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# REPRESENTATIONS FOR THREE-BODY $T$-MATRIX ON UNPHYSICAL SHEETS: PROOFS[2]

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## 1. INTRODUCTION

The paper is a continuation of the author's work [1] devoted to studying a structure of the $T$-matrix, scattering matrices and resolvent of three-body Hamiltonian continued analytically on unphysical sheets of the energy Riemann surface.

A central result of the paper [1] consists in construction of the explicit representations for the continuation of three-body $T$-matrix on unphysical sheets in terms of this matrix itself taken on the physical one, as well as the scattering matrices. There were outlined only schemes to prove the representations above in Ref. [1]. Main goal of the present work is to present a full proof. With the representations for $T$-matrix we base also analogous representations for analytical continuation of the scattering matrices and resolvent (see Ref. [1]).

As in [1] we suppose that interaction potentials are pairwise ones which decrease in the coordinate space not slower than exponentially. All the analysis is carried out on the base of the momentum space Faddeev integral equations [2], [3] for components of the $T$-matrix. At that we find analytical continuation of the Faddeev equations as on neighboring unphysical sheets as on remote ones belonging to a certain part of the total three-body Riemann surface. A full description of the part under consideration see in Ref. [1]. The representations for the components of $T$-matrix on unphysical sheets arise as a result of explicit solving the Faddeev equations continued in terms of the physical sheet.

Note that a continuation of the s-wave Faddeev equations on unphysical sheets neighboring with physical one, was made previously in the work [4] (see also Ref. [5]) in the case of separable pair potentials.

In the paper, we discuss also a practical meaning of the representations obtained. According to the representations [see Eqs. (4.34), (5.1) and (6.1)], the nontrivial singularities of the $T$-matrix as well as the scattering matrices and resolvent are determined, after the continuation of them on unphysical sheets by singularities of the operators inverse to truncated scattering matrices on the physical sheet. Thus, the three-body resonances (i.e. the poles of the resolvent as well the $T$- and scattering matrices) are actually those values of energy for which the scattering matrices, truncated in accordance with the index (number) of the unphysical sheet under consideration, have zero eigenvalue. These properties of three-body scattering matrices are quite analogous to the familiar properties of the scattering matrices in problems of two particles and multichannel scattering problems with binary channels (see e.g., Refs. [6]-[8] or [5], [9], [10]). For computations of three-body resonances as zeros of the truncated scattering matrices above, one can apply in particular, the differential formulation of the scattering problem [3], [11] going on the complex plane of energy (physical sheet).

The paper is organized as follows.

In Sec. 2 we remember main notations of Ref. [1]. The analytical continuation of the Faddeev equations on unphysical sheets is carried out in Sec. 3. Sec. 4 is devoted to deriving the explicit representations for the Faddeev components of the three-body $T$-matrix continued on unphysical sheets. The analogous representations are constructed in Sec. 5 for the scattering matrices and in Sec. 6, for the resolvent. In Sec. 7 we formulate an algorithm to calculate the three-body resonances on the base of the Faddeev differential equations in configuration space.

## 2. NOTATIONS

Throuhout the paper we follow strictly by the conventions and notations adopted in Ref. [1]. Therefore we restrict ourselves here only to presenting for them a brief summary. Note at once that at using formulae of the paper [1] (it will take place rather often) we supply their number in Ref. [1] with the reference "[1]".

For the description of the system of three particles concerned in the momentum representation, we use the standard sets of reduced relative momenta ([1].2.1) $k_\alpha$, $p_\alpha$, $\alpha = 1,2,3$, which are usually combined into six-vectors $P = \{k_\alpha, p_\alpha\}$. Transition from the pair $\{k_\alpha, p_\alpha\}$ to another one, $\{k_\beta, p_\beta\}$, corresponds to the rotation transform in $\mathbf{R}^6$, $k_\alpha = c_{\alpha\beta}k_\beta + s_{\alpha\beta}p_\beta$, $p_\alpha = -s_{\alpha\beta}k_\beta + c_{\alpha\beta}p_\beta$ with coefficients $c_{\alpha\beta}$, $s_{\alpha\beta}$ [3] depending on the particle masses only.

The Hamiltonian $H$ of the system is given by $(Hf)(P) = P^2 f(P) + \sum_{\alpha=1}^{3}(v_\alpha f)(P)$, $P^2 = k_\alpha^2 + p_\alpha^2$, $f \in \mathcal{H}_0 \equiv L_2(\mathbf{R}^6)$, where $v_\alpha$, $\alpha = 1,2,3$, are pair potentials assumed for the sake of definiteness, to be local. This means that the kernel of each $v_\alpha$ depends only on the difference of variables $k_\alpha$ and $k'_\alpha$, $v_\alpha(k_\alpha, k'_\alpha) = v_\alpha(k_\alpha - k'_\alpha)$.

We deal with two variants of the potentials $v_\alpha$. In the first one, $v_\alpha(k)$ are holomorphic functions of variable $k \in \mathbf{C}^3$ satisfying the estimate ([1].2.2). In the second variant, the potentials $v_\alpha(k)$ are holomorphic in $k$ in the strip $W_{2b} = \{k : k \in \mathbf{C}^3, |\mathrm{Im} k| < 2b\}$ only and obey at $k \in W_{2b}$ the estimate ([1].2.3). In the both variants $v_\alpha(-k) = \overline{v_\alpha(k)}$, and this guarantees self-adjointness of the Hamiltonian $H$.

In the paper, the exposition is given for example of the second variant of potentials. Respective statements for the first one may be obtained from the statements of this work if to put in them, $b = +\infty$.

By $h_\alpha$ we denote the Hamiltonians of the pair subsystems $\alpha$, $\alpha = 1,2,3$. Eigenvalues $\lambda_{\alpha,j} \in \sigma_d(h_\alpha)$ of $h_\alpha$, $\lambda_{\alpha,j} < 0$, $j = 1,2,...,n_\alpha$, $n_\alpha < \infty$, are enumerated taking into account their multiplicity: number of times to meet an eigenvalue in the numeration equals to its multiplicity. Maximal of these numbers is denoted by $\lambda_{\max}$, $\lambda_{\max} = \max\limits_{\alpha,j} \lambda_{\alpha,j} < 0$. The notation $\psi_{\alpha,j}(k_\alpha)$ is used for respective eigenfunctions.

We understand by $\sigma_d(H)$ and $\sigma_c(H)$ respectively the discrete and continuous components of the spectrum $\sigma(H)$ of the Hamiltonian $H$. Note that $\sigma_d(H) = (\lambda_{\min}, +\infty)$ with $\lambda_{\min} = \min\limits_{\alpha,j} \lambda_{\alpha,j}$.

The notation $H_0$ is adopted for the kinetic energy operator, $(H_0 f)(P) = P^2 f(P)$. By $R_0(z)$ and $R(z)$ we denote the resolvents of $H_0$ and $H$, respectively: $R_0(z) = (H_0 - zI)^{-1}$ and $R(z) = (H - zI)^{-1}$ with $I$, the identity operator in $\mathcal{H}_0$.

Let $M_{\alpha\beta}(z) = \delta_{\alpha\beta}v_\alpha - v_\alpha R(z) v_\beta$, $\alpha, \beta = 1,2,3$, be the components [2], [3] of the $T$-matrix $T(z) = V - VR(z)V$ where $V = v_1 + v_1 + v_3$. The Faddeev equations [2], [3] for operators $M_{\alpha\beta}$ read in matrix form as

$$M(z) = \mathbf{t}(z) - \mathbf{t}(z)\mathbf{R}_0(z)\Upsilon M(z) \tag{2.1}$$

where $\mathbf{R}_0(z) = \mathrm{diag}\{R_0(z), R_0(z), R_0(z)\}$ and by $\Upsilon$ we understand the $3 \times 3$-matrix with elements $\Upsilon_{\alpha\beta} = 1 - \delta_{\alpha\beta}$. Besides we use the notations $\mathbf{t}(z) = \mathrm{diag}\{\mathrm{t}_1(z), \mathrm{t}_2(z), \mathrm{t}_3(z)\}$. Here, the operators $\mathbf{t}_\alpha(z)$, $\alpha = 1,2,3$, have the kernels $\mathbf{t}_\alpha(P, P', z) = t_\alpha(k_\alpha, k'_\alpha, z - p_\alpha^2)\delta(p_\alpha - p'_\alpha)$ where $t_\alpha(k, k', z)$ stand for the respective pair $T$-matrices $t_\alpha(z)$. These $M$, $\mathbf{t}$, $\mathbf{R}_0$ and $\Upsilon$ are considered as operators in the Hilbert space $\mathcal{G}_0 = \bigoplus\limits_{\alpha=1}^{3} L_2(\mathbf{R}^6)$.

The resolvent $R(z)$ of $H$ is expressed by the matrix $M(z)$ as [3]

$$R(z) = R_0(z) - R_0(z)\Omega M(z)\Omega^\dagger R_0(z) \tag{2.2}$$

where $\Omega$, $\Omega : \mathcal{G}_0 \to \mathcal{H}_0$, denotes operator defined as the matrix-row $\Omega = (1,\ 1,\ 1)$. At the same time $\Omega^\dagger = \Omega^* = (1,\ 1,\ 1)^\dagger$.

Everywhere by $\sqrt{z-\lambda}$, $z \in \mathbf{C}$, $\lambda \in \mathbf{R}$, we understand the main branch of the function $(z-\lambda)^{1/2}$. Usually, by $\hat{q}$ we denote the unit vector in the direction $q \in \mathbf{R}^N$, $\hat{q} = q/|q|$, and by $S^{N-1}$ the unit sphere in $\mathbf{R}^N$, $\hat{q} \in S^{N-1}$. The inner product in $\mathbf{R}^N$ is denoted by $(\cdot,\cdot)$. Notation $\langle\cdot,\cdot\rangle$ is used for inner products in Hilbert spaces.

Let $\mathcal{H}^{(\alpha,j)} = L_2(\mathbf{R}^3)$ and $\mathcal{H}^{(\alpha)} = \bigoplus\limits_{j=1}^{n_\alpha} \mathcal{H}^{(\alpha,j)}$. Notation $\Psi_\alpha$ is used for operator acting from



$\mathcal{H}^{(\alpha)}$ to $\mathcal{H}_0$ as $(\Psi_\alpha f)(P) = \sum_{j=1}^{n_\alpha} \psi_{\alpha,j}(k_\alpha) f_j(p_\alpha)$. By $\Psi$ we understand the matrix-diagonal operator combined of $\Psi_\alpha$ as $\Psi = \mathrm{diag}\{\Psi_1, \Psi_2, \Psi_3\}$, and acting from $\mathcal{H}_1 = \bigoplus\limits_{\alpha=1}^{3} \mathcal{H}^{(\alpha)}$ to $\mathcal{G}_0$.

The operators $\Phi_\alpha$ and $\Phi$ are obtained of $\Psi_\alpha$ and $\Psi$ by the replacement of functions $\psi_{\alpha,j}(k_\alpha)$ with form-factors $\phi_{\alpha,j}(k_\alpha) = (v_\alpha\psi_{\alpha,j})(k_\alpha)$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$.

By $\mathcal{O}(\mathbf{C}^N)$ we denote the Fourier transform of the space $C_0^\infty(\mathbf{R}^N)$.

The operator $\mathrm{J}_{\alpha,j}(z)$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, realizes the restriction of functions $f(p_\alpha)$, $p_\alpha \in \mathbf{R}^3$, on the energy shell $\lambda_{\alpha,j} + |p_\alpha|^2 = E$ at $z = E \pm i0$, $E > \lambda_{\alpha,j}$, and then if possible, continues them analytically into a domain of complex values of energy $z$. On $\mathcal{O}(\mathbf{C}^3)$, this operator acts as $(\mathrm{J}_{\alpha,j}(z)f)(\hat{p}_\alpha) = f(\sqrt{z - \lambda_{\alpha,j}}\hat{p}_\alpha)$. Notation $\mathrm{J}^\dagger_{\alpha,j}(z)$ is used for the operator "transposed" with respect to $\mathrm{J}_{\alpha,j}(z)$ (see. Ref. [1]).

The operator $\mathrm{J}_0(z)$ is defined on $\mathcal{O}(\mathbf{C}^6)$ analogously to $\mathrm{J}_{\alpha,j}(z)$ by $(\mathrm{J}_0(z)f)(\hat{P}) = f(\sqrt{z}\hat{P})$. The notation $\mathrm{J}_0^\dagger(z)$ is used for respective "transposed" operator [1].

The operators $\mathrm{J}_{\alpha,j}$ and $\mathrm{J}^\dagger_{\alpha,j}$ are combined in the diagonal matrices $\mathrm{J}^{(\alpha)}(z) = \mathrm{diag}\{\mathrm{J}_{\alpha,1}(z), ..., \mathrm{J}_{\alpha,n_\alpha}(z)\}$ and $\mathrm{J}^{(\alpha)\dagger}(z) = \mathrm{diag}\{\mathrm{J}^\dagger_{\alpha,1}(z), ..., \mathrm{J}^\dagger_{\alpha,n_\alpha}(z)\}$. In their turn, we construct of the latter, the operators $\mathrm{J}_1(z) = \mathrm{diag}\{\mathrm{J}^{(1)}(z), \mathrm{J}^{(2)}(z), \mathrm{J}^{(3)}(z)\}$ and $\mathrm{J}_1^\dagger(z) = \mathrm{diag}\{\mathrm{J}^{(1)\dagger}(z), \mathrm{J}^{(2)\dagger}(z), \mathrm{J}^{(3)\dagger}(z)\}$. Besides the listed ones, we use in the work, the block-diagonal operator 3×3-matrices $\mathbf{J}_0(z) = \mathrm{diag}\{\mathrm{J}_0(z), \mathrm{J}_0(z), \mathrm{J}_0(z)\}$ and $\mathbf{J}_0^\dagger(z) = \mathrm{diag}\{\mathrm{J}_0^\dagger(z), \mathrm{J}_0^\dagger(z), \mathrm{J}_0^\dagger(z)\}$ as well as operators $\mathbf{J}(z) = \mathrm{diag}\{\mathrm{J}_0(z), \mathrm{J}_1(z)\}$ and $\mathbf{J}^\dagger(z) = \mathrm{diag}\{\mathrm{J}_0^\dagger(z), \mathrm{J}_1^\dagger(z)\}$.

Along with $\mathcal{H}_0$, $\mathcal{G}_0$ and $\mathcal{H}_1$ described above, we consider the Hilbert spaces $\hat{\mathcal{H}}_0 = L_2(S^5)$, $\hat{\mathcal{G}}_0 = \bigoplus\limits_{\alpha=1}^{3} \hat{\mathcal{H}}_0$ and $\hat{\mathcal{H}}_1 = \bigoplus\limits_{\alpha=1}^{3} \hat{\mathcal{H}}^{(\alpha)}$ where $\hat{\mathcal{H}}^{(\alpha)} \equiv \bigoplus\limits_{j=1}^{n_\alpha} \hat{\mathcal{H}}^{(\alpha,j)}$, $\hat{\mathcal{H}}^{(\alpha,j)} = L_2(S^2)$. The identity operators in $\hat{\mathcal{H}}_0$, $\hat{\mathcal{G}}_0$, $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ are denoted by $\hat{I}_0$, $\hat{\mathbf{I}}_0$, $\hat{I}_1$ and $\hat{\mathbf{I}}$, respectively.

The operator-valued function $\mathcal{T}(z)$, $\mathcal{T}(z) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$, of the variable $z \in \mathbf{C} \setminus \overline{\sigma(H)}$ is defined by

$$\mathcal{T}(z) \equiv \begin{pmatrix} \Omega M(z)\Omega^\dagger & \Omega M(z)\Upsilon\Psi \\ \Psi^*\Upsilon M(z)\Omega^\dagger & \Psi^*(\Upsilon\mathrm{v} + \Upsilon M(z)\Upsilon)\Psi \end{pmatrix} \tag{2.3}$$

with $\mathrm{v} = \mathrm{diag}\{v_1, v_2, v_3\}$. The *truncated* three-body scattering matrices are expressed by $\mathcal{T}(z)$ as

$$S_l(z) \equiv \hat{\mathbf{I}} + (\tilde{L}\hat{\mathcal{T}}LA)(z) \quad \text{and} \quad S_l^\dagger(z) \equiv \hat{\mathbf{I}} + (AL\hat{\mathcal{T}}\tilde{L})(z) \tag{2.4}$$

where $\hat{\mathcal{T}}(z) = (\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z)$, $\hat{\mathcal{T}}(z) : \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$. The multi-index

$$l = (l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}) \tag{2.5}$$

has the components $l_0 = 0$ or $l_0 = \pm 1$ and $l_{\alpha,j} = 0$ or $l_{\alpha,j} = 1$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$. Notations $L$ and $\tilde{L}$ are used for the diagonal matrices corresponding to the multi-index $l$: $L = \mathrm{diag}\{L_0, L_1\}$, $\tilde{L} = \mathrm{diag}\{|L_0|, L_1\}$, $L_0 = l_0$ and $L_1 = \mathrm{diag}\{l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}\}$. By $A(z)$ we understand the diagonal matrix-function $A(z) = \mathrm{diag}\{A_0(z), A_{\alpha,j}(z), \alpha = 1,2,3, \ j = 1,2,...,n_\alpha\}$ with the elements $A_0(z) = -\pi i z^2$ and $A_{\alpha,j}(z) = -\pi i\sqrt{z - \lambda_{\alpha,j}}$.

The notation $\Pi^{\mathrm{hol}}_{ll'}$ is used for the domain in variable $z \in \mathbf{C}$ where $(L\hat{\mathcal{T}}L')(z)$ is a holomorphic operator-valued function. The matrices $S_l(z)$ and $S_l^\dagger(z)$ as well as the products $(L_0\mathbf{J}_0 M)(z)$, $(L_1\mathrm{J}_1\Psi^*\Upsilon M)(z)$ and $(M\mathbf{J}_0^\dagger L_0)(z)$, $(M\Upsilon\Psi\mathrm{J}_1^\dagger L_1)(z)$ are holomorphic functions of $z$ on domains $\Pi_l^{(\mathrm{hol})} = \Pi^{\mathrm{hol}}_{ll}$. A description of the domains $\Pi^{\mathrm{hol}}_{ll'}$ and $\Pi_l^{(\mathrm{hol})}$ see in Ref. [1], Sec 4.

We consider only a part of the total three-body Riemann surface. This part is denoted by $\Re$. Sheets $\Pi_l \subset \Re$ are generated by branching in the two-body, $z = \lambda_{\alpha,j}$, $\alpha = 1,2,3$, $j =$

$1,2,...,n_\alpha$, and three–body, $z = 0$, thresholds. When enumerating the sheets, the multi-index $l$ given by (2.5) is used. At $l_0 = 0$ its components $l_{\alpha,j}$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, can get arbitrary value among two numbers 0 and 1. In this case, $\Pi_l$ represents a copy of the complex plane $\mathbf{C}$ cut along the ray $[\lambda_{\min}, +\infty)$. If $l_0 = \pm 1$ then the rest of components $l_{\alpha,j}$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, of $l$ are assumed be equal to 1. There is accepted that at $l_0 = +1$ the sheet $\Pi_l$ coincides with the upper half-plane $\mathbf{C}^+ = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ and at $l_0 = -1$, with the lower one, $\mathbf{C}^- = \{z \in \mathbf{C} : \operatorname{Im} z < 0\}$. We suppose additionally that the sheets $\Pi_l$ with $l_0 = \pm 1$ are cut along the rays constituting together the set $Z_{\text{res}} = \bigcup_{\alpha=1}^{3} Z_{\text{res}}^{(\alpha)}$. Here, $Z_{\text{res}}^{(\alpha)} = \{z : z = z_r\rho, 1 \le \rho < +\infty, z_r \in \sigma_{\text{res}}^{(\alpha)}\}$ is a totality of the rays beginning in the resonance points $\sigma_{\text{res}}^{(\alpha)}$ of subsystem $\alpha$ and going to infinity along the directions $\hat{z}_r = z_r/|z_r|$, $z_r \in \sigma_{\text{res}}^{(\alpha)}$. A more detailed description of the surface $\Re$ and in particular, the way of sticking the sheets $\Pi_l$ see in Sec. 5 of Ref. [1].

If all the components of the multi-index $l$ are zero, $l_0 = l_{\alpha,j} = 0$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, the sheet $\Pi_l$ is called the physical one, $\Pi_0$. The unphysical sheets $\Pi_l$ with $l_0 = 0$ are called the two–body ones since these sheets may be reached from $\Pi_0$ rounding the two-body thresholds $z = \lambda_{\alpha,j}$ only, with no rounding the breakup threshold $z = 0$. The sheets $\Pi_l$ at $l_0 = \pm 1$ are called the three–body ones.

## 3. ANALYTICAL CONTINUATION OF FADDEEV EQUATIONS FOR COMPONENTS OF $T$–MATRIX ON UNPHYSICAL SHEETS

Goal of the present section consists in continuation on unphysical sheets of the surface $\Re$, of the absolute terms and kernels of the Faddeev equations (2.1) and their iterations. The continuation is realized in a sense of generalized functions (distributions) over $\mathcal{O}(\mathbf{C}^6)$. Results of the continuation are represented in terms related with the physical sheet only.

By $L^{(\alpha)}$, $L^{(\alpha)} = L^{(\alpha)}(l)$, we denote the diagonal matrices formed of the components $l_{\alpha,1}, l_{\alpha,2}, ..., l_{\alpha,n_\alpha}$ of the multi-index $l$ of the sheet $\Pi_l \subset \Re$: $L^{(\alpha)}) = \operatorname{diag}\{l_{\alpha,1}, l_{\alpha,2}, ..., l_{\alpha,n_\alpha}\}$. At that $L_1(l) = \operatorname{diag}\{L^{(1)}, L^{(2)}, L^{(3)}\}$ and $L(l) = \operatorname{diag}\{L_0, L_1\}$ c $L_0 \equiv l_0$. Analogously, $A^{(\alpha)}(z) = \operatorname{diag}\{A_{\alpha,1}(z), A_{\alpha,2}, ..., A_{\alpha,n_\alpha}(z)\}$ and $A_1(z) = \operatorname{diag}\{A^{(1)}(z), A^{(2)}(z), A^{(3)}(z)\}$. Thus $A(z) = \operatorname{diag}\{A_0(z), A_1(z)\}$.

By $\mathbf{s}_{\alpha,l}(z)$ we understand the operator defined in $\hat{\mathcal{H}}_0$ as

$$\mathbf{s}_{\alpha,l}(z) = \hat{I}_0 + \mathrm{J}_0(z)\mathbf{t}_\alpha(z)\mathrm{J}_0^\dagger(z)A_0(z)L_0, \quad z \in \Pi_0. \tag{3.1}$$

It follows from Eq. (3.1) that $\mathbf{s}_{\alpha,l} = \hat{I}_0$ at $l_0 = 0$. If $l_0 = \pm 1$ then according to Eqs. ([1].4.42)–[1].4.44), the operator $\mathbf{s}_{\alpha,l}(z)$ is defined for $z \in \mathcal{P}_b \bigcap \mathbf{C}^\pm$, $\mathcal{P}_b = \{z : \operatorname{Re} z > -b^2 + (\operatorname{Im} z)^2/(4b^2)\}$, and acts on $f \in \hat{\mathcal{H}}_0$ as

$$(\mathbf{s}_{\alpha,l}(z)f)(\hat{P}) = \int_{S^2} d\hat{k}' s_\alpha(\hat{k}_\alpha, \hat{k}'_\alpha, z\cos^2\omega) f(\cos\omega_\alpha \hat{k}'_\alpha, \sin\omega_\alpha \hat{p}_\alpha) \tag{3.2}$$

where $\omega_\alpha, \hat{k}_\alpha, \hat{p}_\alpha$ stand for coordinates [3] of the point $\hat{P}$ on the hypersphere $S^5$, $\omega_\alpha \in [0, \pi/2]$, $\hat{k}_\alpha, \hat{p}_\alpha \in S^2$. For all this, $\hat{P} = \{\cos\omega_\alpha \hat{k}_\alpha, \sin\omega_\alpha \hat{p}_\alpha\}$. By $s_\alpha$ we denote the scattering matrix ([1].2.16) for the pair subsystem $\alpha$. As a matter of fact, $\mathbf{s}_{\alpha,l}$ represents the scattering matrix $s_\alpha$ rewritten in the three–body momentum space.

It follows immediately from Eq. (3.2) that if $z \in \mathcal{P}_b \bigcap \mathbf{C}^\pm \setminus Z_{\text{res}}^{(\alpha)}$ then there exists the bounded inverse operator $\mathbf{s}_{\alpha,l}^{-1}(z)$,



$(\mathbf{s}_{\alpha,l}^{-1}(z)f)(\hat{P}) = \int_{S^2} d\hat{k}' s_\alpha^{-1}(\hat{k}_\alpha, \hat{k}'_\alpha, z\cos^2\omega_\alpha) f(\cos\omega_\alpha \hat{k}'_\alpha, \sin\omega_\alpha \hat{p}_\alpha)$ with $s_\alpha^{-1}(\hat{k}, \hat{k}', \zeta)$, the kernel of the inverse pair scattering matrix $s_\alpha^{-1}(\zeta)$.

The operator $\mathbf{s}_{\alpha,l}^{-1}(z)$ becomes unbounded one at the boundary points $z$ situated on rims of the cuts (the "resonance" rays) included in $Z_{\text{res}}^{(\alpha)}$.

THEOREM 1. *The absolute terms $\mathbf{t}_\alpha(P, P', z)$ and kernels $(\mathbf{t}_\alpha R_0)(P, P', z)$ of the Faddeev equations (2.1) admit the analytical continuation in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$ both on two-body and three–body sheets $\Pi_l$ of the Riemann surface $\Re$. The continuation on the sheet $\Pi_l$, $l = (l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3})$, $l_0 = 0$, $l_{\beta,j} = 0, 1$, or $l_0 = \pm 1$, $l_{3,j} = 1$ (in both cases $\beta = 1,2,3$, $j = 1,2,...,n_\beta$), is written as*

$$\mathbf{t}_\alpha^l(z) \equiv \mathbf{t}_\alpha(z)\big|_{\Pi_l} = \mathbf{t}_\alpha - L_0 A_0 \mathbf{t}_\alpha \mathrm{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1} \mathrm{J}_0 \mathbf{t}_\alpha - \Phi_\alpha \mathrm{J}^{(\alpha)t} L^{(\alpha)} A^{(\alpha)} \mathrm{J}^{(\alpha)} \Phi_\alpha^*, \tag{3.3}$$

$$[\mathbf{t}_\alpha(z) R_0(z)]\big|_{\Pi_l} = \mathbf{t}_\alpha^l(z) R_0^l(z) \tag{3.4}$$

*where $R_0^l(z) \equiv R_0(z)\big|_{\Pi_l} = R_0(z) + L_0 A_0(z) \mathrm{J}_0^\dagger(z) \mathrm{J}_0(z)$ is the continuation [9] on $\Pi_l$ of the free Green function $R_0(z)$. If $l_0 = 0$ (and hence $\Pi_l$ is a two–body unphysical sheet). the continuation in the form (3.3), (3.4) is possible on the whole sheet $\Pi_l$. At $l_0 = \pm 1$ (i.e. in the case when $\Pi_l$ is a three-body sheet) the form (3.3), (3.4) continuation is possible on the domain $\mathcal{P}_b \bigcap \Pi_l$. All the kernels in the right–hand parts of Eqs. (3.3) are taken on the physical sheet.*

PROOF of the theorem we give for example of the most intricate continuation on three-body unphysical sheets $\Pi_l$ with $l_0 = \pm 1$. For the sake of definiteness we consider the case $l_0 = +1$. For $l_0 = -1$, the proof is quite analogous.

Let us consider at $z \in \Pi_0$, $\operatorname{Im} z < 0$ the bilinear form

$$(f, \mathbf{t}_\alpha R_0(z) f') = \int_{\mathbf{R}^3} dk \int_{\mathbf{R}^3} dk' \int_{\mathbf{R}^3} dp \, \frac{t_\alpha(k, k', z - p^2)}{k'^2 + p^2 - z} \tilde{f}(k, k'p) \tag{3.5}$$

with $\tilde{f}(k, k'p) = f(k, p) f'(k', p)$, $f, f' \in \mathcal{O}(\mathbf{C}^6)$, $k = k_\alpha$, $k' = k'_\alpha$, $p = p_\alpha$. Making replacements of variables $|k'| \to \rho = |k'|^2$, $|p| \to \lambda = z - |p|^2$ we find that the integral (3.5) turns into

$$\frac{1}{4} \int_{\mathbf{R}^3} dk \int_{S^2} d\hat{k}' \int_{S^2} d\hat{p} \int_{z-\infty}^{z} d\lambda \sqrt{z - \lambda} \int_0^\infty d\rho \sqrt{\rho} \, \frac{t_\alpha(k, \sqrt{\rho}\hat{k}', \lambda)}{\rho - \lambda} \tilde{f}(k, \sqrt{\rho}\hat{k}', \sqrt{z - \lambda}\hat{p}). \tag{3.6}$$

Existence of the analytical continuation of the kernel $(\mathbf{t}_\alpha R_0)(z)$ on the sheet $\Pi_l$, $l_0 = \pm 1$. follows from a possibility to deform continuously the contour of integration over variable $\rho$ to arbitrary sector of the analyticity domain $\mathcal{P}_b \bigcap \sigma_{\text{res}}^{(\alpha)}$ of the integrand in variable $\lambda$ in the way demonstrated in Fig. 1. Besides, this is connected with a possibility at moving of $z$ from $\Pi_0$ to $\Pi_l$, $l_0 = +1$, to make a necessary deformation of the integration path in variable $\lambda$ in such a way that this path is separated from the integration contour in variable $\rho$.

To obtain the representation (3.4) at a concrete point $z = z_0$, we choose a special final location of the integration contours in variables $\lambda$ and $\rho$ after consistent deforming them (see Fig. 2). Singularity of inner integral (over variable $\rho$) remains integrable after such deformation

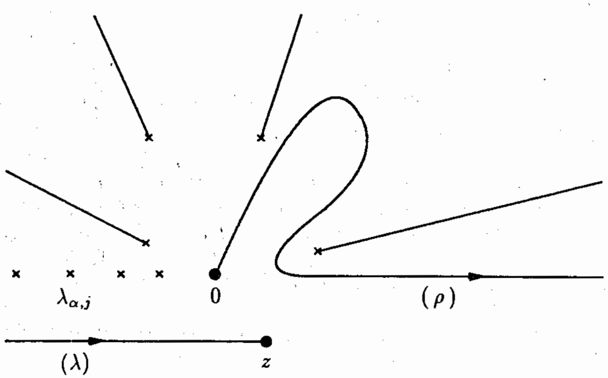


Figure 1: Deformation of the integration contour over variable $\rho$. The integration contours over $\rho$ and $\lambda$ are denoted by letters in brackets. The cross "$\times$" denotes the eigenvalues $\lambda_{\alpha,j}$ of $h_\alpha$ on the negative half-axis of the physical sheet and the pair resonances belonging to the set $\sigma^{(\alpha)}_{\text{res}}$ on the sheet $\Pi_l$, $l_0 = +1$. Also, there are denoted the cuts on $\Pi_l$, $l_0 = +1$, beginning at the points of $\sigma^{(\alpha)}_{\text{res}}$.

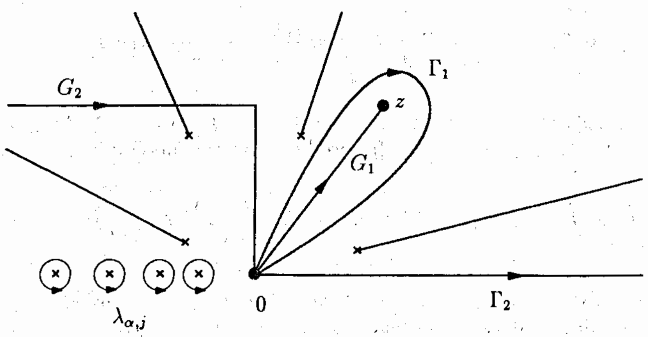


Figure 2: The final location of the integration contours over variables $\rho$ $(\Gamma_1 \bigcup \Gamma_2)$ and $\lambda$ $(G_1 \bigcup G_2)$. The contour $\Gamma_1$ represents a loop going clockwise around the countour $G_1$, the line segment $[0,\ z]$; $\Gamma_2 = [0, +\infty)$; $G_2 = (z - \infty,\ i\,\mathrm{Im}\, z] \bigcup [i\,\mathrm{Im}\, z,\ 0)$.



due to presence of the factor $\sqrt{\rho}$. As a whole the integral (3.6) turns into

$$\frac{1}{4}\int\limits_{\mathbf{R}^3} dk \int\limits_{S^2} d\hat{k}' \int\limits_{S^2} d\hat{p}\times$$

$$\times\left\{\int\limits_{G_1} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_1\bigcup\Gamma_2} d\rho\sqrt{\rho}\,\frac{t'_\alpha(k,\sqrt{\rho}\hat{k}',\lambda)}{\rho-\lambda}\,\tilde{f}(k,\sqrt{\rho}\hat{k}',\sqrt{z-\lambda}\hat{p}) + \right.$$

$$(3.7)\qquad + \int\limits_{G_2} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_1\bigcup\Gamma_2} d\rho\sqrt{\rho}\,\frac{t_\alpha(k,\sqrt{\rho}\hat{k}',\lambda)}{\rho-\lambda}\,\tilde{f}(k,\sqrt{\rho}\hat{k}',\sqrt{z-\lambda}\hat{p}) +$$

$$\left. + \sum_{j=1}^{n_\alpha} 2\pi i\sqrt{z-\lambda_{\alpha,j}} \int\limits_0^{+\infty} d\rho\sqrt{\rho}\,\frac{\phi_{\alpha,j}(k)\overline{\phi}_{\alpha,j}(k')}{\rho-\lambda_{\alpha,j}}\,\tilde{f}(k,\sqrt{\rho}\hat{k}',\sqrt{z-\lambda_{\alpha,j}}\hat{p})\right\}$$

where $t'_\alpha$ denotes the pair $T$-matrix $t_\alpha(z)$ continued on the second sheet (as regards $t_\alpha(\lambda)$, the contour $G_1 \ni \lambda$ belongs to its second sheet). The last term arises as a result of taking residues in the points $\lambda_{\alpha,j} \in \sigma_d(h_\alpha)$.

Evidently, the domain of variable $z \in \Pi_l$, $l_0 = +1$, where one can continue analytically the function (3.5) in the form (3.7) to, is determined by the conditions $\Gamma_1 \subset \mathcal{P}_b$ and $\Gamma_1 \bigcap Z^{(\alpha)}_{\text{res}} = \emptyset$. These conditions may be satisfied at $z \in \mathcal{P}_b$ only.

Note that value of the inner integrals over $\Gamma_1$ at $\lambda \in G_1$ are determined by residues at the points $\rho = \lambda$. At the same time $\int_{G_2} d\lambda \ldots \int_{\Gamma_1} \ldots = 0$ since at $\lambda \in G_2$ the functions under the integration sign are holomorphic in $\rho \in \mathrm{Int}\,\Gamma_1$. Therefore

$$(f, \mathrm{t}_\alpha R_0(z) f')\big|_{z\in\Pi_l, l_0=+1} = \frac{1}{4}\int\limits_{\mathbf{R}^3} dk \int\limits_{S^2} d\hat{k}' \int\limits_{S^2} d\hat{p}\times$$

$$\times\left\{\int\limits_{G_1} d\lambda\sqrt{z-\lambda}(-2\pi i)\sqrt{\lambda}\,t'_\alpha(k,\sqrt{\lambda}\hat{k}',\lambda)\tilde{f}(k,\sqrt{\lambda}\hat{k}',\sqrt{z-\lambda}\,\hat{p}) + \right.$$

$$(3.8)\qquad + \int\limits_{G_1} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_2} d\rho\sqrt{\rho}\,\frac{t_\alpha(k,\sqrt{\rho}\hat{k}',\lambda) + \pi i\sqrt{\lambda}\,\tau_\alpha(k,\sqrt{\rho}\hat{k}',\lambda)}{\rho-\lambda}\,\tilde{f}(k,\sqrt{\rho}\hat{k}',\sqrt{z-\lambda}\hat{p}) +$$

$$+ \int\limits_{G_2} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_2} d\rho\sqrt{\rho}\,\frac{t_\alpha(k,\sqrt{\rho}\hat{k}',\lambda)}{\rho-\lambda}\,\tilde{f}(k,\sqrt{\rho}\hat{k}',\sqrt{z-\lambda}\,\hat{p}) +$$

$$\left. + \sum_{j=1}^{n_\alpha} 2\pi i\sqrt{z-\lambda_{\alpha,j}} \int\limits_0^{+\infty} d\rho\sqrt{\rho}\,\frac{\phi_{\alpha,j}(k)\overline{\phi}_{\alpha,j}(k')}{\rho-\lambda_{\alpha,j}}\,\tilde{f}(k,\sqrt{\rho}\hat{k}',\sqrt{z-\lambda_{\alpha,j}}\,\hat{p})\right\}.$$

In the second summand of Eq. (3.8) we have used the representation ([1].3.2) of for the pair $T$-matrix continued on the second sheet. Look at the expression for $\tau_\alpha(k,k',\zeta)$ in Ref. [1], Sec. 3. Remember that $t'_\alpha(\zeta) = t_\alpha(\zeta) + \pi i\sqrt{\zeta}\,\tau_\alpha(\zeta)$.

Joining the summands including $t_\alpha$ on the physical sheet, in the alone integral $\int_{G_1\bigcup G_2}\cdots$ and using then the holomorphness of the function under the integration sign in variable $\lambda$, we straighten the contour $G_1 \bigcup G_2$ turning it into the ray $(z-\infty, z]$. As a result we get the bilinear form corresponding to the product $(\mathrm{t}_\alpha R_0)(z)$ taken on the physical sheet.

The last term of the expression (3.8) corresponds to the kernel of $-\Phi_\alpha \mathrm{J}^{(\alpha)\dagger} L^{(\alpha)} A^{(\alpha)} \mathrm{J}^{(\alpha)} \Phi^* R_0$.

Backing in the rest of summands including $t'_\alpha$ and $\tau_\alpha$ to the initial variables $k'$, $p'$ and utilizing then the definition (3.1), we find that these summands correspond to the expression

$$L_0 A_0 \left[\mathbf{t}_\alpha - L_0 A_0 \mathbf{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1} \mathbf{J}_0 \mathbf{t}_\alpha\right] \mathbf{J}_0^\dagger \mathbf{J}_0 - L_0 A_0 \mathbf{t}_\alpha \mathbf{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1} \mathbf{J}_0 \mathbf{t}_\alpha R_0.$$

Gathering the results obtained we reveal that the analytical continuation of $\mathbf{t}_\alpha R_0$ on the sheet $\Pi_l$, $l_0 = +1$, looks as

$$[\mathbf{t}_\alpha R_0(z)]\big|_{\Pi_l} = (\mathbf{t}_\alpha - L_0 A_0 \mathbf{t}_\alpha \mathbf{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1} \mathbf{J}_0 \mathbf{t}_\alpha - \Phi_\alpha \mathbf{J}^{(\alpha)\dagger} L^{(\alpha)} A^{(\alpha)} \mathbf{J}^{(\alpha)} \Phi_\alpha^*) \times \tag{3.9}$$
$$\times (R_0 + L_0 A_0 \mathbf{J}_0^\dagger \mathbf{J}_0) + L_0 A_0 \Phi_\alpha \mathbf{J}^{(\alpha)\dagger} L^{(\alpha)} A^{(\alpha)} \mathbf{J}^{(\alpha)} \Phi_\alpha^* \mathbf{J}_0^\dagger \mathbf{J}_0.$$

To be convinced in the factorization (3.4), is sufficient to note that the last summand of (3.9) equals to zero. Indeed, one can check easily that at $\operatorname{Im} z \neq 0$ or $\operatorname{Im} z = 0$ and $z > \max_j \lambda_{\alpha,j}$ the following equalities take place

$$(\mathbf{J}^{(\alpha)} \Phi_\alpha^* \mathbf{J}_0^\dagger)(z) = 0, \quad (\mathbf{J}_0 \Phi_\alpha \mathbf{J}^{(\alpha)\dagger})(z) = 0. \tag{3.10}$$

Thereby the last term of (3.9) disappears and hence, Eq. (3.4) is true. This completes the proof.

REMARK 1. As a matter of fact, the kernel $[\mathbf{t}_\alpha R_0](z)\big|_{\Pi_l}$ corresponds to the two-body problem and thereby it has to be translationally invariant with respect to variable $p_\alpha$. This fact may be understood if one introduces the generalized function (distribution) $\theta_z(p)$ over $\mathcal{O}(\mathbf{C}^3)$ acting as $(\theta_z, f) = \dfrac{1}{2} \displaystyle\int_{\gamma_z} d\xi \sqrt{\xi} \int_{S^2} d\hat{p} f(\sqrt{\xi}\hat{p})$ where $\gamma_z$ is the line segment connecting the points $\xi = 0$ and $\xi = z$. It follows from the representation (3.8) that the kernel of $[\mathbf{t}_\alpha R_0](z)\big|_{\Pi_l}$ may be rewritten as

$$(\mathbf{t}_\alpha R_0)^l(P, P', z) = \delta(p - p') \left\{ \frac{t_\alpha(k, k', z - p^2)}{k'^2 + p^2 - z} + \right.$$
$$+ \pi i L_0 \left[ \theta_z(p) \sqrt{z - p^2} \, \frac{\tau_\alpha(k, k', z - p^2)}{k'^2 + p^2 - z} - \theta_z(k') \sqrt{z - k'^2} \, t'_\alpha(k, k', k'^2) \frac{\delta(\sqrt{z - k'^2} - |p|)}{|p|^2} \right] +$$
$$\left. + \sum_{j=1}^{n_\alpha} 2\pi i \, l_{\alpha,j} \sqrt{z - \lambda_{\alpha,j}} \, \frac{\phi_{\alpha,j}(k) \overline{\phi}_{\alpha,j}(k')}{k^2 - \lambda_{\alpha,j}} \cdot \frac{\delta(|p| - \sqrt{z - \lambda_{\alpha,j}})}{|p|^2} \right\},$$
$$k = k_\alpha, \; k' = k'_\alpha, \; p = p_\alpha, \; p' = p'_\alpha,$$

where due to the presence of the factor $\delta(p - p')$, the translation invariance is emphasized explicitly. Analogously

$$\mathbf{t}_\alpha^l(P, P', z) = \left\{ t_\alpha(k, k', z - p^2) + \pi i \, L_0 \, \theta_z(p) \sqrt{z - p^2} \, \tau_\alpha(k, k' z - p^2) + \right.$$
$$\left. + \sum_{j=1}^{n_\alpha} 2\pi i \, l_{\alpha,j} \sqrt{z - \lambda_{\alpha,j}} \, \phi_{\alpha,j}(k) \overline{\phi}_{\alpha,j}(k') \cdot \frac{\delta(|p| - \sqrt{z - \lambda_{\alpha,j}})}{|p|^2} \right\} \delta(p - p').$$

Using Eqs. (3.3) and (3.4) one can present the Faddeev equations (2.1) continued on the sheet $\Pi_l$ in the matrix form

$$M^l(z) = \mathbf{t}^l(z) - \mathbf{t}^l(z) \mathbf{R}_0^l(z) \Upsilon M^l(z) \tag{3.11}$$



where

$$\mathbf{t}^l(z) = \mathbf{t} - L_0 A_0 \mathbf{t} \mathbf{J}_0^\dagger \mathbf{s}_l^{-1} \mathbf{J}_0 \mathbf{t} - \Phi \mathbf{J}_1^\dagger L_1 A_1 \mathbf{J}_1 \Phi^*, \tag{3.12}$$

$$\mathbf{R}_0^l(z) = \mathbf{R}_0(z) + L_0 A_0(z) \mathbf{J}_0^\dagger(z) \mathbf{J}_0(z). \tag{3.13}$$

Here, $\mathbf{s}_l(z) = \operatorname{diag}\{\mathbf{s}_{1,l}(z), \mathbf{s}_{2,l}(z), \mathbf{s}_{3,l}(z)\}$. By $M^l(z)$ we understand the supposed analytical continuation on the sheet $\Pi_l$ of the matrix $M(z)$.

LEMMA 1. *For each two-body unphysical sheet $\Pi_l$ of the surface $\Re$ there exists such a path from the physical sheet $\Pi_0$ to the domain $\Pi_l^{(\mathrm{hol})}$ in $\Pi_l$ going only on two-body unphysical sheets $\Pi_{l'}$ that moving by this path, the parameter $z$ stays always in respective domains $\Pi_{l'}^{(\mathrm{hol})} \subset \Pi_{l'}$.*

PROOF. Let us use the principle of mathematical induction. To make this, at the beginning we arrange the branching points $\lambda_{\alpha,j}$, $\alpha = 1, 2, 3;$ $j = 1, 2, ..., n_\alpha$, in nondecreasing order redenoting them as $\lambda_1, \lambda_2, ..., \lambda_m$, $m \leq \sum_\alpha n_\alpha$, $\lambda_1 < \lambda_2 < ... < \lambda_m$, and putting $\lambda_{m+1} = 0$. Let the multi-index $l = (l_1, l_2, ..., l_m)$ correspond temporarily namely to this enumeration. As previously, $l_j = 0$ if the sheet $\Pi_l$ is related to the main branch of the function $(z - \lambda_j)^{1/2}$ else $l_j = 1$. The index $l_0$ is omitted in these temporary notations.

It is clear that the transition of z from the physical sheet $\Pi_0$ across the segment $(\lambda_1, \lambda_2)$ on the neighboring unphysical theet $\Pi_{l^{(1)}}$ (into the domain $\Pi_{l^{(1)}}^{\mathrm{hol}}$), $l^{(1)} = (l_1^{(1)}, l_2^{(1)}, ...., l_m^{(1)})$ with $l_1^{(1)} = 1$ and $l_j^{(1)} = 0$ at $j \neq 1$, is possible by definition of the domain $\Pi_{l^{(1)}}^{\mathrm{hol}}$ (see Ref. [1]. Sec 4). According to Lemmas [1].1 and [1].2, if $z$ belongs to $\Pi_{l^{(1)}}^{(\mathrm{hol})}$, it may be lead to the real axis in the interval $(\lambda^{(1)}, +\infty)$ with certain $\lambda^{(1)} < \lambda_1$. Remaining in $\Pi_{l^{(1)}}^{(\mathrm{hol})}$ the point $z$ may even go around the threshold $\lambda_1$ crossing the real axis in the segment $(\lambda^{(1)}, \lambda_1)$. Thus, the parameter $z$ may be lead from the sheet $\Pi_{l^{(1)}}$ on the each neighboring unphysical sheet and in particular, on the sheet $\Pi_l$ related to $l_1 = 0$, $l_2 = 1$, $l_j = 0$, $j \geq 3$. Transition of $z$ from $\Pi_0$ across the segment $(\lambda_2, \lambda_3)$ on the sheet $\Pi_l$ with $l_1 = l_2 = 1$, $l_j = 0$, $j \geq 3$, is always possible.

We suppose further that the parameter $z$ may be carried in this manner from $\Pi_0$ on all the two-body unphysical sheets $\Pi_{l^{(k)}}$ defined by the conditions $l_j^{(k)} = 0$, $j > k$. It assumed also that during the carrying, $z$ always remains in the domains $\Pi_{l^{(k)}}^{(\mathrm{hol})}$ of these sheets and does not visit different sheets. It follows from Lemmas [1].1 and [1].2 that if $z$ stays in the domain $\Pi_{l^{(k)}}^{(\mathrm{hol})}$ of each sheet of the type described then wittingly, it can be lead to the real axis in the segment $(\lambda^{(k)}, +\infty)$ with certain $\lambda^{(k)} < \lambda_k$. Hence the parameter $z$ from each of the sheets $\Pi_{l^{(k)}}$ may be carried across the interval $(\lambda_k, \lambda_{k+1})$ on the neighboring unphysical sheet $\Pi_{l^{(k+1)}}$ with $l_j^{(k+1)} = 1 - l_j^{(k)}$, $j \leq k$, $l_{k+1}^{(k+1)} = 1$ and $l_j^{(k+1)} = 0$, $j > k + 1$. This means actually that $z$ may be carried from $\Pi_0$ on all the two-body unphysical sheets $\Pi_{l^{(k+1)}}$ with $l_j^{(k+1)} = 0$, $j > k + 1$. For all this the parameter $z$ remains in the holomorphness domains $\Pi_{l^{(k+1)}}^{(\mathrm{hol})}$ and does not visit the sheets $\Pi_{l^{(s)}}$ with $s > k + 1$. By the principle of mathematical induction we conclude that the parameter $z$ may be carried really on all the two-body unphysical sheets.

Proof is completed.

Using results of Sec. 4 of the paper [1] and Lemma 1, one can prove the following important statement.

THEOREM 2. *The iterations $\mathcal{Q}^{(n)}(z) = ((-\mathbf{t}\mathbf{R}_0\Upsilon)^n \mathbf{t})(z)$, $n \geq 1$, of absolute terms of the Faddeev equations (2.1) admit in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$, the analytical continuation on the domain $\Pi_l^{(\mathrm{hol})}$ of each unphysical sheet $\Pi_l \subset \Re$. This continuation is described by the equalities $\mathcal{Q}^{(n)}(z)\big|_{\Pi_l} = ((-\mathbf{t}^l \mathbf{R}_0^l \Upsilon)^n \mathbf{t}^l)(z)$.*

Remark 2. The products $L_1\mathrm{J}_1\Psi^*\Upsilon Q^{(m)}$, $Q^{(m)}\Upsilon\Psi\mathrm{J}_1^\dagger L_1$, $\tilde{L}_0\mathbf{J}_0Q^{(m)}$, $Q^{(m)}\mathbf{J}_0^\dagger\tilde{L}_0$, $L_1\mathrm{J}_1\Psi^*\Upsilon Q^{(m)}\Upsilon\Psi\mathrm{J}_1^\dagger L_1$, $\tilde{L}_0\mathbf{J}_0Q^{(m)}\mathbf{J}_0^\dagger\tilde{L}_0$, $L_1\mathrm{J}_1\Psi^*\Upsilon Q^{(m)}\mathbf{J}_0^\dagger\tilde{L}_0$ and $\tilde{L}_0\mathbf{J}_0Q^{(m)}\Upsilon\Psi\mathrm{J}_1^\dagger L_1$, $0 \leq m < n$, arising at substitution of the relations (3.12) and (3.13) into $Q^{(n)}(z)\big|_{\Pi_l}$, have to be understood in a sense of definitions from Sec. 4 of the paper [1].

Remark 3. Theorem 2 means that one can pose the continued Faddeev equations (3.11) only in the domains $\Pi_l^{(\mathrm{hol})} \subset \Pi_l$.

## 4. REPRESENTATIONS FOR THE FADDEEV COMPONENTS OF THREE-BODY $T$-MATRIX

In the present section, using the Faddeev equations (3.11) continued, we shall obtain the representations for the matrix $M^l(z)$ in the domains $\Pi_l^{(\mathrm{hol})}$ of unphysical sheets $\Pi_l \subset \Re$. The representations will be given in terms of the matrix $M(z)$ components taken on the physical sheet, or more precisely, in terms of the half-on-shell matrix $M(z)$ as well as the operators inverse to the truncated scattering matrices $S_l(z)$ and $S_l^\dagger(z)$. As a matter of fact, the construction of the representations for $M^l(z)$ consists in explicit "solving" the continued Faddeev equations (3.11) in the same way as in [9], [10] where the type ([1].3.2) representations had been found for analytical continuation of the $T$-matrix in the multichannel scattering problem with binary channels. We consider derivation of the representations for $M^l(z)$ as a constructive proof of the existence (in a sense of distributions over $\mathop{\times}\limits_{\alpha=1}^{3} \mathcal{O}(\mathbf{C}^6)$) of the analytical continuation of the matrix $M(z)$ on unphysical sheets $\Pi_l$ of the surface $\Re$.

So, let us consider the Faddeev equations (3.11) on the sheet $\Pi_l$ with $l_0 = 0$ or $l_0 = \pm 1$ and $l_{\beta,j} = 0$ or $l_{\beta,j} = 1$, $\beta = 1,2,3$, $j = 1,2,...,n_\beta$. Using the expressions (3.12) for $\mathbf{t}^l(z)$ and (3.13) for $\mathbf{R}_0^l(z)$, we transfer all the summands including $M^l(z)$ but not $\mathbf{J}_0$ and $\mathrm{J}_1$, to the left-hand part of Eqs. (3.11). Making then a simple transformation based on the identity $\mathbf{s}_l^{-1}(z) = \hat{\mathbf{I}}_0 - \mathbf{s}_l^{-1}(z)\mathbf{J}_0(z)\mathbf{t}(z)\mathbf{J}_0^\dagger(z)A_0(z)L_0$ we rewrite (3.11) in the form

$$(\mathbf{I}+\mathbf{t}\mathbf{R}_0\Upsilon)M^l = \mathbf{t}\left[\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t} - A_0^{(l)}\mathbf{J}_0\mathbf{X}_0^{(l)}\right] - \Phi\mathrm{J}_1^\dagger A_1^{(l)}(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}) \tag{4.1}$$

where $A_0^{(l)}(z) = L_0A_0(z)$, $A_1^{(l)}(z) = L_1A_1(z)$. Besides we denote

$$\begin{aligned} \mathbf{X}_0^{(l)} &= |L_0|\mathbf{s}_l^{-1}\mathbf{J}_0(\mathbf{I}-\mathbf{t}\mathbf{R}_0)\Upsilon M^l, \\ \mathbf{X}_1^{(l)} &= -L_1\left[\mathrm{J}_1\Phi^*\mathbf{R}_0 + A_0^{(l)}\mathrm{J}_1\Phi^*\mathbf{J}_0^\dagger\mathbf{J}_0\right]\Upsilon M^l. \end{aligned} \tag{4.2}$$

It follows from Eq. ([1].3.5) that

$$\mathrm{J}_1\Phi^*\mathbf{R}_0 = -\mathrm{J}_1\Psi^*. \tag{4.3}$$

Together with (4.3) the equalities

$$\left(\mathrm{J}_1\Phi^*\mathbf{J}_0^\dagger\right)(z) = 0, \quad \left(\mathbf{J}_0\Phi\mathrm{J}_1^\dagger\right)(z) = 0, \tag{4.4}$$

take place being true in accordance with (3.10) for all $z \in \mathbf{C}\setminus(-\infty,\lambda_{\max}]$.

Using Eq. (4.3) and first of Eqs. (4.4) one can rewrite $\mathbf{X}_1^{(l)}$ in the form

$$\mathbf{X}_1^{(l)} = L_1\mathrm{J}_1\Psi^*\Upsilon M^l, \tag{4.5}$$

too. Note that the condition $z \notin (-\infty,\lambda_{\max})$ necessary for Eq. (4.4) to be valid, does not touch the two-body unphysical sheets $\Pi_l$, $l_0 = 0$, since in this case $A_0^{(l)}(z) = 0$ and consequently, the terms including the products $\mathbf{J}_0^\dagger\mathbf{J}_0$, are plainly absent in (4.1). Meanwhile the points $z \in (-\infty,\lambda_{\max}]$ were excluded from the three-body sheets $\Pi_l$, $l_0 = \pm 1$, by definition.

Notice further that the operator $\mathbf{I}+\mathbf{t}\mathbf{R}_0\Upsilon$ admits the explicit inversion in terms of $M(z)$,

$$(\mathbf{I}+\mathbf{t}\mathbf{R}_0\Upsilon)^{-1} = \mathbf{I} - M\Upsilon\mathbf{R}_0, \tag{4.6}$$

for all $z \in \Pi_0$ which do not belong to the discrete spectrum $\sigma_d(H)$ of the Hamiltonian $H$, and

$$(\mathbf{I} - M\Upsilon\mathbf{R}_0)\mathbf{t} = M. \tag{4.7}$$

The equality (4.6) is a simple consequence of the Faddeev equations (2.1) and the identity $\mathbf{R}_0\Upsilon = \Upsilon\mathbf{R}_0$. The relation (4.7) represents an alternative variant of these equations. Now, we can rewrite Eqs. (4.1) in the equivalent form

$$\begin{aligned} M^l = M&\left(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{X}_0^{(l)}\right) - \\ &-(\mathbf{I}-M\Upsilon\mathbf{R}_0)\Phi\mathrm{J}_1^\dagger A_1^{(l)}(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}). \end{aligned} \tag{4.8}$$

Eq. (4.8) means that the matrix $M^l(z)$ is expressed in terms of the quantities $\mathbf{X}_0^{(l)}(z)$ and $\mathbf{X}_1^{(l)}(z)$. Main goal of the section consists really in presenting these quantities in terms of the matrix $M(z)$ considered on the physical sheet.

To obtain for $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$ a closed system of equations we use the definitions (4.2) and (4.5) and act on the both parts of Eq. (4.8) by the operators $\mathbf{s}_l^{-1}\mathbf{J}_0(\mathbf{I}-\mathbf{t}\mathbf{R}_0)\Upsilon$ and $\mathrm{J}_1\Psi^*$. At this moment we use also the identities

$$[\mathbf{I}-\mathbf{t}\mathbf{R}_0]\Upsilon M = M_0 - \mathbf{t}, \quad [\mathbf{I}-\mathbf{t}\mathbf{R}_0]\Upsilon[\mathbf{I}-M\Upsilon\mathbf{R}_0] = [\mathbf{I}-M_0\mathbf{R}_0]\Upsilon \tag{4.9}$$

where $M_0 = \Omega^\dagger\Omega M = (\mathbf{I}+\Upsilon)M$. The relations (4.9) are another easily checked consequence of the Faddeev equations (2.1). Along with Eq. (4.9) we apply second of the equalities (4.4). As a result we come to the following system of equations for $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$:

$$\begin{aligned} \mathbf{X}_0^{(l)} &= |L_0|\mathbf{s}_l^{-1}\mathbf{J}_0\left[(M_0-\mathbf{t})(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{X}_0^{(l)}\right] - \\ &\quad -|L_0|\mathbf{s}_l^{-1}\mathbf{J}_0M_0\Upsilon\Psi\mathrm{J}_1^\dagger A_1^{(l)}(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}), \end{aligned} \tag{4.10}$$

$$\begin{aligned} \mathbf{X}_1^{(l)} &= L_1\mathrm{J}_1\Psi^*\Upsilon M(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0^\dagger\mathbf{t} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{X}_0^{(l)}) - \\ &\quad -L_1\mathrm{J}_1\Psi^*\Upsilon[\Phi + M\Upsilon\Psi]\mathrm{J}_1^\dagger A_1^{(l)}(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}). \end{aligned} \tag{4.11}$$

It is convenient to write this system in the matrix form $\tilde{B}^{(l)}\mathbf{X}^{(l)} = \tilde{D}^{(l)}$, $\mathbf{X}^{(l)} = (\mathbf{X}_0^{(l)}, \mathbf{X}_1^{(l)})^\dagger$ with $\tilde{B}^{(l)} = \{\tilde{B}_{ij}^{(l)}\}$, $i,j = 0,1$, the matrix consisting of operators standing at unknown $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$. By $\tilde{D}^{(l)}$, $\tilde{D}^{(l)} = (\tilde{D}_0^{(l)}, \tilde{D}_1^{(l)})^\dagger$), we understand a column constructed of the absolute terms of Eqs. (4.10) and (4.11). Since $\mathbf{s}_l = \hat{\mathbf{I}}_0 + A_0^{(l)}\mathbf{J}_0\mathbf{t}\mathbf{J}_0^\dagger$ we find $\tilde{B}_{00}^{(l)} = \mathbf{s}_l^{-1}(\hat{\mathbf{I}}_0 + A_0^{(l)}\mathbf{J}_0M_0\mathbf{J}_0^\dagger)$, At the same time $\tilde{B}_{01}^{(l)} = |L_0|\mathbf{s}_l^{-1}\mathbf{J}_0M_0\Upsilon\Psi\mathrm{J}_1^\dagger A_1^{(l)}$, $\tilde{B}_{10}^{(l)} = L_1\mathrm{J}_1\Psi^*\Upsilon M\mathbf{J}_0^\dagger A_0^{(l)}$ and $\tilde{B}_{11}^{(l)} = \hat{I}_1 + L_1\mathrm{J}_1\Psi^*U\Psi\mathrm{J}_1^\dagger A_1^{(l)}$ because $\Upsilon(\Phi + M\Upsilon\Psi) = \Upsilon\mathbf{v}\Psi + \Upsilon M\Upsilon\Psi = (\Upsilon\mathbf{v} + \Upsilon M\Upsilon)\Psi = U\Psi$ (see [1], Sec. 4).

The absolute terms look as

$$\begin{aligned} \tilde{D}_0^{(l)} &= |L_0|\mathbf{s}_l^{-1}[\mathbf{J}_0(M_0-\mathbf{t})(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}) - |L_0|\mathbf{J}_0M_0\Upsilon\Psi\mathrm{J}_1^\dagger A_1^{(l)}\mathrm{J}_1\Phi^*], \\ \tilde{D}_1^{(l)} &= L_1\mathrm{J}_1\Psi^*\Upsilon M(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}) - L_1\mathrm{J}_1\Psi^*U\Psi\mathrm{J}_1A_1^{(l)}\mathrm{J}_1\Phi^*. \end{aligned}$$

The operator $\mathbf{s}_l(z)$, $l_0 = \pm 1$, has inverse one for all $z \in \mathbf{C}$. If $z \notin Z_{\mathrm{res}}$ then $\mathbf{s}_l^{-1}(z)$ is a bounded operator in $\hat{\mathcal{G}}_0$. That is why, acting on the both parts of the first equation

$\tilde{B}_{00}^{(l)}\mathbf{X}_0^{(l)} + \tilde{B}_{01}^{(l)}\mathbf{X}_1^{(l)} = \tilde{D}_0^{(l)}$ of the system $\tilde{B}^{(l)}\mathbf{X}^{(l)} = \tilde{D}^{(l)}$ by the operator $s_l$, and not changing its second equation, we come to the equivalent system

$$B^{(l)}\mathbf{X}^{(l)} = D^{(l)} \tag{4.12}$$

with the (operator) matrix

$$B^{(l)} = \begin{pmatrix} \hat{\mathbf{I}}_0 + |L_0|\mathbf{J}_0 M_0 \mathbf{J}_0^\dagger A_0^{(l)} & |L_0|\mathbf{J}_0 M_0 \Upsilon \Psi \mathbf{J}_1^\dagger A_1^{(l)} \\ L_1 \mathbf{J}_1 \Psi^* \Upsilon M \mathbf{J}_0^\dagger A_0^{(l)} & \hat{I}_1 + L_1 \mathbf{J}_1 \Psi^* U \Psi \mathbf{J}_1^\dagger A_1^{(l)} \end{pmatrix}, \tag{4.13}$$

$B^{(l)}(z) : \hat{\mathcal{G}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{G}}_0 \oplus \hat{\mathcal{H}}_1$, and the absolute term $D^{(l)}$ having the components $D_0^{(l)} = s_l \tilde{D}_0^{(l)}$ and $D_1^{(l)} = \tilde{D}_1^{(l)}$.

LEMMA 2. *The operator $\left(B^{(l)}(z)\right)^{-1}$ exists for all $z \in \mathbf{C}$ such that there exists the operator $S_l^{-1}(z)$ inverse to the truncated three-body scattering matrix $S_l(z)$ given by first of the equalities (2.4) with $L = \mathrm{diag}\{L_0, L_1\}$, $\tilde{L} = \mathrm{diag}\{|L_0|, L_1\}$, and such that there exist the operators $[S_l(z)]_{00}^{-1}$ and $[S_l(z)]_{11}^{-1}$ inverse to $[S_l(z)]_{00} = \hat{I}_0 + \mathbf{J}_0 T \mathbf{J}_0^\dagger A_0 L_0$ and $[S_l(z)]_{11} = \hat{I}_1 + L_1 \mathbf{J}_1 \Psi^* U \Psi \mathbf{J}_1^\dagger A_1 L_1$, respectively. The components $\left[\left(B^{(l)}(z)\right)^{-1}\right]_{ij}$, $i,j = 0,1$, of the operator $\left(B^{(l)}(z)\right)^{-1}$ admit the representation*

$$\left[\left(B^{(l)}(z)\right)^{-1}\right]_{00} = \hat{\mathbf{I}}_0 - \Omega^\dagger [S_l^{-1}]_{00} \left\{|L_0|\mathbf{J}_0 T_0 - [S_l]_{01}[S_l]_{11}^{-1} L_1 \mathbf{J}_1 \Psi^* \Upsilon M\right\} \mathbf{J}_0^\dagger A_0^{(l)}, \tag{4.14}$$

$$\left[\left(B^{(l)}(z)\right)^{-1}\right]_{01} = \Omega^\dagger [S_l^{-1}]_{01}, \tag{4.15}$$

$$\left[\left(B^{(l)}(z)\right)^{-1}\right]_{10} = -[S_l^{-1}]_{11} L_1 \mathbf{J}_1 \Psi^* \Upsilon M \mathbf{J}_0^\dagger A_0^{(l)} \left\{\hat{\mathbf{I}}_0 - \Omega^\dagger [S_l]_{00}^{-1} |L_0| \mathbf{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)}\right\}, \tag{4.16}$$

$$\left[\left(B^{(l)}(z)\right)^{-1}\right]_{11} = [S_l^{-1}]_{00} \tag{4.17}$$

*with $T_0 \equiv \Omega M$.*

Note that since $|L_0|$ and $A_0^{(l)}$ are numbers turning into zero at $l_0 = 0$ simultaneously, the factors $|L_0|$ in (4.14) and (4.16) may be omitted.

PROOF. Let us find at the beginning, the components $\left[\left(B^{(l)}(z)\right)^{-1}\right]_{00}$ and $\left[\left(B^{(l)}(z)\right)^{-1}\right]_{10}$, which will be denoted temporarily (for the sake of contracting the writing) by $Y_{00}$ and $Y_{10}$. Using Eq. (4.13) we write the equation system for these components,

$$[B^{(l)}]_{00}\, Y_{00} + [B^{(l)}]_{01}\, Y_{10} = \hat{\mathbf{I}}_0 \tag{4.18}$$

$$[B^{(l)}]_{10}\, Y_{00} + [B^{(l)}]_{11}\, Y_{10} = 0 \tag{4.19}$$

Eliminating the unknown $Y_{10}$ from the first equation (4.18) with a help of (4.19) we come to the following equation including the element $Y_{00}$ only,

$$\left\{\hat{\mathbf{I}}_0 + \Omega^\dagger \left[|L_0|\mathbf{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)} - [S_l]_{01}[S_l]_{11}^{-1} L_1 \mathbf{J}_1 \Psi^* \Upsilon M \mathbf{J}_0^\dagger A_0^{(l)}\right]\right\} Y_{00} = \hat{\mathbf{I}}_0. \tag{4.20}$$

At intermediate transforms we used the equality $M_0 = \Omega^\dagger T_0$.

The operator matrix in the left-hand part of Eq. (4.20) complementary to $\hat{\mathbf{I}}_0$, has three the same rows. Thus one can apply to Eq. (4.20) the inversion formula

$$\left[\hat{\mathbf{I}}_0 + \Omega^\dagger (C_1, C_2, C_3)\right]^{-1} = \hat{\mathbf{I}}_0 - \Omega^\dagger \left[\hat{I}_0 + C_1 + C_2 + C_3\right]^{-1} (C_1, C_2, C_3), \tag{4.21}$$



which is true for a wide class of the operators ($C_1$, $C_2$ and $C_3$). A single essential requirement to $C_1$, $C_2$ and $C_3$ evidently, is the existence of $(\hat{I}_0 + C_1 + C_2 + C_3)^{-1}$.

In the case concerned

$$C_\beta(z) \equiv \left\{|L_0|\mathbf{J}_0 T_{0\beta} \mathbf{J}_0^\dagger - [S_l]_{01}[S_l]_{11}^{-1} L_1 \mathbf{J}_1 \Psi^* \Upsilon [M]_\beta \mathbf{J}_0^\dagger\right\} A_0^{(l)}$$

where $[M]_\beta$ is the $\beta$-th column of the matrix $M$, $[M]_\beta = \left(M_{1\beta}, M_{2\beta}, M_{3\beta}\right)^\dagger$. Thus

$$\hat{I}_0 + C_1 + C_2 + C_3 = \hat{I}_0 + \mathbf{J}_0 T \mathbf{J}_0^\dagger A_0^{(l)} - [S_l]_{01}[S_l]_{11}^{-1} \mathbf{J}_1 \Psi^* U_0^\dagger \mathbf{J}_0^\dagger A_0^{(l)} \equiv [S_l]_{00} - [S_l]_{01}[S_l]_{11}^{-1}[S_l]_{10}.$$

Note that elements $\left[S_l^{-1}\right]_{ij}$, $i,j = 0,1$, of $S_l^{-1}$ may be present by the components $[S_l]_{ij}$ as

$$\left[S_l^{-1}\right]_{00} = \left([S_l]_{00} - [S_l]_{01}\, [S_l]_{11}^{-1}\, [S_l]_{10}\right)^{-1} \tag{4.22}$$

$$\left[S_l^{-1}\right]_{11} = \left([S_l]_{11} - [S_l]_{10}\, [S_l]_{00}^{-1}\, [S_l]_{01}\right)^{-1} \tag{4.23}$$

$$\left[S_l^{-1}\right]_{10} = -[S_l]_{11}^{-1}\, [S_l]_{10}\, \left[S_l^{-1}\right]_{00} \tag{4.24}$$

$$\left[S_l^{-1}\right]_{01} = -[S_l]_{00}^{-1}\, [S_l]_{01}\, \left[S_l^{-1}\right]_{11} \tag{4.25}$$

It follows from (4.22) that $\hat{I}_0 + C_1 + C_2 + C_3 = \left(\left[S_l^{-1}\right]_{00}\right)^{-1}$. Therefore in the conditions of Lemma, the operator $(\hat{I}_0 + C_1 + C_2 + C_3)^{-1}$ invertible. Now, a use of Eq. (4.21) in (4.20) leads us immediately to the representation (4.14) for $\left[\left(B^{(l)}\right)^{-1}\right]_{00}$.

When calculating $Y_{10} = \left[\left(B^{(l)}\right)^{-1}\right]_{10}$ we eliminate from the second equation (4.19) vice versa, the quantity $Y_{00}$ using Eq. (4.18). For all this, we need to calculate the operator inverse to $\hat{\mathbf{I}}_0 + \mathbf{J}_0 M_0 \mathbf{J}_0^\dagger A_0^{(l)}$. Here, we apply again the relation (4.21) and obtain that

$$\begin{aligned} \left(\hat{\mathbf{I}}_0 + |L_0|\mathbf{J}_0 M_0 \mathbf{J}_0^\dagger A_0^{(l)}\right)^{-1} &= \left(\hat{\mathbf{I}}_0 + \Omega^\dagger |L_0| \mathbf{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)}\right)^{-1} = \\ &= \hat{\mathbf{I}}_0 - \Omega^\dagger\, [S_l]_{00}^{-1}\, |L_0| \mathbf{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)}. \end{aligned} \tag{4.26}$$

With a help of (2.4) we can write the resulting equation for $Y_{10}$ as

$$\begin{aligned} &\left\{[S_l]_{11} - [S_l]_{10}\, [S_l]_{00}^{-1}\, [S_l]_{01}\right\} Y_{10} = \\ &= -\mathbf{J}_1 \Psi^* \Upsilon M \mathbf{J}_0^\dagger A_0^{(l)} \left[\hat{\mathbf{I}}_0 + \mathbf{J}_0 M_0 \mathbf{J}_0^\dagger A_0^{(l)}\right]^{-1}. \end{aligned} \tag{4.27}$$

According to (4.23) the expression in braces in the left hand part of Eq. (4.27) coincides with $\left[S_l^{-1}\right]_{11}^{-1}$. Then from (4.27) we get immediately (4.15).

System of the equations

$$[B^{(l)}]_{00}\, Y_{01} + [B^{(l)}]_{01}\, Y_{11} = 0 \tag{4.28}$$

$$[B^{(l)}]_{10}\, Y_{01} + [B^{(l)}]_{11}\, Y_{11} = \hat{I}_1 \tag{4.29}$$

for the components $Y_{01} = [(B^{(l)})^{-1}]_{01}$ and $Y_{11} = [(B^{(l)})^{-1}]_{11}$ is solved analogously. Search for $Y_{11}$ is at all a simple problem because the use of the inversion formula (4.26) for Eq. (4.28) immediately gives $Y_{01} = \Omega^\dagger\, [S_l]_{00}^{-1}\, [S_l]_{01}\, Y_{11}$. Substituting this $Y_{01}$ in (4.29) we find

$$\left\{[S_l]_{11} - [S_l]_{10}\, [S_l]_{00}^{-1}\, [S_l]_{01}\right\} Y_{11} = \hat{I}_1.$$

Here, one can see in the left-hand part as in (4.27), the operator $\left[S_l^{-1}\right]_{11}^{-1}$. Inverting it, we come to Eq. (4.17).

When calculating the unknown $Y_{01}$, we begin with expressing by it the unknown $Y_{11}$. Using Eq. (4.29) we get

$$Y_{11} = [S_l]_{11}^{-1} \left(\hat{I}_1 - L_1 \mathrm{J}_1 \Psi \Upsilon M \mathbf{J}_0 A_0^{(l)} Y_{01}\right). \tag{4.30}$$

Substituting (4.30) into Eq. (4.28) we obtain an equation with operator standing at $Y_{01}$, which may be inverted with a help of Eq. (4.21). Then we use also the chain of equalities

$$|L_0|\mathbf{J}_0 M_0 \Upsilon \Psi \mathrm{J}_1^\dagger A_1^{(l)} = |L_0|\mathbf{J}_0 \Omega^\dagger \Omega M \Upsilon \Psi \mathrm{J}_1^\dagger A_1^{(l)} =$$
$$= \Omega^\dagger |L_0| \mathrm{J}_0 \Omega M \Upsilon \Psi \mathrm{J}_1^\dagger A_1^{(l)} = \Omega^\dagger [S_l]_{01},$$

simplifying the absolute term as well as the summand in the left-hand part, engendered there due to (4.30) by the element $\left[B^{(l)}\right]_{01}$. Completing the transforms we find

$$Y_{01} = -\Omega^\dagger \left\{[S_l]_{00} - [S_l]_{01} [S_l]_{11}^{-1} [S_l]_{10}\right\}^{-1} [S_l]_{01} [S_l]_{11}^{-1}.$$

In view of (4.25), the expression standing after $\Omega^\dagger$ in the right-hand part of the last equation, coincides exactly with that for $\left[S_l^{-1}\right]_{01}$. Therefore finally, we obtain Eq. (4.15). Thus, all the components of the inverse operator $\left(B^{(l)}\right)^{-1}$ have already been calculated.

It follows from the representations (4.14) – (4.17) that $\left(B^{(l)}(z)\right)^{-1}$ exists for such $z \in \mathbf{C}$ that there exist the operators inverse to $S_l(z)$, $[S_l(z)]_{00}$ and $[S_l(z)]_{11}$.

The lemma has been proved.

Let us back to Eq. (4.12) and inverse in it, using the relations (4.14) – (4.17), the operator $B^{(l)}(z)$. Thereby we find the unknowns $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$ which express $M^l(z)$ [see Eq. (4.8)].

When carrying a concrete calculation of $\mathbf{X}_0^{(l)} = \left[\left(B^{(l)}\right)^{-1}\right]_{00} D_0^{(l)} + \left[\left(B^{(l)}\right)^{-1}\right]_{01} D_1^{(l)}$ we use the relation $|L_0| \left[\left(B^{(l)}\right)^{-1}\right]_{00} \mathbf{J}_0 M_0 = \Omega^\dagger |L_0| \left[S_l^{-1}\right]_{00} \mathrm{J}_0 T_0$ that can be checked with a help of (2.4) and (2.3). Along with the identity

$$\mathbf{J}_0 \mathrm{t} \left(\hat{\mathbf{I}}_0 - A_0^{(l)} \mathbf{J}_0^\dagger \mathrm{s}_l^{-1} \mathbf{J}_0 \mathrm{t}\right) = \mathrm{s}_l^{-1} \mathbf{J}_0 \mathrm{t}, \tag{4.31}$$

this relation simplifies essentially the transform of the product $\left[\left(B^{(l)}\right)^{-1}\right]_{00} D_0^{(l)}$. Besides when calculating $\mathbf{X}_0^{(l)}$, we use the equalities (4.4). As a result we find

$$\mathbf{X}_0^{(l)} = \Omega^\dagger \left\{|L_0| \left[S_l^{-1}\right]_{00} \mathrm{J}_0 T_0 + \left[S_l^{-1}\right]_{01} L_1 (\mathrm{J}_1 \Psi^* \Upsilon M + \mathrm{J}_1 \Phi^*)\right\} - |L_0| \mathrm{s}_l^{-1} \mathbf{J}_0 \mathrm{t}. \tag{4.32}$$

Now, to find $\mathbf{X}_1^{(l)} = \left[\left(B^{(l)}\right)^{-1}\right]_{10} D_0^{(l)} + \left[\left(B^{(l)}\right)^{-1}\right]_{11} D_1^{(l)}$, we observe additionally that the equality $\left\{\hat{\mathbf{I}}_0 - \Omega^\dagger \left[S_l^{-1}\right]_{00}^{-1} \mathrm{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)}\right\} \mathbf{J}_0 M_0 = \Omega^\dagger \left[S_l^{-1}\right]_{00}^{-1} \mathrm{J}_0 T_0$ simplifying the product $\left[\left(B^{(l)}\right)^{-1}\right]_{10} D_0^{(l)}$, is valid. The final expression for $\mathbf{X}_1^{(l)}$ read as

$$\mathbf{X}_1^{(l)} = L_1 \left\{\left[S_l^{-1}\right]_{10} |L_0| \mathrm{J}_0 T_0 + \left[S_l^{-1}\right]_{11} L_1 \mathrm{J}_1 \Psi^* \Upsilon M - \left(\hat{I}_1 - \left[S_l^{-1}\right]_{11}\right) L_1 \mathrm{J}_1 \Psi^*\right\}. \tag{4.33}$$

To obtain now a representation for $M^l(z)$, one needs at the moment only to substitute the found expressions (4.32) for $\mathbf{X}_0^{(l)}$ and (4.33) for $\mathbf{X}_1^{(l)}$ in Eq. (4.8). Carrying out series of simple but rather cumbersome transformations of Eq. (4.8) we come as a result to the statement analogous to Theorem 1 of Ref. [1] concerning analytical continuation of the two-body $T$-matrix. The statement is following.



THEOREM 3. *The matrix $M(z)$ admits in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$, the analytical continuation in $z$ on the domains $\Pi_l^{(\mathrm{hol})}$ of unphysical sheets $\Pi_l$ of the surface $\Re$. The continuation is described by*

$$M^l = M - \left(M \Omega^\dagger \mathrm{J}_0^\dagger, \quad \Phi \mathrm{J}_1^\dagger + M \Upsilon \Psi \mathrm{J}_1^\dagger\right) L A \, S_l^{-1} \tilde{L} \begin{pmatrix} \mathrm{J}_0 \Omega M \\ \mathrm{J}_1 \Psi^* \Upsilon M + \mathrm{J}_1 \Phi^* \end{pmatrix} \tag{4.34}$$

*where $S_l(z)$ is a truncated scattering matrix (2.4), $L = \mathrm{diag}\{l_0, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}\}$ and $\tilde{L} = \mathrm{diag}\{|l_0|, l_{1,1}, ..., l_{1,n_1}, l_{2,1}, ..., l_{2,n_2}, l_{3,1}, ..., l_{3,n_3}\}$. Kernels of all the operators in the right-hand part of Eq. (4.34) are taken on the physical sheet.*

Note that $LA \, S_l^{-1}(z) \tilde{L} = \tilde{L}[S_l^\dagger(z)]^{-1} AL$. This means that the relations (4.34) may be rewritten also in terms of the scattering matrices $S_l^\dagger(z)$.

## 5. ANALYTICAL CONTINUATION OF THE SCATTERING MATRICES

Let $l = \{l_0, l_{1,1}, ..., l_{1,n_1}, \; l_{2,1}, ..., l_{2,n_2}, \; l_{3,1}, ..., l_{3,n_3}\}$ with certain $l_0$, $l_0 = 0$ or $l_0 = \pm 1$ and $l_{\alpha,j}$, $l_{\alpha,j} = 0$ or $l_{\alpha,j} = +1$, $\alpha = 1, 2, 3$, $j = 1, 2, ..., n_\alpha$. The truncated scattering matrices $S_l(z) : \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ and $S_l^\dagger(z) : \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$, given by formulae (2.4), are operator-valued functions of variable $z$ being holomorphic in the domain $\Pi_l^{(\mathrm{hol})}$ of the physical sheet $\Pi_0$. At $l_0 = 1$ and $l_{\alpha,j} = 1$, $\alpha = 1, 2, 3$, $j = 1, 2, ..., n_\alpha$, these matrices coincide with the respective total three-body scattering matrices: $S_l(z) = S(z)$, $S_l^\dagger(z) = S^\dagger(z)$.

We describe now the analytical continuation of $S_{l'}(z)$ and $S_{l'}^\dagger(z)$ with a certain multi-index $l'$ on unphysical sheets $\Pi_l \in \Re$. We shall base here on the representations (4.34) for $M(z)\big|_{\Pi_l}$. As mentioned above, our goal is to find the explicit representations for $S_l(z)\big|_{\Pi_{l'}}$ and $S_l^\dagger(z)\big|_{\Pi_{l'}}$ again in terms of the physical sheet.

First of all, we remark that the function $A_0(z)$ is univalent. It looks as $A_0(z) = -\pi i z^2$ on all the sheets $\Pi_l$. At the same time after continuing from $\Pi_0$ on $\Pi_l$, the function $A_{\beta,j}(z) = -\pi i \sqrt{z - \lambda_{\beta,j}}$ keeps its form if only $l_{\beta,j} = 0$. If $l_{\beta,j} = 1$ this function turns into $A'_{\beta,j}(z) = -A_{\beta,j}(z)$. Analogous inversion takes (or does not take) place for arguments $\hat{P}$, $\hat{P}'$, $\hat{p}_\alpha$ and $\hat{p}'_\beta$ of kernels of the operators $\mathrm{J}_0 \Omega M \Omega^\dagger \mathrm{J}_0^\dagger$, $\mathrm{J}_0 \Omega M \Upsilon \Psi \mathrm{J}_1^\dagger$, $\mathrm{J}_1 \Psi^* \Upsilon M \Omega^\dagger \mathrm{J}_0^\dagger$ and $\mathrm{J}_1 \Psi^* (\Upsilon \mathrm{v} + \Upsilon M \Upsilon) \Psi \mathrm{J}_1^\dagger$, too. Remember that on the physical sheet $\Pi_0$, the action of $\mathrm{J}_0(z)$ $(\mathrm{J}_0^\dagger(z))$ transforms $P \in \mathbf{R}^6$ in $\sqrt{z}\hat{P}$ $(P' \in \mathbf{R}^6$ in $\sqrt{z}\hat{P}')$. At the same time, $p_\alpha \in \mathbf{R}^3$ $(p'_\beta \in \mathbf{R}^3)$ turns under $\mathrm{J}_{\alpha,i}(z)$ $(\mathrm{J}_{\beta,j}^\dagger(z))$ into $\sqrt{z - \lambda_{\alpha,i}}\,\hat{p}_\alpha$ $(\sqrt{z - \lambda_{\beta,j}}\,\hat{p}'_\beta)$. That is why we introduce the operators $\mathcal{E}(l) = \mathrm{diag}\{\mathcal{E}_0, \mathcal{E}_1\}$ where $\mathcal{E}_0$ is the identity operator in $\hat{\mathcal{H}}_0$ if $l_0 = 0$, and $\mathcal{E}_0$, the inversion, $(\mathcal{E}_0 f)(\hat{P}) = f(-\hat{P})$ if $l_0 = \pm 1$. Analogously $\mathcal{E}_1(l) = \mathrm{diag}\{\mathcal{E}_{1,1}, ..., \mathcal{E}_{1,n_1}; \mathcal{E}_{2,1}, ..., \mathcal{E}_{2,n_2}; \mathcal{E}_{3,1}, ..., \mathcal{E}_{3,n_3}\}$ where $\mathcal{E}_{\beta,j}$ is the identity operator in $\hat{\mathcal{H}}^{(\beta,j)}$ if $l_{\beta,j} = 0$, and $\mathcal{E}_{\beta,j}$, the inversion $(\mathcal{E}_{\beta,j} f)(\hat{p}_\beta) = f(-\hat{p}_\beta)$ if $l_{\beta,j} = 1$. By $\mathrm{e}_1(l)$ we denote the diagonal matrix $\mathrm{e}_1(l) = \mathrm{diag}\{\mathrm{e}_{1,1}, ..., \mathrm{e}_{1,n_1}; \mathrm{e}_{2,1}, ..., \mathrm{e}_{2,n_2}; \mathrm{e}_{3,1}, ..., \mathrm{e}_{3,n_3}\}$ with elements $\mathrm{e}_{\beta,j} = 1$ if $l_{\beta,j} = 0$ and $\mathrm{e}_{\beta,j} = -1$ if $l_{\beta,j} = 1$. Let $\mathrm{e}(l) = \mathrm{diag}\{\mathrm{e}_0, \mathrm{e}_1\}$ where $\mathrm{e}_0 = +1$.

THEOREM 4. *If there exists a path on the surface $\Re$ such that at moving by it from the domain $\Pi_{l'}^{(\mathrm{hol})}$ on $\Pi_0$ to the domain $\Pi_{l'}^{(\mathrm{hol})} \bigcap \Pi_{l'l}^{(\mathrm{hol})}$ on $\Pi_l$, the parameter $z$ stays on intermediate sheets $\Pi_{l''}$ always in the domains $\Pi_{l'}^{(\mathrm{hol})} \bigcap \Pi_{l''l'}^{(\mathrm{hol})}$, then the truncated scattering matrices $S_{l'}(z)$ and $S_{l'}^\dagger(z)$ admit the analytical continuation in $z$ on the domain $\Pi_{l'}^{(\mathrm{hol})} \bigcap \Pi_{l'l}^{(\mathrm{hol})}$ of the sheet $\Pi_l$. The continuation is described by*

$$S_{l'}(z)\big|_{\Pi_l} = \mathcal{E}(l) \left[\hat{\mathbf{I}} + \tilde{L}' \hat{\mathcal{T}} L' \, A\mathrm{e}(l) - \tilde{L}' \hat{\mathcal{T}} L \, A \, S_l^{-1} \, \tilde{L} \hat{\mathcal{T}} L' \, A\mathrm{e}(l)\right] \mathcal{E}(l), \tag{5.1}$$

$$S_{l'}^{\dagger}(z)\Big|_{\Pi_l} = \mathcal{E}(l)\left[\hat{\mathbf{I}} + \mathrm{e}(l)A\, L'\hat{\mathcal{T}}\tilde{L}' - \mathrm{e}(l)A\, L\hat{\mathcal{T}}\tilde{L}\, [S_l^{\dagger}]^{-1} A\, L\hat{\mathcal{T}}\tilde{L}'\right]\mathcal{E}(l), \tag{5.2}$$

*where* $L' = \{l'_0, l'_{1,1}, \ldots, l'_{1,n_1},\ l'_{2,1}, \ldots, l'_{2,n_2},\ l'_{3,1}, \ldots, l'_{3,n_3}\}$ *and* $\tilde{L}' = \{|l'_0|, l'_{1,1}, \ldots, l'_{1,n_1},\ l'_{2,1}, \ldots, l'_{2,n_2},\ l'_{3,1}, \ldots, l'_{3,n_3}\}$.

PROOF. We give the proof for example of $S_{l'}(z)$. Using the definition (2.3) of the operator $\mathcal{T}(z)$ we rewrite $S_{l'}(z)$ in the form

$$S_{l'}(z) = \hat{\mathbf{I}} + \tilde{L}'\left[\begin{pmatrix} \mathrm{J}_0\Omega \\ \mathrm{J}_1\Psi^*\Upsilon \end{pmatrix} M\left(\Omega^{\dagger}\mathrm{J}_0^{\dagger},\ \ \Upsilon\Psi\mathrm{J}_1^{\dagger}\right) + \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{J}_1\Psi^*\Upsilon\mathbf{v}\Psi\mathrm{J}_1^{\dagger} \end{pmatrix}\right] L'A.$$

Note that when continuing on the sheet $\Pi_{l''}$, the operators $\mathrm{J}_0(z)$, $\mathrm{J}_0^{\dagger}(z)$, $\mathrm{J}_1(z)$ and $\mathrm{J}_1^{\dagger}(z)$ turn into $\mathcal{E}_0(l'')\mathrm{J}_0(z)$, $\mathrm{J}_0^{\dagger}(z)\mathcal{E}_0(l'')$, $\mathcal{E}_1(l'')\mathrm{J}_1(z)$ and $\mathrm{J}_1^{\dagger}(z)\mathcal{E}_1(l'')$, respectively. At the same time the matrix–function $A(z)$ turns into $A(z)\mathrm{e}(l'')$. Then using Theorem 3, for the domains $\Pi_{l'}^{(\mathrm{hol})}\bigcap\Pi_{l'l''}^{(\mathrm{hol})}$ of intermediate sheets $\Pi_{l''}$ we have

$$\begin{aligned} S_{l'}(z)\big|_{\Pi_{l''}} &= \hat{\mathbf{I}} + \mathcal{E}(l'')\tilde{L}'\hat{\mathcal{T}}L'\mathcal{E}(l'')A\mathrm{e}(l'') - \\ &-\mathcal{E}(l'')\tilde{L}'\begin{pmatrix} \mathrm{J}_0\Omega \\ \mathrm{J}_1\Psi^*\Upsilon \end{pmatrix}\left(M\Omega^{\dagger}\mathrm{J}_0^{\dagger},\ [\mathbf{v} + M\Upsilon]\Psi\mathrm{J}_1^{\dagger}\right) L''A\,S_{l''}^{-1}\,\times \\ &\times\tilde{L}''\begin{pmatrix} \mathrm{J}_0\Omega M \\ \mathrm{J}_1\Psi^*[\mathbf{v} + \Upsilon M] \end{pmatrix}\left(\Omega^{\dagger}\mathrm{J}_0^{\dagger}, \Upsilon\Psi\mathrm{J}_1^{\dagger}\right) L'\mathcal{E}(l'')A\mathrm{e}(l'') \end{aligned} \tag{5.3}$$

where the summand following immediately by $\hat{\mathbf{I}}$, is engendered by the term $M(z)$ of the right-hand part of (4.34). The last summand of (5.3) is originated from the second summand of (4.34).

In view of (4.4) we have $\mathrm{J}_1\Psi^*\mathbf{v}\Omega^{\dagger}\mathrm{J}_0^{\dagger} = \mathrm{J}_1\Phi^*\mathrm{J}_0^{\dagger}\Omega^{\dagger} = 0$. Analogously, $\mathrm{J}_0\Omega\mathbf{v}\Psi\mathrm{J}_1^{\dagger}$ equals to zero, too. Thus, taking into account (2.3) we find

$$S_{l'}(z)\big|_{\Pi_{l''}} = \hat{\mathbf{I}} + \mathcal{E}(l'')\ \tilde{L}'\hat{\mathcal{T}}L'\ \mathcal{E}(l'')A\mathrm{e}(l'') - \mathcal{E}(l'')\ \tilde{L}'\hat{\mathcal{T}}L''\ A\,S_{l''}^{-1}\ \tilde{L}''\hat{\mathcal{T}}L'\ \mathcal{E}(l'')\,A\mathrm{e}(l''). \tag{5.4}$$

By the supposition, the parameter $z$ moves along such a path that on the sheet $\Pi_{l''}$ it is situated in the domain $\Pi_{l'}^{(\mathrm{hol})}\bigcap\Pi_{l'l''}^{(\mathrm{hol})}$. In this domain, the operators $(\tilde{L}'\hat{\mathcal{T}}L')(z)$, $(\tilde{L}'\hat{\mathcal{T}}L'')(z)$ and $(\tilde{L}''\hat{\mathcal{T}}L')(z)$ are defined and depending on $z$ analytically. Consequently, the same may be said also about the function $S_{l'}(z)\big|_{\Pi_{l''}}$. In equal degree, this statement is related to the sheet $\Pi_l$. Replacing the values of multi-index $l''$ in the representations (5.3) and (5.4) with $l$, we come to the assertion of theorem for $S_{l'}(z)\big|_{\Pi_l}$. Truth of the representations (5.2) for $S_{l'}^{\dagger}(z)\Big|_{\Pi_l}$ is established in the same way.

The proof is completed.

REMARK 4. If $l_0 = 0$ then the representation (5.1) for the analytical continuation of $S_l(z)$ on the (its "own") sheet $\Pi_l$ acquires the simple form [cf. ([1].3.6)],

$$S_l(z)\big|_{\Pi_l} = \mathcal{E}(l)\left[\hat{\mathbf{I}} + \mathrm{e}(l) - S_l^{-1}(z)\mathrm{e}(l)\right]\mathcal{E}(l) = \mathcal{E}(l)\,S_l^{-1}(z)\,\mathcal{E}(l).$$

Just so $S_l^{\dagger}(z)\Big|_{\Pi_l} = \mathcal{E}(l)\,[S_l^{\dagger}(z)]^{-1}\,\mathcal{E}(l)$.



## 6. REPRESENTATIONS FOR ANALYTICAL CONTINUATION OF RESOLVENT

The resolvent $R(z)$ of the Hamiltonian $H$ for three-body system concerned is expressed by $M(z)$ according to Eq. (2.2). As we established, kernels of all the operators included in the right-hand part of (2.2) admit in a sense of distributions over $\mathcal{O}(\mathbf{C}^6)$, the analytical continuation on the domains $\Pi_l^{(\mathrm{hol})}$ of unphysical sheets $\Pi_l \subset \Re$. So, such continuation is admitted as well for the kernel $R(P, P', z)$ of $R(z)$. Moreover, there exists an explicit representation for this continuation analogous to the representation ([1].3.7) for two-body resolvent.

THEOREM 5. *The analytical continuation, in a sense of distributions over* $\mathcal{O}(\mathbf{C}^6)$, *of the resolvent* $R(z)$ *on the domain* $\Pi_l^{(\mathrm{hol})}$ *of unphysical sheet* $\Pi_l \subset \Re$ *is described by*

$$\begin{aligned} R(z)\big|_{\Pi_l} &= R + \\ &+([I - RV]\mathrm{J}_0^{\dagger},\ \ \Omega[\mathbf{I} - \mathbf{R}_0 M\Upsilon]\Psi\mathrm{J}_1^{\dagger})LAS_l^{-1}\tilde{L}\begin{pmatrix} \mathrm{J}_0[I - VR] \\ \mathrm{J}_1\Psi^*[\mathbf{I} - \Upsilon M\mathbf{R}_0]\Omega^{\dagger} \end{pmatrix}. \end{aligned} \tag{6.1}$$

*Kernels of all the operators present in the right-hand part of Eq. (6.1) are taken on the physical sheet.*

PROOF. For analytical continuation $R^l(z)$ of the kernel $R(P, P', z)$ of $R(z)$ on the sheet $\Pi_l$ we have according to (2.2),

$$R^l(z) = R_0^l(z) - R_0^l(z)\Omega M^l(z)\Omega^{\dagger}R_0^l(z). \tag{6.2}$$

For $M^l(z)$ we have found already the representation (4.34). Since $R_0^l = R_0 + L_0A_0\mathrm{J}_0^{\dagger}\mathrm{J}_0$ we can rewrite Eq. (6.2) in the form

$$\begin{aligned} R^l &= R_0 - R_0\Omega M^l\Omega^{\dagger}R_0 + A_0L_0\mathrm{J}_0^{\dagger}\left(\hat{I}_0 - \mathrm{J}_0\Omega M^l\Omega^{\dagger}\mathrm{J}_0^{\dagger}L_0A_0\right)\mathrm{J}_0 - \\ &-A_0L_0\mathrm{J}_0^{\dagger}\mathrm{J}_0\Omega M^l\Omega^{\dagger}R_0 - R_0\Omega M^l\Omega^{\dagger}\mathrm{J}_0^{\dagger}\mathrm{J}_0L_0A_0. \end{aligned} \tag{6.3}$$

Consider separately the contributions of each summand of (6.3). Doing this we shall use the notations

$$\mathbf{B} = \left(\Omega M\Omega^{\dagger}\mathrm{J}_0^{\dagger},\ \ \Omega M\Upsilon\Psi\mathrm{J}_1^{\dagger} + \Omega\Phi\mathrm{J}_1^{\dagger}\right) \text{ and } \mathbf{B}^{\dagger} = \begin{pmatrix} \mathrm{J}_0\Omega M\Omega^{\dagger} \\ \mathrm{J}_1\Psi^*\Upsilon M\Omega^{\dagger} + \mathrm{J}_1\Phi^*\Omega^{\dagger} \end{pmatrix}.$$

It follows from (4.34) that $\Omega M^l\Omega^{\dagger} = \Omega M\Omega^{\dagger} - \mathbf{B}LA\ S_l^{-1}\ \tilde{L}\mathbf{B}^{\dagger}$. Hence two first summands of (6.3) give together

$$R_0 - R_0\Omega M\Omega^{\dagger}R_0 + R_0\mathbf{B}L\,AS_l^{-1}\,\tilde{L}\mathbf{B}^{\dagger}R_0 = R + R_0\mathbf{B}L\,AS_l^{-1}\,\tilde{L}\mathbf{B}^{\dagger}R_0.$$

Transforming the third term of (6.3) we use again the representation (4.34). We find

$$\begin{aligned} \mathrm{J}_0\Omega M^l\Omega^{\dagger}\mathrm{J}_0^{\dagger}L_0A_0 &= \hat{\mathcal{T}}_{00}L_0A_0 - \left(\hat{\mathcal{T}}_{00},\ \ \hat{\mathcal{T}}_{01}\right)L\,AS_l^{-1}\,\tilde{L}\begin{pmatrix} \hat{\mathcal{T}}_{00} \\ \hat{\mathcal{T}}_{10} \end{pmatrix}L_0A_0 = \\ &= \omega_0\hat{\mathcal{T}}L\,A\omega_0^* - \omega_0\hat{\mathcal{T}}L\ A\,S_l^{-1}\,\tilde{L}\hat{\mathcal{T}}L\,A\omega_0^* = \omega_0\hat{\mathcal{T}}L\,A\left(\hat{\mathbf{I}} - S_l^{-1}\tilde{L}\hat{\mathcal{T}}L\ A\right)\tilde{L}\omega_0^* \end{aligned}$$

where $\omega_0$ stands for the projector acting from $\hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ to $\hat{\mathcal{H}}_0$ as $\omega_0\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = f_0$, $f_0 \in \hat{\mathcal{H}}_0$, $f_1 \in \hat{\mathcal{H}}_1$. By $\omega_0^*$ we understand as usually the operator adjont to $\omega_0$. So far as $S_l = \hat{\mathbf{I}} + \tilde{L}\hat{\mathcal{T}}L\,A$

we have $\hat{\mathbf{I}} - S_l^{-1}\tilde{L}\hat{\mathcal{T}}LA = S_l^{-1}\left(\hat{\mathbf{I}} + \tilde{L}\hat{\mathcal{T}}LA - \tilde{L}\hat{\mathcal{T}}LA\right) = S_l^{-1}$. Taking in account that $L = L\cdot\tilde{L}$ we find

$$L_0A_0(\hat{I}_0 - \mathrm{J}_0\Omega M^l\Omega^\dagger \mathrm{J}_0^\dagger L_0A_0) = \omega_0 AL(\hat{\mathbf{I}} - \tilde{L}\hat{\mathcal{T}}LAS_l^{-1})\tilde{L}\omega_0^* = \omega_0 LA\,S_l^{-1}\,\tilde{L}\omega_0^*.$$

This means that the third term of (6.3) may be present as $\mathrm{J}_0^\dagger\omega_0 L\,AS_l^{-1}\,\tilde{L}\omega_0^*$.

When studying the fourth summand of (6.3) we begin with transforming the product $A_0L_0\mathrm{J}_0\Omega M^l\Omega^\dagger$ to more convenient form. It follows from (4.34) that

$$A_0L_0\mathrm{J}_0\Omega M^l\Omega^\dagger = A_0L_0\mathrm{J}_0\Omega M\Omega^\dagger - A_0L_0\left(\hat{\mathcal{T}}_{00},\ \hat{\mathcal{T}}_{01}\right)LA\,S_l^{-1}\,\tilde{L}\mathbf{B}^\dagger.$$

In view of $A_0L_0\mathrm{J}_0\Omega M\Omega^\dagger = \omega_0 AL\mathbf{B}^\dagger$ and $A_0L_0\left(\hat{\mathcal{T}}_{00}, \hat{\mathcal{T}}_{01}\right)L = \omega_0 AL\hat{\mathcal{T}}L$ we have

$$A_0L_0\mathrm{J}_0\Omega M^l\Omega^\dagger = \omega_0\left(AL - A\,L\hat{\mathcal{T}}L\,AS_l^{-1}\tilde{L}\right)\mathbf{B}^\dagger = \omega_0 L\,AS_l^{-1}\,\tilde{L}\mathbf{B}^\dagger.$$

Analogously, in the fifth term of (6.3), $\Omega M^l\Omega^\dagger \mathrm{J}_0^\dagger L_0A_0 = \mathbf{B}\tilde{L}\left[S_l^\dagger\right]^{-1}A\,L\omega_0^* = \mathbf{B}L\,AS_l^{-1}\,\tilde{L}\omega_0^*$. Thus two last summands of (6.3) give together $-\mathrm{J}_0^\dagger\omega_0 L\,AS_l^{-1}\,\tilde{L}\mathbf{B}^\dagger R_0 -\ R_0\mathbf{B}L\,AS_l^{-1}\,\tilde{L}\omega_0^*\mathrm{J}_0$. Substituting the expressions obtained into Eq. (6.3) we find

$$R^l = R + \left(\mathrm{J}_0^\dagger\omega_0 - R_0\mathbf{B}\right)L\,AS_l^{-1}\tilde{L}\left(\omega_0^*\mathrm{J}_0 - \mathbf{B}^\dagger R_0\right).$$

Taking into account the definitions of $\mathbf{B}$ and $\mathbf{B}^\dagger$ as well as the fact that $R_0\Omega M\Omega^\dagger = RV$, $\Omega^\dagger M\Omega R_0 = VR$ (see [2], [3]) and $R_0\Omega\Phi\mathrm{J}_1 = -\Omega\Psi\mathrm{J}_1$, $\mathrm{J}_1\Phi^*\Omega^\dagger R_0 = -\mathrm{J}_1\Psi^*\Omega^\dagger$, we come finally to Eq. (6.4) and this completes the proof.

## 7. ON USE OF THE DIFFERENTIAL FADDEEV EQUATIONS FOR COMPUTATION OF THREE-BODY RESONANCES

As follows from the representations (4.34), (5.1) and (6.1), the matrices $M(z)\big|_{\Pi_l}$, $S_{l'}(z)\big|_{\Pi_l}$ and the Green function $R(z)\big|_{\Pi_l}$ may have poles at points belonging to the discrete spectrum $\sigma_d(H)$ of the Hamiltonian $H$. Nontrivial singularities of $M(z)\big|_{\Pi_l}$, $S_{l'}(z)\big|_{\Pi_l}$ and $R(z)\big|_{\Pi_l}$ correspond to those points $z \in \Pi_0\bigcap\Pi_l^{(\mathrm{hol})}$ where the inverse truncated scattering matrix $[S_l(z)]^{-1}$ (or $[S_l^\dagger(z)]^{-1}$ and it is the same) does not exist or where it represents an unbounded operator. The points $z$ where $[S_l(z)]^{-1}$ does not exist, engender poles for $M(z)\big|_{\Pi_l}$, $S_{l'}(z)\big|_{\Pi_l}$ and $R(z)\big|_{\Pi_l}$. Such points are called (three-body) resonances.

The necessary and sufficient condition [12] of irreversibility of the operator $S_l(z)$ for given $z$ consists in existence of non-trivial solution $\mathcal{A}^{(\mathrm{res})} \in \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1$ to the equation

$$S_l(z)\mathcal{A}^{(\mathrm{res})} = 0. \tag{7.1}$$

Investigation of this equation may be carried out on the base of the results of Sec. 4 of the paper [1] concerning properties of kernels of the operator $\hat{\mathcal{T}}(z)$. In view of the space shortage we postpone this investigation for another paper.

The equation (7.1) may be applied for a practical computations of resonances situated in the domains $\Pi_l^{(\mathrm{hol})} \subset \Pi_l$. The resonances have to be considered as those values of $z \in \Pi_0\bigcap\Pi_l^{(\mathrm{hol})}$ for which the operators $S_l(z)$ and $S_l^\dagger(z)$ have eigenvalue zero.



Elements of the scattering matrices $S_l(z)$ and $S_l^\dagger(z)$ are expressed in terms of the amplitudes (continued in the energy $z$ on the physical sheet) for different processes taking place in the three-body system under consideration. Respective formulae [3] written for the components of $\hat{\mathcal{T}}$, read as

$$\begin{aligned}
\hat{\mathcal{T}}_{\alpha,j;\beta,k}(\hat{p}_\alpha,\hat{p}'_\beta,z) &= C_0^{(3)}(z)\ \mathcal{A}_{\alpha,j;\beta,k}(\hat{p}_\alpha,\hat{p}'_\beta,z),\\
\hat{\mathcal{T}}_{\alpha,j;0}(\hat{p}_\alpha,\hat{P}',z) &= C_0^{(3)}(z)\ \mathcal{A}_{\alpha,j;0}(\hat{p}_\alpha,\hat{P}',z)\\
\hat{\mathcal{T}}_{0;\beta,k}(\hat{P},\hat{p}'_\beta,z) &= C_0^{(6)}(z)\ \mathcal{A}_{0;\beta,k}(\hat{P},\hat{p}'_\beta,z),\\
\hat{\mathcal{T}}_{00}(\hat{P},\hat{P}',z) &= C_0^{(6)}(z)\ \mathcal{A}_{00}(\hat{P},\hat{P}',z)
\end{aligned}$$

with $C_0^{(N)}(z) = -\dfrac{e^{i\pi(N-3)/4}}{2^{(N-1)/2}\,\pi^{(N+1)/2}\,z^{(N-3)/4}}$ where for the function $z^{(N-3)/4}$ one takes the main branch. The functions $\mathcal{A}_{\alpha,j;\beta,k}$ represent amplitudes of elastic ($\alpha = \beta$; $j = k$) or inelastic ($\alpha = \beta$; $j \neq k$) scattering and rearrangement ($\alpha \neq \beta$) for the ($2 \to 2,3$) process, in the initial state of which the pair subsystem is in the $k$-th bound state and the complementary particle is asymptotically free. The function $\mathcal{A}_{0;\beta,k}$ represents for this process, a breakup amplitude of the system into three particles. The amplitudes $\mathcal{A}_{\alpha,j;0}$ and $\mathcal{A}_{00}$ correspond to processes respectivly. ($3 \to 2$) and ($3 \to 3$) in the state where initially, all three particles are asymptotically free. Remember that contributions to $\mathcal{A}_{00}$ from the single and double rescattering represent singular distributions (cм. [1]).

Describing in Sec. 4 of the paper [1] the analytical properties in variable $z$ and the smoothness properties in angular variables $\hat{P}$ or $\hat{p}_\alpha$ and $\hat{P}'$ or $\hat{p}'_\beta$, of the matrix $\hat{\mathcal{T}}$ kernels we have described thereby as well the properties of the amplitudes $\mathcal{A}(z)$.

To search for the amplitudes $\mathcal{A}(z)$ continued on the physical sheet, one can use e.g., the formulation [3], [11] of three-body scattering problem based on the Faddeev differential equations for components of the scattering wave functions considered in the coordinate space. It is necessary only to come in this formulation, to complex values of energy $z$. The square roots $z^{1/2}$ and $(z - \lambda_{\alpha,j})^{1/2}$, $\alpha = 1,2,3$, $j = 1,2,...,n_\alpha$, presenting in the formulae of [3], [11] determining asymptotical boundary conditions at the infinity, have to be considered as the main branches of $\sqrt{z}$ and $\sqrt{z - \lambda_{\alpha,j}}$. Solving the Faddeev differential equations with such conditions one finds really the analytical continuation on the physical sheet for the wave functions and consequently, for the amplitudes $\mathcal{A}(z)$. Knowing the amplitudes $\mathcal{A}(z)$, one can construct a necessary truncated scattering matrix $S_l(z)$ and then find those values of $z$ for which there exits a nontrivial solution $\mathcal{A}^{(\mathrm{res})}$ to Eq. (7.1). As mentioned above these values of $z$ represent the three-body resonances on respective sheet $\Pi_l$.

Concluding the paper we make the following remark.

It is well known [3] that a generalization of the Faddeev equations [2] on the case of systems with arbitrary number of particle is represented by the Yakubovsky equations [13]. The latter have the same structure as the Faddeev equations. Thus the scheme used in the present paper, may be applied as well to construction of the type (4.34), (5.1) and (6.1) explicit representations for analytical continuation of the $T$- and scattering matrices and resolvent on unphysical part of the energy Riemann surface in the $N$-body problems with arbitrary $N$.

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Представления для трехчастичной $T$-матрицы
на нефизических листах: доказательства


Дается доказательство явных представлений, сформулированных в предыдущей работе автора для аналитического продолжения компонент Фаддеева трехчастичной $T$-матрицы на нефизические листы римановой поверхности энергии. Проводится обоснование аналогичных представлений для аналитического продолжения трехчастичных матриц рассеяния и резольвенты. Обсуждается алгоритм нахождения резонансов в системе трех квантовых частиц на основании дифференциальных уравнений Фаддеева.



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Motovilov A.K. E5-95-46

Representations for Three-Body $T$-matrix
on Unphysical Sheets: Proofs


A proof is given for the explicit representations which have been formulated in the author's previous work for the Faddeev components of three-body $T$-matrix continued analytically on unphysical sheets of the energy Riemann surface. Also, the analogous representations for analytical continuation of the three-body scattering matrices and resolvent are proved. An algorithm to search for the three-body resonances on the base of the Faddeev differential equations is discussed.



The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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