in Sec.I : calculation of the geometric phase for an arbitrary (and with a finite rate) line of the excursion of a magnetic field on Poincare's sphere? The first consideration connected with this problem, as was mentioned, is dealt with

in [3]. It is not an abstraction: one arrives at the setting of this question from the analysis of experiments, as well [12].

By the way, the conclusions that arise relating to the basic assumption of Aharonov–Anandan approach:

Conclusion 1. To be exact, from the cyclicity on the level of vector-parameter

$$\vec{R}(T) = \vec{R}(0)$$

the corresponding quantum cyclic analog does not follow:

$$\Psi(T) = e^{i\alpha(T)}\Psi(0)$$

but instead,

$$\Psi(T) = \sum_m C_m(0) e^{i\alpha_m(T)} \Psi_m(0). \tag{21}$$

So, the classic cycle does not directly mean the corresponding quantum cycle.

Conclusion 2. Moreover, for an arbitrary moment of time t, the expression

$$e^{i\alpha(t)}\Psi(t)$$

can not be accepted as the general form for the quantum system under non-adiabatic evolution. So, the Aharonov–Anandan time integration [7] is valid when the general form

$$\Psi(t) = \sum_m C_m(0) e^{i\alpha_m(t)} \Psi_m(t) \tag{22}$$

is reduced, as in the case of precession, to the separate evolutions of the partial (with fixed quantum number) basic states.

Let us consider the general setting. Dirac's standard substitution

$$\Psi(t) = \sum_m C_m(t) \Psi_m(t),$$

for the Schrödinger equation (the corresponding $\Psi(t)$ spectrum is assumed to be non-degenerate) gives a system of first-degree linear equations:

$$\dot{C}_n = \sum_m C_m A_{mn}.$$

Having the aim of understanding the case of neutron spin evolutions in detail, we shall limit our consideration to a two-level systems analysis. As a consequence, we obtain well-known, standard expressions, i.e, a system of two first-degree linear equations

$$-i\dot{C}_1 = A(t)C_1(t) + B(t)C_2(t), \tag{23}$$

$$-i\dot{C}_2 = C(t)C_1(t) + D(t)C_2(t). \tag{24}$$

A, B, C, D are equal

$$A(t) = i(\Psi_1, \dot{\Psi}_1) - \frac{(\Psi_1, \hat{H}\Psi_1)}{\hbar}, \; B(t) = i(\Psi_1, \dot{\Psi}_2), \tag{25}$$

$$C(t) = i(\Psi_2, \dot{\Psi}_1), \; D(t) = i(\Psi_2, \dot{\Psi}_2) - \frac{(\Psi_2, \hat{H}\Psi_2)}{\hbar}, \tag{26}$$

where $i(\Psi_m, \dot{\Psi}_n)$ are the so-called coefficients of the holonomy connection induced by $\Psi_n(q, \vec{R}(t))$. As is known, we can separate the equations for C_1 and C_2 by passing to second-degree differential equations. They are easy to obtain:

$$B\ddot{C}_1 - \Big(\dot{B} + i(A+D)B\Big)\dot{C}_1 + \Big(i(\dot{B}A - B\dot{A}) + B^2C\Big)C_1 = 0, \tag{27}$$

$$D\ddot{C}_2 - \Big(\dot{D} + i(C+B)D\Big)\dot{C}_2 + \Big(i(\dot{D}C - D\dot{C}) + D^2A\Big)C_2 = 0. \tag{28}$$

At last, with the exponential substitutions

$$C_s = e^{i\alpha_s} = e^{i(\gamma_s+\beta_s)} = e^{i(\int^t g_s d\tau - \int^t \omega_s d\tau)}, \; s = 1, 2,$$

the above equations can be rewritten in the following conventional form,

$$p_s\ddot{C}_s + q_s\dot{C}_s + r_sC_s = 0, \; s = 1, 2, \tag{29}$$

which leads to the following first-degree quadratic equation for g:

$$p_s(\dot{g}_s + ig_s^2) + (q_s - 2ip_s\omega_s)g_s - (p_s\dot{\omega}_s - ip_s{\omega_s}^2 + q_s\omega_s + ir_s) = 0 \tag{30}$$

In essence, these are the most general equations determining the geometric phases for two-level systems.

Further steps leading to the solution of these equations (or equations (27), (28)), depend on the behavior of the holonomic coefficients and the expressions above combined from the latter. In a better case, they can be reduced to ones from the known list of common first (or second)-degree differential equations [13].

Nevertheless, it is possible to simplify the above problem with the following conclusions: evidently the wave function Ψ can be represented in a general form some what different from (22):

$$\Psi = e^{iu}(e^{iz}\Psi_1 + e^{-iz}\Psi_2), \tag{31}$$

eliminating of the so-called global phase u. The usefulness of this form becomes evident after its substitution into (23), (24): we arrive to the following system of the first-degree equations

$$2i\dot{z} = A - D + Be^{-2iz} - Ce^{2iz}, \tag{32}$$

$$2i\dot{u} = A + D + Be^{-2iz} + Ce^{2iz} \tag{33}$$

that are generally transcendent, but with a separate equation for one of the unknown phases, z.

If we constrain our consideration, however, to observables which do not contain time derivatives (like the polarization vector), an easier scheme for the conclusions is possible: for the specified class of observables, it is not necessary to know the global phase e^{iu} and, really, we can consider and try to find only the following partial solution

$$\Psi = e^{iz}\Psi_1 + e^{-iz}\Psi_2,$$

or equivalent one

$$\Psi = \sin(z)\tilde{\Psi}_1 + \cos(z)\tilde{\Psi}_2. \tag{34}$$

In Sec.V, we explore the representation (34) in the problem which, of course, can be viewed as an alternative one to the precession problem considered in Sec.III.

V. Nutation

Let us assume that the time-dependence of the magnetic field components has the following unusual form

$$H_x = H\sin(\omega t + \theta)\cos\phi,$$
$$H_y = H\sin(\omega t + \theta)\sin\phi, \tag{35}$$
$$H_z = H\cos(\omega t + \theta),$$

where θ is the initial (axial) polar angle and ϕ is the azimutal angle fixed during the magnetic field changes.

Naturally, (35) associates directly with the motion on sphere as an alternative to precession, i.e, n u t a t i o n. As a consequence, we accept and use this terminology in our consideration.

Is it possible to solve Schrödinger's equation for neutron spin evolution with the time-dependence given in (35)? Are there principal differences between neutron spin evolutions in a nutating field and the well-known case of the precessing field? Here it will be shown that this problem is solvable exactly and the mentioned difference is essential.

The substitution of (35) into Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\Psi = |\mu|\vec{H}\hat{\vec{\sigma}}\Psi$$

gives us the following

$$\begin{pmatrix} \dot{\psi}^1 \\ \dot{\psi}^2 \end{pmatrix} = -\frac{i\omega_L}{2}\begin{pmatrix} \cos(\omega t) & \sin(\omega t)e^{-i\Phi} \\ \sin(\omega t)e^{i\Phi} & -\cos(\omega t) \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \tag{36}$$

where ω is the characteristic frequency of the nutation. For simplicity, we assume $\theta = 0$.

Let us try to find the solution in the following form:

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = a(t)\Psi_1 + b(t)\Psi_2 =$$
$$a(t)\begin{pmatrix} \cos(\frac{\omega t}{2})e^{-i\Phi} \\ \sin(\frac{\omega t}{2}) \end{pmatrix} e^{-\frac{i\omega_L t}{2}} + b(t)\begin{pmatrix} -\sin(\frac{\omega t}{2})e^{-i\Phi} \\ \cos(\frac{\omega t}{2}) \end{pmatrix} e^{\frac{i\omega_L t}{2}} \tag{37}$$

As a result, we obtain

$$\dot{a}(t)\begin{pmatrix} \cos(\frac{\omega t}{2})e^{-i\Phi} \\ \sin(\frac{\omega t}{2}) \end{pmatrix} e^{-\frac{i\omega_L t}{2}} + a(t)\frac{\omega}{2}\begin{pmatrix} -\sin(\frac{\omega t}{2})e^{-i\Phi} \\ \cos(\frac{\omega t}{2}) \end{pmatrix} e^{-\frac{i\omega_L t}{2}} +$$

$$+\dot{b}(t)\begin{pmatrix} -\sin(\frac{\omega t}{2})e^{-i\Phi} \\ \cos(\frac{\omega t}{2}) \end{pmatrix} e^{\frac{i\omega_L t}{2}} - b(t)\frac{\omega}{2}\begin{pmatrix} -\sin(\frac{\omega t}{2})e^{-i\Phi} \\ \cos(\frac{\omega t}{2}) \end{pmatrix} e^{\frac{i\omega_L t}{2}} = 0,$$

or

$$\dot{a}\Psi_1 + a\frac{\omega}{2}\Psi_2 e^{-i\omega_L t} + \dot{b}\Psi_2 - b\frac{\omega}{2}\Psi_1 e^{i\omega_L t} = 0 \tag{38}$$

Further, using the parameterization announced in Sec.IV:

$$a(t) = \sin z(t)\ ,\ b(t) = \cos z(t),$$

we obtain the following:

$$\dot{z}\Big(\cos z\Psi_1 - \sin z\Psi_2\Big) = \frac{\omega}{2}\Big(\cos z\Psi_1 e^{i\omega_L t} - \sin z\Psi_2 e^{-i\omega_L t}\Big).$$

After multiplying on the $\Big(\cos z^*\Psi_1^* - \sin z^*\Psi_2^*\Big)$, we have:

$$\dot{z}\cosh(2Im(z)) = \frac{\omega}{2}\Big(\cosh(2Im(z))\cos(\omega_L t) + i\cos(2Re(z))\sin(\omega_L t)\Big), \tag{39}$$

or

$$\frac{dRe(z)}{d\tau} = \frac{\omega}{2}\cos(\omega_L\tau), \tag{40}$$

$$\frac{dIm(z)}{dt} = \frac{\omega}{2}\frac{\cos(2Re(z))}{\cosh(2Im(z))}\sin(\omega_L\tau). \tag{41}$$

Let us consider the integration of $Re(z)$:

$$Re(z) = \frac{\omega}{2}\int_0^t \cos(\omega_L\tau)d\tau = \frac{\omega}{2\omega_L}\sin(\omega_L t). \tag{42}$$

This result can be written in another useful form, too:

$$Re(z) = \frac{\omega t}{2} - \frac{1}{2}\int_0^t (1-\cos(\omega_L\tau))d(\omega\tau). \tag{43}$$

This expression equals the solid angle drawn by the unit vector in the direction of the nutation around an instantaneous position of the polarization vector projection on the local azimutal ("horizontal") plane. As we will see below, the real part of z describes the quantum corrections to the classical parallel transport law in the meridian plane ("vertical" drift). Integration of the imaginary part gives the following:

$$Im(z) = \frac{1}{2}\operatorname{arcsinh}\int_0^t \cos(2Re(z))\sin(\omega_L\tau)d\tau \tag{44}$$

or

$$Im(z) = \frac{1}{2}\operatorname{arcsinh}\int_0^t \cos\Big(\frac{\omega}{\omega_L}\sin(\omega_L t)\Big)\sin(\omega_L\tau)d\tau. \tag{45}$$

The latter represents an integral which can be expressed due to incomplete cylindrical functions [9]:

$$\epsilon_\nu(i\beta, z) = \frac{1}{\pi i}\int_0^{i\beta} e^{z sh t - \nu t}dt = \frac{1}{\pi}\int_0^{\beta} e^{z sin\theta - \nu\theta}d\theta,$$

or Weber functions

$$B_\nu(\beta, z) = \frac{1}{\pi}\int_0^{\beta}\sin(\nu\theta - z\sin\theta)d\theta.$$

As a result, we obtain

$$Im(z) = \tfrac{1}{2}\,\text{arcsinh}\Big[\tfrac{\omega\pi}{4i\omega_L}\cdot$$

$$\Big(-\epsilon(i\omega_L t, -\tfrac{\omega}{\omega_L}) + \epsilon^*(i\omega_L t, -\tfrac{\omega}{\omega_L}) - \epsilon(i\omega_L t, \tfrac{\omega}{\omega_L}) + \epsilon^*(i\omega_L t, \tfrac{\omega}{\omega_L})\Big)\Big] = \qquad (46)$$

$$= \tfrac{1}{2}\,\text{arcsinh}\Big[\tfrac{\omega\pi}{2\omega_L}\Big(B_1(\omega_L t, -\tfrac{\omega}{\omega_L}) + B_1(\omega_L t, \tfrac{\omega}{\omega_L})\Big)\Big].$$

Let us consider the expressions for the components of the polarization vector P. Under two simplified assumptions: $\phi = 0$ and $t = 2\pi/\omega_L$, it is easy to obtain the following remarkable expressions

$$P_x = \frac{1}{\cosh(2Im(z))}\sin\Big(2Re(z) - 2\pi\frac{\omega}{\omega_L}\Big) \qquad (47)$$

$$P_y = \tanh(2Im(z)) \qquad (48)$$

$$P_z = \frac{1}{\cosh(2Im(z))}\cos\Big(2Re(z) - 2\pi\frac{\omega}{\omega_L}\Big). \qquad (49)$$

So, the variable weight in (39) is a principal value in our conclusions. It is generated by the imaginary part of the geometric phase and can not be ignored (calibrated) because it carries information about the spin evolution. It is important that the quantum phases in the arguments above do not vanish for the elementary cyclic nutation: the excursion of the magnetic field along the one of the meridians during the first half of the period, π/ω, and in the opposite direction for the second half of the period.

Conclusion 3 (Theorem). The lines on Poincare's sphere for a magnetic field can be iterated by the consequence of two alternative types of traverse: precession and nutation. Corresponding evolutions for the polarization vector differ essentially: weak non-adiabatic changes in the case of precession and strong non-adiabatic changes in the case of nutation.

Let us try to apply this approach to the more complex problem discussed by Berry in [4].

VI. Twisted Landau-Zener problem

This problem corresponds to the following magnetic field configuration:

$$H_x = \Delta\cos(\Phi(t)),$$
$$H_y = \Delta\sin(\Phi(t)), \qquad (50)$$
$$H_z = At.$$

The question arising in connection with this problem is the following: if the consideration of the problem correct, an easy correspondence rule must be realized. This problem, for $A \to 0$, ought to transform into the precession problem the discussed above. However, the expression for the geometric phase obtained in [4] has a singularity in this limit:

$$\Gamma_g = -\pi\frac{B\Delta^2}{A^2}sgn(A) \to \infty,$$

where $B = \ddot{\Phi}/2 = const$. This circumstance points out that the quasi-classical conclusions founded on the representation

$$\Gamma_g = -2Im\int_0^{i\Delta/A} d\tau\dot{\Phi}\cos\theta$$

can not be accepted as satisfactory and must be improved.

Let us search for the solution in the following form:

$$\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \sin z(t)\begin{pmatrix}\cos(\frac{\Theta(t)}{2})e^{-i\Phi(t)}\\ \sin(\frac{\Theta(t)}{2})\end{pmatrix}e^{i(\beta_+ + \gamma_+)} +$$
$$\cos z(t)\begin{pmatrix}-\sin(\frac{\Theta(t)}{2})e^{-i\Phi(t)}\\ \cos(\frac{\Theta(t)}{2})\end{pmatrix}e^{i(\beta_- + \gamma_-)}, \qquad (51)$$

where

$$\beta_\pm = \mp\frac{1}{2}\int_0^t \omega_L(\tau)\cos(\theta(\tau) - \Theta(\tau))d\tau, \qquad (52)$$

$$\gamma_\pm = \frac{1}{2}\int_0^t (1 \pm \cos\Theta(\tau))d\Phi, \qquad (53)$$

and

$$\omega_L(t) = 2\mu H(t)/\hbar\,,\ H(t) = \sqrt{\Delta^2 + A^2t^2}\,, \qquad (54)$$

$$\theta(t) = \arctan\Big(\frac{\Delta}{At}\Big), \qquad (55)$$

$$\cos\frac{\Theta(t)}{2} = \sqrt{\frac{\Lambda(t) + \omega_L(t)\cos\theta - \omega(t)}{2\Lambda(t)}}, \tag{56}$$

$$\omega(t) = \frac{d\Phi(t)}{dt}. \tag{57}$$

$\Lambda(t)$ is determined as in Sec.III.

Steps similar to the ones in Sec.V give us the following expressions:

$$Re(z) = \frac{1}{2}\int_0^t \cos(\Delta\alpha(\tau))d\Theta, \tag{58}$$

$$Im(z) = \frac{1}{2}\,\mathrm{arcsinh}\int_0^t \cos(2Re(z))\sin(\Delta\alpha(\tau))d\Theta, \tag{59}$$

where $\Delta\alpha = \alpha_- - \alpha_+ = \beta_- - \beta_+ + \gamma_- - \gamma_+$.

These values vanish when $A \to 0, \dot{\Phi} \to 0$ and, as a result, we obtain the proper limit: the state corresponding to the evolution in the precessing field.

The most general expressions (58) and (59) do not reduce to known incomplete functions, but they can be simplified and reduced simplified and can be reduced to the latter in two important limiting modes: the plane accelerated mode

$$A \to 0\,,\ \dot{\Phi} \neq 0$$

and the untwisted mode or axial lift

$$\Phi(t) = const.$$

This problem, of course, deserves a more detailed description, but we are constrained here, it seems, by the most essential remarks above.

Summary

So, generally, the picture of quantum evolution differs from the (quasi) classical parallel transport. Particularly, as we have seen, the picture of neutron spin evolution naturally contains the nutation mode, and accounting for the nutation mode in the geometric phase iterative calculation in the general case, i.e., for the arbitrary configuration of the magnetic field traversed by a finite rate, can not be ignored. Conceptually, we have had to accept, for logical completeness, that Berry's anzats (the existence of the holonomy connection) must be arranged over the amplitudes and constants of normalization, too. As a consequence, we summarized that the general consideration ought to be non-unitary. Generally, in a quantum setting, the term "topological phase" is incorrect: the precise notion is definitely "geometric phase."

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