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DYNAMIC PROPERTIES OF LOCALIZED SOLUTIONS OF ANISOTROPIC CHIRAL FIELD OF UNIT 3-COMPONENT ISOVECTOR. I. ON COMPLETE INTEGRABILITY OF ANISOTROPIC HEISENBERG ANTIFERROMAGNET IN 2D SPACE-TIME

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1. The present paper launches a series of publications in which dynamic properties of classical localized solutions will be considered within a model of the 3-component "easy-axis" chiral field of unit isovector $s_a(\mathbf{x},t), a = 1,2,3, s_a s_a = 1$ (we shall call it "the A3-field" for brevity to underline an anisotropy of this field in internal space - on the unit sphere S^2). The Lorentz-invariant A3-field emerges, for instance, when going over to the continuous limit in lattice models of Heisenberg antiferromagnet (AFM) [1] and ferroelectrics [2], which have easy-axis anisotropy. Futhermore the A3-field (and possibly other anisotropic scalar isovector fields taking their values on unit spheres S^n) may prove to be useful side by side with vector gauge and spinor fields when developing theoretical schemes beyond the Standard Model of the high-energy physics; we mean by that the A3-field may be used as an alternative of the Higgs field. In particular, the Maxwell field becomes massive as a result of its "minimal" interaction with the A3-field [3], and within the system of the interacting Maxwell and A3-fields there exist non-one-dimensional particle-like solutions (in [3] two-dimensional axisymmetric solitons have been found). Thus, to our opinion the A3-field model may be of interdisciplinary interest for theoretical physics.

Note that in the A3-field model (easy-axis Heisenberg AFM) there exist both one-dimensional (1D) [1,5] and two-dimensional (2D) [4] dynamic (having periodic dependence of the isovector $s_a(\mathbf{x})$ on time t) solitons with nonzero topological indices ("topological charges", for their definitions see, e.g., [1], [6]).

We shall start investigation of dynamic properties of localized A3-field solutions within the framework of the 1D model, studying the stability and interaction under collisions of the DT solitons of the A3-field. In this paper we put forward the hypothesis and present some arguments that the 1D A3-field model and its 1D generalizations are completely integrable.

2.Lagrangian and Hamiltonian densities of the A3-field model in standard field theory notation are as follows (recall that $s_a s_a = 1$):

$$\mathcal{L}=\frac{1}{2}[\partial_\mu s_a\partial^\mu s_a+(s_3^2-1)]=\frac{1}{2}[(\partial_0 s_a)^2-(\partial_1 s_a)^2+(s_3^2-1)], \quad (1)$$

$$\mathcal{H}=\frac{1}{2}[(\partial_0 s_a)^2+(\partial_1 s_a)^2+(1-s_3^2)], \quad (2)$$

here $a=1,2,3,\mu=0,1,...,D(D=1$ in the present paper), $\partial_0=\frac{\partial}{\partial t},\partial_1=\frac{\partial}{\partial x},\partial_\mu\partial^\mu=\partial_0^2-\partial_1^2=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}$, and summation over repeated indices μ,a is meant. The Euler-Lagrange equations derived from (1) under taking into account the constraint $s_a s_a=1$ have the form:

$$\partial_\mu\partial^\mu s_i+(\partial_\mu s_a\partial^\mu s_a)s_i-s_3(\delta_{i3}-s_i s_3)=0, \quad (3)$$
$$i=1,2,3$$

(notice, that the second term in Eq.(3) may be rewritten as $-s_a(\partial_\mu\partial^\mu s_a)s_i$).

Using representation of components s_i of the unit isovector $\mathbf{s}(\mathbf{x},t)$ in terms of angular variables θ,ϕ on the sphere S^2,

$$s_1=\sin\theta\cos\phi, s_2=\sin\theta\sin\phi, s_3=\cos\theta, \quad (4)$$

and in terms of u,v variables (or equivalently in terms of $z=u+iv$), arising as a result of stereographic projection of S^2 to R^2_{comp},

$$z=u+iv=\frac{s_1+is_2}{1+s_3}=\frac{\sin\theta\exp(i\phi)}{1+\cos\theta}=\tan\frac{\theta}{2}\exp(i\phi), \quad (5)$$

we derive from (1),(2) expressions for $\mathcal{L}$ and $\mathcal{H}$ in terms of θ,ϕ

$$\mathcal{L}=\frac{1}{2}[\partial_\mu\theta\partial^\mu\theta+\sin^2\theta\partial_\mu\phi\partial^\mu\phi-\sin^2\theta]=$$
$$=\frac{1}{2}\left\{\left(\frac{\partial\theta}{\partial t}\right)^2-\left(\frac{\partial\theta}{\partial x}\right)^2+\sin^2\theta\left[\left(\frac{\partial\phi}{\partial t}\right)^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right]-\sin^2\theta\right\}, \quad (6)$$

$$\mathcal{H}=\frac{1}{2}\left\{\left(\frac{\partial\theta}{\partial t}\right)^2+\left(\frac{\partial\theta}{\partial x}\right)^2+\sin^2\theta\left[\left(\frac{\partial\phi}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial x}\right)^2\right]+\sin^2\theta\right\}, \quad (7)$$

and in terms of complex z

$$\mathcal{L}=\frac{2(\partial_\mu z^*\partial^\mu z-z^*z)}{(1+z^*z)^2}, \quad (8)$$

$$\mathcal{H}=\frac{2(\partial_0 z^*\partial_0 z+\partial_1 z^*\partial_1 z+z^*z)}{(1+z^*z)^2}. \quad (9)$$

The dynamic system defined by relations (1),(3) possesses the following densities of momentum $\mathcal{P}_x$ and of "isospin" $\mathcal{S}_3$

$$\mathcal{P}_x=\frac{\partial s_a}{\partial t}\frac{\partial s_a}{\partial x}, \quad (10)$$

$$\mathcal{S}_3=s_1\frac{\partial s_2}{\partial t}-s_2\frac{\partial s_1}{\partial t}. \quad (11)$$

The Noether invariants of the system (1),(3) can be calculated by means of integration of the densities $\mathcal{H},\mathcal{P},\mathcal{S}_3$:

$$H=\int\mathcal{H}dx, \quad (12)$$

$$P_x=\int\mathcal{P}_x dx, \quad (13)$$

$$S_3=\int\mathcal{S}_3 dx. \quad (14)$$

In particular

$$S_3=2\int\sin^2\theta\frac{\partial\phi}{\partial t}dx. \quad (15)$$

Euler-Lagrange equations which follow from (6) are:

$$2\partial_\mu\partial^\mu\theta+\sin 2\theta(1-\partial_\mu\partial^\mu\phi)=$$
$$=2\left(\frac{\partial^2\theta}{\partial t^2}-\frac{\partial^2\theta}{\partial x^2}\right)+\sin 2\theta\left[1-\left(\frac{\partial\phi}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial x}\right)^2\right]=0, \quad (16)$$

$$2\cos\theta\partial_\mu\theta\partial^\mu\phi+\sin\theta\partial_\mu\partial^\mu\phi=$$

$$2\cos\theta\left(\frac{\partial\theta}{\partial t}\frac{\partial\phi}{\partial t}-\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial x}\right)+\sin\theta\left(\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}\right)=0. \qquad (17)$$

These equations are equivalent to Eqs. (6) of the paper [5] (see also [1]).

Notice that by setting $\phi=0$ (i.e., $s_2=0$) we get the famous completely integrable sine-Gordon equation (SGE) [8-13] for 2θ variable:

$$\frac{\partial^2(2\theta)}{\partial t^2}-\frac{\partial^2(2\theta)}{\partial x^2}+\sin 2\theta=0. \qquad (18)$$

It is quite natural to put the question: is the system described by (16),(17) completely integrable as well?

When varying Lagrangian defined by (8) we arrive at an equivalent equation for the complex variable $z(x,t)$ (it was apparently written out first in [7]):

$$(1+z^*z)\partial_\mu\partial^\mu z-2z^*\partial_\mu z\partial^\mu z+(1-z^*z)z=$$

$$=(1+z^*z)\left(\frac{\partial^2 z}{\partial t^2}-\frac{\partial^2 z}{\partial x^2}\right)-2z^*\left[\left(\frac{\partial z}{\partial t}\right)^2-\left(\frac{\partial z}{\partial x}\right)^2\right]+(1-z^*z)z=0. \qquad (19)$$

Making substitution $z=u\exp(i\phi), u=\tan\frac{\theta}{2}$ and setting $\phi=0$, we find once again the SGE for $\theta(x,t)$. Therefore, Eq.(19) may be viewed as complexification of the SGE emerging in the quasiclassical models of theoretical physics. Moreover, the model under consideration possesses soliton solutions which prove to be direct generalizations of the SGE solitons. Zero-velocity soliton of the model can be easily found in terms of z (denote it $z_0(x,t)$):

$$z_0(x,t)=\exp\left[\substack{+\\-}(1-\omega^2)^{\frac{1}{2}}x\right]\exp(i\omega t+i\phi_0), |\omega|<1, \qquad (20)$$

in terms of θ,ϕ:

$$\theta(x,t)=2\arctan\left\{\exp\left[\substack{+\\-}(1-\omega^2)^{\frac{1}{2}}\right]\right\}, \qquad (21)$$

$$\phi(x,t)=\phi_0+\omega t, \quad \phi_0=const,$$

and finally by using (4),(21) in terms of $s_i(x,t)$, $i=1,2,3$

Making Lorentz transformation

$$x\to\gamma(x-vt), t\to\gamma(t-vx), \gamma=(1-v^2)^{\frac{1}{2}}, \qquad (22)$$

we can find from (20),(21) the soliton moving at velocity $v, -1<v<+1$. To illustrate this, we write it down in $z(x,t)$ terms:

$$z_v(x,t)=z_0[\gamma(x-vt),\gamma(t-vx)]=.$$

$$=\exp[\substack{+\\-}P\gamma(x-vt)]\exp[i\omega\gamma(t-vx)+i\phi_0]=$$

$$=\exp(p_0t-p_1x+i\phi_0)=\exp(p_\mu x_\mu+i\phi_0),$$

$$P=(1-\omega^2)^{\frac{1}{2}}, \qquad (23)$$

where we use notation

$$p_0=i\omega\gamma\substack{-\\+}\gamma vP, \quad p_1=i\omega\gamma v\substack{-\\+}\gamma P. \qquad (24)$$

The values p_0 and p_1 are related by the identity:

$$p_\mu p^\mu=p_0^2-p_1^2=-1. \qquad (25)$$

Notice that the 1-soliton solution in the easy-axis Heisenberg AFM has been first found in [5] (see also [1]). However as far as we know the issue of complete integrability of the system (1),(3) was not till now discussed in the literature.

3. One of the most important characteristic properties of completely integrable PDEs is that if such systems possess 1-soliton solution, then they possess N-soliton solutions as well. These N-soliton solutions can be found [8-13]:

1) by inverse scattering method (ISM), if Lax or zero-curvature representation is available for the system under consideration,

2) by means of the Backlund transformation if it is known for the system under study,

3) by the direct Hirota technique, which allowed to find N-soliton solutions to the SGE [14].

Below we show that Eqs.(3), describing the A3-field, admit Hirota representations which are the generalizations of the bilinear Hirota representation for the SGE [14]. Here we use the same ansatz

$$z = \frac{G}{F}, \tag{26}$$

as was used for the SGE, but for the A3-field model we assume functions $G(x,t)$ and $F(x,t)$ to be complex-valued. By using $\mathcal{D}_\mu$ operators, defined by the relations [15,16]

$$\mathcal{D}^2_\mu[G(X) * F(X)] =$$

$$= \left\{ \left(\frac{\partial}{\partial X_\mu} - \frac{\partial}{\partial X'_\mu} \right) \left(\frac{\partial}{\partial X^\mu} - \frac{\partial}{\partial X'^\mu} \right) [G(X) F(X')] \right\}_{X=X'} =$$

$$= F\partial_\mu \partial^\mu G - 2\partial_\mu G \partial^\mu F + G\partial_\mu \partial^\mu F, \tag{27}$$

$$X = (x,t), X' = (x', t'),$$

we obtain from Eq. (19):

$$\mathcal{D}^2_\mu(F * G) + FG = 0, \tag{28}$$

$$F^* G \mathcal{D}^2_\mu(F * F) = G^* F \mathcal{D}^2_\mu(G * G), \tag{29}$$

or in standard notation:

$$F\partial_\mu \partial^\mu G - 2\partial_\mu F \partial^\mu G + G\partial_\mu \partial^\mu F + FG = 0, \tag{30}$$

$$F^* G(\partial_\mu F \partial^\mu F - F\partial_\mu \partial^\mu F) = G^* F(\partial_\mu G \partial^\mu G - G\partial_\mu \partial^\mu G). \tag{31}$$

In contrast to Hirota form of the SGE, the representation (30), (31) is not bilinear due to complex-valuedness of $F(X)$ and $G(X)$. However by introducing $H(X)$ and $E(X)$ functions which satisfy equations:

$$\mathcal{D}^2_\mu(F * F) = FH, \tag{32}$$

$$\mathcal{D}^2_\mu(G * G) = GE, \tag{33}$$

we obtain a bilinear representation of the Eq.(19) and hence of the Eq.(3):

$$\mathcal{D}^2_\mu(F * G) + FG = 0, \tag{34}$$

$$\mathcal{D}^2_\mu(F * F) = FH, \tag{35}$$

$$\mathcal{D}^2_\mu(G * G) = GE, \tag{36}$$

$$F^* H = G^* E. \tag{37}$$

The search for N-soliton solutions of the A3-field model by using representations (28), (29) and (34)-(37) is presently in progress.

4. In order to investigate dynamic properties of localized solutions to Eqs.(3) and in particular the issue of complete integrability of the Eqs.(3) by means of computer experiments we have elaborated a numerical technique which is based on using independent variables, arising as a result of stereographic projection of the unit sphere S^2 to the plane R^2. Namely, we made the following substitution of variables: if $s_3 \geqq 0$,

$$z = \frac{s_1 + is_2}{1 + s_3}, \tag{38}$$

and

$$w = \frac{s_1 + is_2}{1 - s_3}, \tag{39}$$

if $s_3 < 0$, and solved numerically Eq. (19) (if $s_3 \geqq 0$) or the identical equation for the w variable,

$$(1 + w^* w) \left(\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} \right) - 2w^* \left[\left(\frac{\partial w}{\partial t} \right)^2 - \left(\frac{\partial w}{\partial x} \right)^2 \right] + (1 - w^* w) w = 0, \tag{40}$$

(if $s_3 < 0$). To do this we employed a stable explicit finite-difference scheme of the second accuracy order, which ensured conservation of the motion invariants H, P_x and S_3 with high accuracy. This numerical algorithm was thoroughly checked in test experiments with initial data set at $t = 0$ by known analytical soliton and bion (breather) solutions of the SGE and by more general soliton solutions (20)-(24). Our computer experiments confirmed that 1D solitons of the A3-field are stable [5].

Then we considered head-on collisions of kink (k) moving at velocity $v_1 = v$ and antikink (a) with $v_2 = -v$ (in contrast to the case of SGE only kink-antikink (ka) collisions are admissible in the case of the A3-field model). We accomplished computer experiments on head-on ka-collisions for various sets of parameters characterizing ka-pair, and we have not seen generation of small-amplitude waves under these interactions; this supports our hypothesis that the Eqs.(3) are completely integrable. More detailed presentation of these experiments supplied with graphical computer output will be given elsewhere.

In the next section we describe some results of search for zero-curvature representation for a more wide class of models, comprising the A3-field model as a particular case.

5. Consider N-component field $s_a(x,t)$, $s_a s_a = 1$, $a = 1, 2, ..., N$, with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left\{(\partial_\mu s_a \partial^\mu s_a) + J_a \delta_{ab}[s_b(x,t)s_a(x,t) - s_b(\infty,t)s_a(\infty,t)]\right\} =$$

$$= \frac{1}{2}\left[(\partial_\mu s_a \partial^\mu s_a) + (\mathbf{J}\mathbf{s}, \mathbf{s}) - \mathbf{J}_N\right], \tag{41}$$

here $\mathbf{J}$ is the diagonal matrix, $\mathbf{J} = diag(J_1, J_2, ..., J_N)$; we assume that $J_1 \leqq J_2 \leqq ... < J_N$, here δ_{ab} is the Kronecker symbol.

Consider the chiral field g(x,t) defined by

$$g_{ik} = \delta_{ik} - 2s_i s_k, \quad i, k = 1, ..., N. \tag{42}$$

It is easy to check that

$$g^{-1}_{ik} = g_{ik} = \delta_{ik} - 2s_i s_k, \quad (\mathbf{g}^{-1}\mathbf{g})_{lm} = (\mathbf{g}\mathbf{g}^{-1})_{lm} = \delta_{lm}. \tag{43}$$

Define "left" currents [8],[13]

$$\mathbf{l}_0(x,t) = \mathbf{g}_t\mathbf{g}^{-1}, \mathbf{l}_1(x,t) = \mathbf{g}_x\mathbf{g}^{-1}, \tag{44}$$

$$\mathbf{g}_t = \frac{\partial \mathbf{g}}{\partial t}, \mathbf{g}_x = \frac{\partial \mathbf{g}}{\partial x}.$$

These currents satisfy the identity [8], [13]

$$\mathbf{l}_{1t} - \mathbf{l}_{0x} + [\mathbf{l}_1, \mathbf{l}_0] = 0. \tag{45}$$

Then it is straightforward to check that

$$(\mathbf{l}_{0t} - \mathbf{l}_{1x} + [\mathbf{J}, \mathbf{g}])_{ik} = s_k(s_{itt} - s_{ixx}) - s_i(s_{ktt} - s_{kxx}) + s_k s_i (J_k - J_i) =$$

$$= E(\mathbf{s}, \mathbf{J}). \tag{46}$$

Multiplying the r.h.s. of Eq. (46) by s_k, summing over k and equating the result to zero, we find

$$\left(\frac{\partial^2 s_i}{\partial t^2} - \frac{\partial^2 s_i}{\partial x^2}\right) - s_k\left(\frac{\partial^2 s_k}{\partial t^2} - \frac{\partial^2 s_k}{\partial x^2}\right) s_i + s_i\left(\sum_{k=1}^{N} J_k s_k s_k - J_i\right) = 0. \tag{47}$$

On the other hand, Eq.(47) is the Euler-Lagrange equation derived from Eq. (41). Note that Eq.(3) describing the easy-axis A3-field arises from Eq.(47) in the particular case $N = 3, J_1 = J_2 < J_3$ after simple scaling transformation of $x, t : x' = (\Delta J)^{\frac{1}{2}} x, t' = (\Delta J)^{\frac{1}{2}} t, \Delta J = J_3 - J_1$.

Our goal is to find a zero-curvature representation for the equation $E = 0$ (see(46)), because if $E = 0$, then Eq. (47) is valid.

Introduce the matrix $\mathbf{N}$:

$$N_{ik} = 2s_i s_k = \delta_{ik} - g_{ik}. \tag{48}$$

It is easy to check that

$$[\mathbf{J}, \mathbf{g}]_{ik} = 2s_i s_k (J_k - J_i) = [\mathbf{N}, \mathbf{J}]_{ik} \tag{49}$$

(no summation over i, k!).

Hence Eq. $E = 0$ can be rewritten as follows

$$\mathbf{l}_{0t} - \mathbf{l}_{1x} + [\mathbf{N}, \mathbf{J}] = 0\,. \tag{50}$$

Notice also that

$$(\mathbf{l}_\alpha)_{ik} = 2\left(\frac{\partial n_i}{\partial x_\alpha} n_k - \frac{\partial n_k}{\partial x_\alpha} n_i\right), \quad \alpha = 1, 2, \tag{51}$$

and

$$[\mathbf{N}, \mathbf{l}_\alpha] = -2\frac{\partial \mathbf{N}}{\partial x_\alpha}. \tag{52}$$

Assume that the zero-curvature representation

$$\frac{\partial U}{\partial t}-\frac{\partial V}{\partial x}+[U,V]=0 \tag{53}$$

of the Eq.(47) can be found by setting

$$\mathbf{U}=a\mathbf{l}_0+b\mathbf{l}_1+c\mathbf{J}+d\mathbf{N},$$

$$\mathbf{V}=e\mathbf{l}_0+f\mathbf{l}_1+q\mathbf{J}+h\mathbf{N}, \tag{54}$$

where a,b,c,d,e,f,q,h are, generally speaking, matrix-valued functions of spectral parameter λ. First we considered the possibility that $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, $d(\lambda)$, $e(\lambda)$, $f(\lambda)$, $q(\lambda)$, $h(\lambda)$ are commutative (complex-valued) functions of λ. Substituting (53) in (52) we find

$$a\frac{\partial \mathbf{l}_0}{\partial t}+b\frac{\partial \mathbf{l}_1}{\partial t}+d\frac{\partial \mathbf{N}}{\partial t}-e\frac{\partial \mathbf{l}_0}{\partial x}-f\frac{\partial \mathbf{l}_1}{\partial x}-h\frac{\partial \mathbf{N}}{\partial x}+$$
$$(be-af)[\mathbf{l}_1,\mathbf{l}_0]+(ce-aq)[\mathbf{J},\mathbf{l}_0]+(cf-bq)[\mathbf{J},\mathbf{l}_1]+$$
$$(de-ah)[\mathbf{N},\mathbf{l}_0]+(df-bh)[\mathbf{N},\mathbf{l}_1]+(dq-ch)[\mathbf{N},\mathbf{J}]=0. \tag{55}$$

Note that an equation

$$b\frac{\partial \mathbf{l}_1}{\partial t}-e\frac{\partial \mathbf{l}_0}{\partial t}+(be-af)[\mathbf{l}_1,\mathbf{l}_0]=0 \tag{56}$$

is equivalent to Eq.(45) if the following Eqs. are valid:

$$b=e, \tag{57}$$

$$b=be-af. \tag{58}$$

Then, Eq.(50) is equivalent to

$$a\frac{\partial \mathbf{l}_0}{\partial t}-f\frac{\partial \mathbf{l}_1}{\partial x}+(dq-ch)[\mathbf{N},\mathbf{J}]=0 \tag{59}$$

if

$$a=f,\quad a(\lambda)\neq 0, \tag{60}$$

$$a=dq-ch. \tag{61}$$

Then by means of Eq.(52) we obtain that

$$d\frac{\partial \mathbf{N}}{\partial t}+(de-ah)[\mathbf{N},\mathbf{l}_0]=\frac{\partial \mathbf{N}}{\partial t}[d-2(de-ah)], \tag{62}$$

and

$$-h\frac{\partial \mathbf{N}}{\partial x}+(df-bh)[\mathbf{N},\mathbf{l}_1]=-\frac{\partial \mathbf{N}}{\partial x}[h+2(df-bh)]. \tag{63}$$

Equating to zero square brackets in the r.h.s.of (62) and (63) and the coefficients at $[\mathbf{J},\mathbf{l}_0]$ and $[\mathbf{J},\mathbf{l}_1]$ in Eq.(55) we find

$$d-2(de-ah)=0, \tag{64}$$

$$h+2(df-bh)=0, \tag{65}$$

$$ce-aq=0, \tag{66}$$

$$cf-bq=0. \tag{67}$$

One can easily prove that the set of Eqs. (57), (58), (60), (61), (64), (65), (66), (67) does not have solutions with $a(\lambda)\neq 0$. Therefore it is quite natural to consider matrix-valued functions $A(\lambda),...,H(\lambda)$ instead of $a(\lambda),...,h(\lambda)$. Having in mind that the SGE can be represented in the form of Eq.(53) with 2×2 matrices U,V [8-13], we shall look for U,V matrices of zero-curvature representation of Eq.(47) in the form

$$\mathbf{U}=\begin{pmatrix}\mathbf{U}_{11} & \mathbf{U}_{12}\\ \mathbf{U}_{21} & \mathbf{U}_{22}\end{pmatrix},$$

$$\mathbf{V}=\begin{pmatrix}\mathbf{V}_{11} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}, \tag{68}$$

$$\mathbf{U}_{11}=a_1(\lambda)\mathbf{l}_0+b_1(\lambda)\mathbf{l}_1+c_1(\lambda)\mathbf{J}+d_1(\lambda)\mathbf{N},$$
$$\mathbf{U}_{12}=e_1(\lambda)\mathbf{l}_0+f_1(\lambda)\mathbf{l}_1+q_1(\lambda)\mathbf{J}+h_1(\lambda)\mathbf{N},$$
$$\mathbf{V}_{11}=a_2(\lambda)\mathbf{l}_0+b_2(\lambda)\mathbf{l}_1+c_2(\lambda)\mathbf{J}+d_2(\lambda)\mathbf{N},$$
$$\mathbf{V}_{12}=e_2(\lambda)\mathbf{l}_0+f_2(\lambda)\mathbf{l}_1+q_2(\lambda)\mathbf{J}+h_2(\lambda)\mathbf{N}, \tag{69}$$

i.e., we shall assume additional symmetries of $\mathbf{U},\mathbf{V}$ matrices (note that the [U,V] matrix retains the symmetries set by Eq.(68)),trying to find a zero-curvature representation of Eq.(47), which comprises the least possible number of unknown functions of the spectral parameter $a_1(\lambda), ..., h_1(\lambda), a_2(\lambda), ..., h_2(\lambda)$.

From (53),(68) we get:

$$\frac{\partial \mathbf{U}_{11}}{\partial t} - \frac{\partial \mathbf{V}_{11}}{\partial x} + [\mathbf{U}_{11}, \mathbf{V}_{11}] + [\mathbf{U}_{12}, \mathbf{V}_{12}] = 0, \tag{70}$$

$$\frac{\partial \mathbf{U}_{12}}{\partial t} - \frac{\partial \mathbf{V}_{12}}{\partial x} + [\mathbf{U}_{12}, \mathbf{V}_{11}] + [\mathbf{U}_{11}, \mathbf{V}_{12}] = 0\,. \tag{71}$$

By using of Eqs. (69) we get from (70):

$$a_1\frac{\partial \mathbf{l}_0}{\partial t} + b_1\frac{\partial \mathbf{l}_1}{\partial t} + d_1\frac{\partial \mathbf{N}}{\partial t} - a_2\frac{\partial \mathbf{l}_0}{\partial x} - b_2\frac{\partial \mathbf{l}_1}{\partial x} - d_2\frac{\partial \mathbf{N}}{\partial x} +$$
$$+(b_1a_2 - a_1b_2 + f_1e_2 - e_1f_2)[\mathbf{l}_1, \mathbf{l}_0] + (c_1a_2 - a_1c_2 + q_1e_2 - e_1q_2)[\mathbf{J}, \mathbf{l}_0] +$$
$$+(c_1b_2 - b_1c_2 + q_1f_2 - f_1q_2)[\mathbf{J}, \mathbf{l}_1] + (d_1a_2 - a_1d_2 + h_1e_2 - e_1h_2)[\mathbf{N}, \mathbf{l}_0] +$$
$$+(d_1b_2 - b_1d_2 + h_1f_2 - f_1h_2)[\mathbf{N}, \mathbf{l}_1] + (d_1c_2 - c_1d_2 + h_1q_2 - q_1h_2)[\mathbf{N}, \mathbf{J}] = 0, \tag{72}$$

and from (71):

$$e_1\frac{\partial \mathbf{l}_0}{\partial t} + f_1\frac{\partial \mathbf{l}_1}{\partial t} + h_1\frac{\partial \mathbf{N}}{\partial t} - e_2\frac{\partial \mathbf{l}_0}{\partial x} - f_2\frac{\partial \mathbf{l}_1}{\partial x} - h_2\frac{\partial \mathbf{N}}{\partial x} +$$
$$+(f_1a_2 - e_1b_2 + b_1e_2 - a_1f_2)[\mathbf{l}_1, \mathbf{l}_0] + (q_1a_2 - e_1c_2 + c_1e_2 - a_1q_2)[\mathbf{J}, \mathbf{l}_0] +$$
$$+(q_1b_2 - f_1c_2 + c_1f_2 - b_1q_2)[\mathbf{J}, \mathbf{l}_1] + (h_1a_2 - e_1d_2 + d_1e_2 - a_1h_2)[\mathbf{N}, \mathbf{l}_0] +$$
$$+(h_1b_2 - f_1d_2 + d_1f_2 - b_1h_2)[\mathbf{N}, \mathbf{l}_1] + (h_1c_2 - q_1d_2 + d_1q_2 - c_1h_2)[\mathbf{N}, \mathbf{J}] = 0, \tag{73}$$

Eqs. (72), (73) may be satisfied in the same way as Eq. (55) was. As a result we arrive at Eq. (50) if the following Eqs. are valid:

$$b_1 = a_2 = b_1a_2 - a_1b_2 + f_1e_2 - e_1f_2,$$
$$a_1 = b_2 = d_1c_2 - c_1d_2 + h_1q_2 - q_1h_2, \qquad a_1(\lambda) \neq 0,$$
$$c_1a_2 - a_1c_2 + q_1e_2 - e_1q_2 = 0,$$
$$c_1b_2 - b_1c_2 + q_1f_2 - f_1q_2 = 0,$$

$$d_1a_2 - a_1d_2 + h_1e_2 - e_1h_2 - \frac{d_1}{2} = 0,$$
$$d_1b_2 - b_1d - 2 + h_1f_2 - f_1h_2 + \frac{d_2}{2} = 0,$$
$$f_1 = e_2 = f_1a_2 - e_1b_2 + b_1e_2 - a_1f_2,$$
$$e_1 = f_2 = h_1c_2 - q_1d_2 + d_1q_2 - c_1h_2, \qquad e_1(\lambda) \neq 0,$$
$$q_1a_2 - e_1c_2 + c_1e_2 - a_1q_2 = 0,$$
$$q_1b_2 - f_1c_2 + c_1f_2 - b_1q_2 = 0,$$
$$h_1a_2 - e_1d_2 + d_1e_2 - a_1h_2 - \frac{h_1}{2} = 0,$$
$$h_1b_2 - f_1d_2 + d_1e_2 - a_1h_2 - \frac{h_2}{2} = 0. \tag{74}$$

We have not yet found the solutions $a_1(\lambda), ... h_2(\lambda)$ to Eqs.(74), the investigation of this problem is in progress. In the separate paper we hope to present higher (non-Noether) integrals of motion of Eqs.(3), which are known [8-13] to be attributes of completely integrable systems. Note in conclusion that Eq. (47) is the field extension of equations of C.Neumann problem [17], describing motion of a point particle on the unit sphere surface with quadratic potential set on it. It is known that C.Neumann problem is completely integrable (see, e.g., [18] and Refs. therein).

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