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STUDYING THE STABILITY OF DIFFERENCE PROBLEMS ON SUBSTRUCTURED DOMAINS

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Сердюкова С.И., Сунэ М. E5-95-381
Исследование устойчивости разностных задач на областях составной структуры

С использованием систем аналитических вычислений REDUCE и MAPLE исследуются схема Русанова и схема Гари с начальными, дополнительными граничными условиями и условиями стыковки на областях составной структуры. Для схемы Русанова доказана устойчивость рассматриваемой задачи в L_2 и в равномерной метрике. В случае схемы Гари доказана степенная неустойчивость в L_2. Разработан алгоритм проверки устойчивости на PC, который может быть использован для исследования других разностных краевых задач.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ и в отделе научных вычислений Университета Уппсала.

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Serdyukova S.I., Thuné M. E5-95-381
Studying the Stability of Difference Problems on Substructured Domains

By using the computing algebra systems REDUCE and MAPLE the Rusanov scheme and the Gary schemes with initial, complementary boundary and overlap conditions on substructured domains are studied. Stability (in L_2 and in uniform metrics) of the resulting difference problem is proved for the Rusanov scheme. In the case of the Gary scheme a power instability in space L_2 is proved. The developed algorithm for stability verification on PC can be used for studying other difference boundary problems as well.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR and at the Department of Scientific Computing of Uppsala University.

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1 Introduction

The stability properties of the numerical method are essential in the numerical solution of time-dependent partial differential equations. Here, we will consider the stability of difference approximations of first order systems. We focus on stability verification, i.e. how to perform the stability analysis in practice.

The general stability theory [1]-[3] is based on normal mode (NM) analysis. To actually perform such an analysis is very difficult even for small problems and low order accurate approximations. Most of the cases reported in the literature concern problems that are small enough to be treated analytically, mostly with considerable effort. The authors that treated more difficult cases (e.g.,[4],[5]) used numerical techniques that do not take advantage of the special properties of the NM analysis problem.

Thuné [6] presented a numerical algorithm for stability verification, which, by being specially tailored for the task, is much more efficient than previous attempts. Moreover, he sketched a software environment for stability analysis, where the numerical algorithm is combined with symbolic algebraic manipulations [7]. In particular, the stability verification is to be performed completely symbolically, for classes of problems simple enough to allow for this.

Attempts with general symbolic algebraic algorithms for the solution of algebraic systems have indicated that such approaches are not viable for the problem of stability verification [8]. Instead, even in the symbolic case, specially tailored algorithms are needed. Promising results in this direction have been presented in [9]-[11]. A number of realistic difference methods for a scalar model problem has been treated. A special method for solving the algebraic systems analytically, using computer algebra systems on PC, is under development.

Here, we discuss the application of Serdyukova's algorithm to two concrete and nontrivial cases. The Rusanov scheme [12] and the Gary scheme [13] are considered. These schemes are used in airflow simulations [14]. Difference problems with initial, complementary boundary and overlap conditions are studied. Such problems arise when domains of complex geometry are substructured and different, overlapping grids are used.

First we consider the Rusanov scheme of the third order accuracy

intended for numerical solution of nonlinear hyperbolic systems

$$\frac{\partial w}{\partial t} = \frac{\partial F(w,x,t)}{\partial x} + f(w,x,t).$$

As usually the stability is studied in the case of constant coefficients. The hyperbolicity means that the matrix of F_w has real different eigenvalues in the region of (w,x,t) under consideration. Such a matrix can be reduced to diagonal form. So we study stability for the primitive hyperbolic equation $u_t = u_x$. In this case the Rusanov scheme has the form

$$u_\nu^{n+1} = u_\nu^n + \frac{\alpha}{12}(-u_{\nu+2}^n + 8u_{\nu+1}^n - 8u_{\nu-1}^n + u_{\nu-2}^n) + \frac{\alpha^2}{8}(u_{\nu+2}^n - 2u_\nu^n + u_{\nu-2}^n) +$$

$$+\frac{\alpha^3}{12}(u_{\nu+2}^n - 2u_{\nu+1}^n + 2u_{\nu-1}^n - u_{\nu-2}^n) - \frac{\omega}{24}(u_{\nu+2}^n - 4u_{\nu+1}^n + 6u_\nu^n - 4u_{\nu-1}^n + u_{\nu-2}^n).$$

Here $\alpha = \tau/h$, where τ, h are the step sizes in t, x respectively. The corresponding Cauchy problem is stable in space L_2 if α, ω lie in the region [12]

$$0 < \alpha \le 1, \quad 4\alpha^2 - \alpha^4 \le \omega \le 3.$$

Inside of this region there is stability in space C[15].

Here we discuss a problem with initial, complementary boundary and overlap conditions. The considered substructured domain [16] consists of two semi-infinite overlapping intervals with different grids $G1$ and $G2$, having steps $h1, h2$ in x respectively.

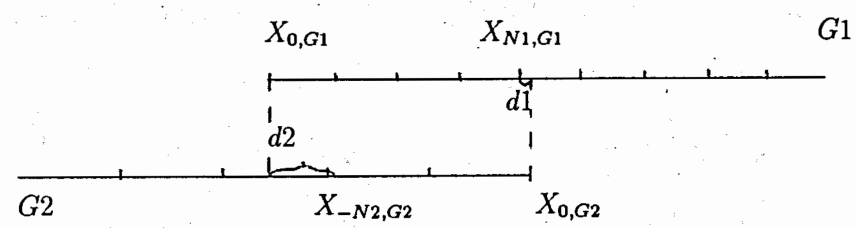


Fig. 1

$X_{0,G1}$ and $X_{0,G2}$ are boundary points of $G1$ and $G2$ respectively. $X_{N1,G1}$ is the rightmost point on $G1$ to the left of $X_{0,G2}$. Let $d1$ denote the distance between these points. We suppose, that $d1 < h1$. Symmetrically $X_{-N2,G2}$ is the leftmost point on $G2$ to the right of $X_{0,G1}$ and $d2$ is the distance between these points, $d2 < h2$.

On this substructured domain a problem with initial data is solved. Let $\mathcal{G}$ be the operator of transition from the layer with $t = n\tau$ to the layer with $t = (n+1)\tau$. What is $\mathcal{G}$ here? On $G1$ the left boundary problem for $u_t = u_x$ is solved. This is an outflow problem. Using the Rusanov scheme with $\alpha = \tau/h1$ we can find u_ν^{n+1} for $\nu \geq 2$. The value of u_1^{n+1} is found from the complementary boundary condition (the Lax-Wendroff scheme of the second-order accuracy)

$$u_1^{n+1} = u_1^n + \frac{\alpha}{2}(u_2^n - u_0^n) + \frac{\alpha^2}{2}(u_2^n - 2u_1^n + u_0^n).$$

The value of u_0^{n+1} is found by interpolation of solution values in points of $G2$ closest to $X_{0,G1}$:we use the points $X_{-N2,G2}, X_{-N2-1,G2}$ in the case of linear interpolation and in addition the point $X_{-N2+1,G2}$ in the case of quadratic interpolation. On $G2$ the right boundary problem is solved for $u_t = u_x$. This inflow problem can be interpreted as the left boundary problem for $v_t = -v_x$. Values of $v_\nu^{n+1}, \nu \geq 2$, are found by the Rusanov scheme with α replaced by $\hat{\alpha} = -\tau/h2 = -\phi\alpha$, $\phi = h1/h2$. The value of v_1^{n+1} is defined by the second complementary boundary condition, which is obtained from the first one by replacing u with v and α with $\hat{\alpha}$. The value of v_0^{n+1} is found by interpolation of solution values in the points of the grid $G1$ closest to the considered boundary point $X_{0,G2}$. Put $\beta 1 = d1/h1$, $\beta 2 = d2/h2$. The overlap conditions are

$$u_0^{n+1} = (1-\beta_2)v_{N2}^{n+1} + \beta_2 v_{N2+1}^{n+1}, \quad v_0^{n+1} = (1-\beta_1)u_{N1}^{n+1} + \beta_1 u_{N1+1}^{n+1}.$$

in the case of linear interpolation; and they are

$$u_0^{n+1} = \frac{1}{2}\beta_2(\beta_2 - 1)v_{N2-1}^{n+1} + (1-\beta_2^2)v_{N2}^{n+1} + \frac{1}{2}\beta_2(\beta_2+1)v_{N2+1}^{n+1},$$

$$v_0^{n+1} = \frac{1}{2}\beta_1(\beta_1 - 1)u_{N1-1}^{n+1} + (1-\beta_1^2)u_{N1}^{n+1} + \frac{1}{2}\beta_1(\beta_1+1)u_{N1+1}^{n+1}.$$

in the case of quadratic interpolation. So the operator $\mathcal{G}$ is presented.

The considered problem with initial, complementary boundary and overlap conditions (i.b.o. problem) is stable if there exists a positive constant $c > 0$ such that $||\mathcal{G}^n|| \leq c$ for all $n \geq 0$.

It is necessary for stability, that the spectrum of $\mathcal{G}$ lies in the unit disk $|z| \leq 1$. The classic spectrum definition is used: a point of the complex plain z_0 is a spectrum point of $\mathcal{G}$, if a nonzero sequence w_0 is found, such that $\mathcal{G}w_0 = z_0 w_0$. Here, w consists of two semi-infinite sequences $\{u_\nu\}, \{v_\nu\}, \nu \geq 0$, satisfying complementary boundary and overlap conditions. When the stability in L_2 is studied these sequences are supposed to be from L_2. And when the stability in C is studied, we suppose that all elements of these sequences are bounded uniformly in ν.

The spectral problem for the considered i.b.o. problem is a system of ordinary difference equations with spectral parameter z :

$$(1-z)u_\nu + \frac{\alpha}{12}(-u_{\nu+2} + 8u_{\nu+1} - 8u_{\nu-1} + u_{\nu-2}) +$$

$$+\frac{\alpha^2}{8}(u_{\nu+2} - 2u_\nu + u_{\nu-2}) + \frac{\alpha^3}{12}(u_{\nu+2} - 2u_{\nu+1} + 2u_{\nu-1} - u_{\nu-2}) -$$

$$-\frac{\omega}{24}(u_{\nu+2} - 4u_{\nu+1} + 6u_\nu - 4u_{\nu-1} + u_{\nu-2}) = 0, \quad \nu \geq 2;$$

$$(1-z)v_\nu - \phi\frac{\alpha}{12}(-v_{\nu+2} + 8v_{\nu+1} - 8v_{\nu-1} + v_{\nu-2}) +$$

$$+\phi^2\frac{\alpha^2}{8}(v_{\nu+2} - 2v_\nu + v_{\nu-2}) - \phi^3\frac{\alpha^3}{12}(v_{\nu+2} - 2v_{\nu+1} + 2v_{\nu-1} - v_{\nu-2}) -$$

$$-\frac{\omega}{24}(v_{\nu+2} - 4v_{\nu+1} + 6v_\nu - 4v_{\nu-1} + v_{\nu-2}) = 0, \quad \nu \geq 2;$$

$$(1-z)u_1 + \frac{\alpha}{2}(u_2 - u_0) + \frac{\alpha^2}{2}(u_2 - 2u_1 + u_0) = 0,$$

$$(1-z)v_1 - \phi\frac{\alpha}{2}(v_2 - v_0) + \phi^2\frac{\alpha^2}{2}(v_2 - 2v_1 + v_0) = 0;$$

in the case of linear interpolation

$$u_0 = (1-\beta_2)v_{N2} + \beta_2 v_{N2+1}, \quad v_0 = (1-\beta_1)u_{N1} + \beta_1 u_{N1+1};$$

in the case of quadratic interpolation

$$u_0 = \frac{1}{2}\beta_2(\beta_2 - 1)v_{N2-1} + (1-\beta_2^2)v_{N2} + \frac{1}{2}\beta_2(\beta_2+1)v_{N2+1},$$

$$v_0 = \frac{1}{2}\beta_1(\beta_1 - 1)u_{N1-1} + (1-\beta_1^2)u_{N2} + \frac{1}{2}\beta_1(\beta_1+1)u_{N1+1}.$$

The solution of the spectral problem can be represented by the roots of the characteristic polynomials:

$$P_1(z,\kappa) = (1-z)\kappa^2 + \frac{\alpha}{12}(-\kappa^4 + 8\kappa^3 - 8\kappa + 1) + \frac{\alpha^2}{8}(\kappa^4 - 2\kappa^2 + 1) +$$

$$+\frac{\alpha^3}{12}(\kappa^4 - 2\kappa^3 + 2\kappa - 1) - \frac{\omega}{24}(\kappa^4 - 4\kappa^3 + 6\kappa^2 - 4\kappa + 1),$$

$$P_2(z,\hat\kappa) = (1-z)\hat\kappa^2 - \phi\frac{\alpha}{12}(-\hat\kappa^4 + 8\hat\kappa^3 - 8\hat\kappa + 1) + \phi^2\frac{\alpha^2}{8}(\hat\kappa^4 - 2\hat\kappa^2 + 1) -$$

$$-\phi^3\frac{\alpha^3}{12}(\hat\kappa^4 - 2\hat\kappa^3 + 2\hat\kappa - 1) - \frac{\omega}{24}(\hat\kappa^4 - 4\hat\kappa^3 + 6\hat\kappa^2 - 4\hat\kappa + 1).$$

It is necessary for the stability of the considered i.b.o. problem that the corresponding Cauchy problems on infinite grids with the steps $h1, h2$ are stable. So we suppose that α, ω, ϕ satisfy inequalities $\alpha^2 \le 1,\ 4\alpha^2 - \alpha^4 \le \omega \le 3;\ \phi^2\alpha^2 \le 1,\ 4\phi^2\alpha^2 - \phi^4\alpha^4 \le \omega \le 3.$ Then [1] for z with $|z| > 1$ each characteristic equation, $P_1(z,\kappa) = 0,\ \ P_2(z,\hat\kappa) = 0$ has exactly two solutions with absolute value less than 1: κ_1, κ_2 and $\hat\kappa_1, \hat\kappa_2$, respectively. In the general case (when $\kappa_1 \ne \kappa_2$, and $\hat\kappa_1 \ne \hat\kappa_2$) the solution of the considered spectral problem has the following form:

$$u_\nu = c_1\kappa_1^\nu + c_2\kappa_2^\nu, \quad v_\nu = c_3\hat\kappa_1^\nu + c_4\hat\kappa_2^\nu, \quad \nu \ge 0.$$

On introducing these into the boundary and overlap conditions we get a linear homogeneous system for the definition of c_1, c_2, c_3, c_4. This system has nonzero solution, if the determinant is equal to zero. As a result we have the determinant equation:

$$D(z) = det(E) = \det\begin{bmatrix} -1 & -1 & e1(\hat\kappa_1) & e1(\hat\kappa_2) \\ e2(\kappa_1) & e2(\kappa_2) & -1 & -1 \\ 0 & 0 & e3(\hat\kappa_1) & e3(\hat\kappa_2) \\ e4(\kappa_1) & e4(\kappa_2) & 0 & 0 \end{bmatrix} = 0.$$

where

$$e1(\hat\kappa) = (1-\beta_2)\hat\kappa^{N2} + \beta_2\hat\kappa^{N2+1}, \quad e2(\kappa) = (1-\beta_1)\kappa^{N1} + \beta_1\kappa^{N1+1},$$

in the case of linear interpolation, and in the case of quadratic interpolation

$$e1(\hat\kappa) = \frac{1}{2}\beta_2(\beta_2 - 1)\hat\kappa^{N2-1} + (1-\beta_2^2)\hat\kappa^{N2} + \frac{1}{2}\beta_2(\beta_2+1)\hat\kappa^{N2+1},$$

$$e2(\kappa) = \frac{1}{2}\beta_1(\beta_1 - 1)\kappa^{N1-1} + (1-\beta_1^2)\kappa^{N1} + \frac{1}{2}\beta_1(\beta_1+1)\kappa^{N1+1}.$$

The elements of the last two rows are defined by

$$e3(\hat\kappa) = (1-z)\hat\kappa - \phi\frac{\alpha}{2}(\hat\kappa^2 - 1) + \phi^2\frac{\alpha^2}{2}(\hat\kappa - 1)^2,$$

$$e4(\kappa) = (1-z)\kappa + \frac{\alpha}{2}(7\kappa^2 - 1) + \frac{\alpha^2}{2}(\kappa - 1)^2.$$

To summarize: we have shown that the spectrum of the considered i.b.o. problem is described by a system of five polynomial equations with five variables:

$$D(z,\kappa_1,\kappa_2,\hat\kappa_1,\hat\kappa_2) = 0, \quad |z| \ge 1,$$

$$P_1(z,\kappa_1) = 0, \quad P_1(z,\kappa_2) = 0,$$
$$P_2(z,\hat\kappa_1) = 0, \quad P_2(z,\hat\kappa_2) = 0.$$

2 Solving for the spectrum symbolically

In the following, we describe the steps in Serdyukova's algorithm for symbolically solving the polynomial equations above. First, reduce this system to a system of two polynomial equations in two variables $x = \kappa_1\kappa_2, y = \hat\kappa_1\hat\kappa_2$. First $det(E(\kappa_1,\kappa_2,\hat\kappa_1,\hat\kappa_2,z)) = 0$ is transformed by a number of elementary manipulations with the columns of E into $det(\hat E(x1, x, \hat x1, y, z)) = 0,\ x1 = \kappa_1 + \kappa_2,\ \hat x1 = \hat\kappa_1 + \hat\kappa_2$. Then by using the Vieta relations for the characteristic equations, we get $x1 = x1(x), z = z(x)$ and $\hat x1 = \hat x1(y), z = z(y)$. All these functions are simple ratios of polynomials. After substituting them into the transformed determinant equation we get one of the polynomial equations. Another equation is obtained from the relation $z(x) = z(y)$:

$$P(x,y) = num(z(x) - z(y)) = 0,$$

$$Q(x,y) = num(det(\hat{E}(x1(x), x, \hat{x}1(y), y, z(x)) = 0.$$

This system automatically describes the spectrum points $z, |z| \geq 1$, when $\kappa_1 = \kappa_2$ or $\hat{\kappa}_1 = \hat{\kappa}_2$. This follows from the structure of the solution in the case of multiple roots of $P_1(\kappa, z), P_2(\kappa, z)$.

Solving this system by using REDUCE [17]: $solve(\{P, Q\}, \{x, y\})$, gave no result. After this we used the resultant method. The reduction to the resultant leads to false solutions. Another reason why false solutions arise is the following. We cannot say a priori in the program, that the new variables $x1, x, \hat{x}1, y$ are symmetric functions of κ_j and $\hat{\kappa}_j$, with absolute value less than 1. Thus, in the program, $x1, x$ depend on two arbitrary κ_j and $\hat{x}1, y$ depend on two arbitrary $\hat{\kappa}_j$. An algorithm for separating the false spectrum points was developed and implemented on a PC, using REDUCE. The separation has two steps.

In the first step, for each $x, |x| \leq 1$, (the solution of the resultant) we find all solutions $y, |y| \leq 1$, of $P(x,y) = 0$ (the first polynomial equation). For each such pair (x, y) the values $x1(x), \hat{x}1(y), z(x)$ are calculated. For the following analysis are kept only the sets $(z, x1, \hat{x}1, x, y)$, satisfying the natural inequalities

$$|x1| \leq 2, \quad |\hat{x}1| \leq 2, \quad |z| \geq 1 - \epsilon.$$

In our program $\epsilon = 10^{-10}$. The points $z, 1 - \epsilon \leq |z| \leq 1 + \epsilon$, from the $\epsilon-$ vicinity of the unit circle are replaced by $z/|z|$, the closest points of the unit circle. After this we separate the false spectrum points, arising in the reduction to the resultant. We must simply verify if the selected (x, y) satisfy $Q(x,y) = 0$ (the second polynomial equation). But x, y were found numerically and $Q(x,y)$ has many terms with rather big coefficients. In order to avoid numerical difficulties we solve the resultant equation with high accuracy, put in our REDUCE calculations PRECISION 15. Consequently, the solutions $x, |x| \leq 1$, lie at a distance less than 10^{-15} from the corresponding exact solutions. In our program x_0, y_0 are assumed to satisfy the second equation if the following inequality holds

$$|Q(x_0, y_0)| < 2 \times 10^{-15} \left| Q_x(x_0, y_0) + Q_y(x_0, y_0) \frac{P_x(x_0, y_0)}{P_y(x_0, y_0)} \right|.$$

The absolute value of the derivative of Q with respect to x is of correct size, since it is reasonable to assume that in such a small (10^{-15}) vicinity

of the exact solution, the absolute value of the derivative can change not more than twice. This completes the first step of the separation.

In the second step the selected sets $(z, x1, \hat{x}1, x, y)$ are analysed. For each set $\kappa_1, \kappa_2, \hat{\kappa}_1, \hat{\kappa}_2$ (the roots of $P_1(\kappa, z), P_2(\hat{\kappa}, z)$ characteristic polynomials with z from the considered set less than one in absolute value) are found. We check if $x1, \hat{x}1$ from the considered set are really the sums of the computed κ_1, κ_2 and $\hat{\kappa}_1, \hat{\kappa}_2$ respectively. In our implementation, this is considered to be the case if the following inequalities hold:

$$|x1 - (\kappa_1 + \kappa_2)| < 10^{-3}|x1|, \quad |\hat{x}1 - (\hat{\kappa}_1 + \hat{\kappa}_2)| < 10^{-3}|\hat{x}1|.$$

All constants and inequalities were chosen in the process of numerical experiments. In what follows we present the results of concrete studies produced on a PC, using REDUCE and MAPLE.

I. The first example is the i.b.o. problem for the Rusanov scheme with the following parameters: $\alpha = 1/2$, $\omega = 2$, $N1 = 2$, $N2 = 1$, $\phi = 1/2$, $\beta_1 = 1/2$, $\beta_2 = 1/4$. The quadratic interpolation is used. The substructured domain here is the following

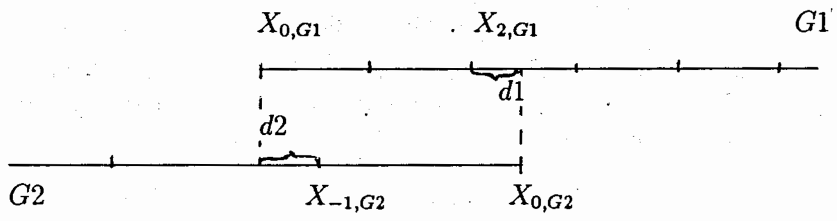


Fig. 2

Here, u_0^{n+1} is found by interpolation of $v_{-2}^{n+1}, v_{-1}^{n+1}, v_0^{n+1}$ values. The last one is the solution value in the boundary point $X_{0,G2}$. But we can find first v_0^{n+1} by interpolation of $u_1^{n+1}, u_2^{n+1}, u_3^{n+1}$ solution values in the inner points of $G1$.

Here P(x,y) is the polynomial of the sixth order with respect to y, while $Q(x,y)$ - of the fifteenth. So we prefer to use P to find y.

The resutant (the polynomial of 106 order) in factored form is

$$R(x) = 1967427222552000000000x^3(x^2 - 7)^4(x^3 + 63x^2 - 9x - 7)^6 S(x).$$

The last factor is the polynomial of 77 order with huge coefficients. The maximal one is the coefficient of x^{37} :

+23776275738933848749928920812229736190814717991598426600493

4853326.

The characteristic equations have not zero solutions for $z, |z| \geq 1$. So only nonzero $x, |x| \leq 1$, are interesting. The second factor gives two such solutions and the last - 37. For each of 39 selected x we find nonzero solutions $y, |y| \leq 1$ by solving $P(x, y)$. After the first step of separation only three sets were kept:

$$[z, \quad x1, \quad \hat{x}1, \quad x, \quad y]$$

$$[1.0370381621219, 1.46132552723516, 1.05246335761816,$$
$$0.417550457853634, 0.158337807785756],$$
$$[1, 1.4107123111035, 1.17793091226657, 0.410712311103496,$$
$$0.177930912266573],$$
$$[1, 0.730070495734075, 1.17793091226657, -0.269929504265927,$$
$$0.177930912266574].$$

On the second step all sets were rejected. Let us consider, for instance, why the sets with $z = 1$ were rejected. $P_1(\kappa, 1)$ (the first characteristic polynomial with $z = 1$) has roots

$$\kappa 1 = -0.269..., \quad \kappa 2 = 0.410..., \quad \kappa 3 = 1, \quad \kappa 4 = -63.14...$$

The third root is the limit value of $\kappa_3(z)$, greater than one in absolute value, when $z \rightarrow 1$,being outside the unit disk. The sum $\kappa_1 + \kappa_2 =$ 0.140... is not close to $x1 = 1.140...$ from the second set and not close to $x1 = 0.730...$ from the third set. But it's clear that $x1$ from the second set is close to $\kappa_2 + \kappa_3$ and $x1$ from the third set is close to $\kappa_1 + \kappa_3$. As a matter of fact these values coincide in all 10 digits printed, which guarantees the correctness of the results computed indepedently on the PC. The values $x1$ in the sets were obtained at the end of long analytical and numerical calculations. To conclude, we have shown that $z = 1$ is not a spectrum point. Thus, it has been proved that there are no spectrum points outside the unit disk and on its boundary. For

the parameter values under consideration, the corresponding Cauchy problems are stable in L_2 and in C. Then the considered i.b.o. problem is stable in L_2 [1] and in C [3].

II. The same result was proved in the case of linear interpolation.

III. The third example is i.b.o. problem for the Gary scheme [13]. The multistage form for $u_t = u_x$ is

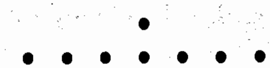
$$\check{u}_\nu = u_\nu^n + \frac{\alpha}{2}\Delta_0 u_\nu^n = u_\nu^n + \frac{\alpha}{2}(u_{\nu+1}^n - u_{\nu-1}^n),$$

$$\check{\check{u}} = u_\nu^n + \frac{\alpha}{4}\Delta_0(u_\nu^n + \check{u}_\nu),$$

$$u_\nu^{n+1} = u_\nu^n + \frac{\alpha}{4}\Delta_0(u_\nu^n + \check{\check{u}}_\nu), \quad \alpha = \tau/h.$$

Here, Δ_0 denotes the second order centered difference operator in space. For $\alpha \leq 2$ the corresponding Cauchy problem is stable in L_2. The Gary scheme has second order accuracy. It is well known that all schemes of even order accuracy are unstable in C (see, e.g., [18]). We consider

$\alpha = 2, \quad N1 = 2, \quad N2 = 1, \quad \phi = 1/2, \quad \beta 1 = \beta 2 = 0$. In contrast to the previous case (Rusanov's scheme), there is no displacement here ($d1 = d2 = 0$.) The overlap conditions here are simply $u_0^{n+1} = v_1^{n+1}, \quad v_0^{n+1} = u_2^{n+1}$. The Gary scheme



requires two complementary boundary conditions [16] to define u_1^{n+1}, u_2^{n+1} :

$$u_2^{n+1} = u_2^n + \frac{\alpha}{4}\Delta_0(u_2^n + \check{\check{u}}_2),$$

$$u_1^{n+1} = u_1^n + \frac{\alpha}{4}\Delta_0(u_1^n + \check{\check{u}}_1),$$

$$\check{u}_0 = \check{\check{u}}_0 = 2u_1^{n+1} - u_2^{n+1}.$$

The complementary boundary conditions, defining v_1^{n+1}, v_2^{n+1}, are obtained from above by replacing u with v and α with $-\phi\alpha$. The Rusanov scheme has on the low layer 5 point, the Gary scheme - 7. This complicates studying. In particular, the resultant $R(x)$ is here a polynomial of order 900. REDUCE was not able to find it. MAPLE did this and factored $R(x)$ in approximately 4 hours. The factored $R(x)$ has 10 different factors-polynomials with maximal order 68 and sufficiently small coefficients. The resultant has 96 nonzero solutions $x, |x| \leq 1$. For each computed x the polynomial equation $P(x, y) = 0$ of fifteenth order is solved with respect to y. So we have to solve a lot of polynomial equations with complex coefficients. All polynomials were solved and all the selected solutions were analysed. As a result the unique (in the region $|z| \geq 1$) spectrum point $z = 1$ was found. Here $\kappa 1(1) = 1$. This leads to a power instability in space L_2. Thus, for the considered problem with initial, complementary boundary and overlap conditions $||\mathcal{G}^n||$ grows as $\sqrt{n}$ when $n \to \infty$. This is in agreement with the results of numerical experiments [16]. It is an example of an interesting instability phenomenon: the instability is observed in calculations only in the case of an even number ($N1$) of full steps ($h1$) on the overlapping interval for the outflow problem. This phenomenon was explained theoretically in [19].

3 Conclusion

We have presented stability analyses of two difference methods, the Rusanov scheme and the Gary scheme, for a nontrivial case with initial, boundary, and overlap conditions on a substructured domain, with two overlapping, structured grids. The analyses were performed analytically, by means of symbolic algebraic manipulations, using REDUCE and MAPLE on a PC. Previous attempts to study the stability of these schemes, under such nontrivial conditions, have been based on numerical approaches.

The conclusions of the analyses are that the Rusanov scheme of third order accuracy is stable in L_2 and in C, whereas the second order Gary scheme (which uses centered differences in space and a Runge-Kutta type time-marching scheme) exhibits an interesting instability phenomenon, for special choices of parameters.

The key to the analyses presented here, is the algorithm invented by Serdyukova. In this paper, and in Serdyukova's previous work, the algorithm was applied to concrete difference problems. In ongoing work, we aim at expressing the algorithm in a general form, and to implement (primarily in MAPLE or REDUCE) of a general software tool for symbolic stability analysis.

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