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CENTRE OF MASS IN SPACES WITH TORSION FREE FLAT LINEAR CONNECTION

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1. INTRODUCTION

In a series of works, which began with [2,3] and is partially summarized in [4] W. G. Dixon developed some methods of dynamics of extended bodies in general relativity. He made an essential usage of the theory of bitensors, first of all for defining some dynamical quantities in curved spaces and their further treatment.

The bitensors theory, originally considered by H.S. Ruse [8] and J. L. Synge [9,10], is deeper investigated in [11], where some its physical applications can be found. This theory is also widely used in [1]. It should be emphasized that in all mentioned physical applications of bitensors mainly are used those obtained by differentiation of the world function, primary introduced by J. L. Synge [10].

The present work, which was inspired by the above references and some purely mathematical considerations, begins an investigation of dynamics in, generally curved, space-times endowed with a structure called a ("parallel") "transport" (along paths), which, when it is linear and acts in the tensor bundles over a given manifold, is equivalent to (a system of) bitensors with a suitable properties (cf. [13]). In particular, here we shall use "flat linear transports" in tensor bundles over the space-time [12] which, as it is proved in [12], are simply parallel transports generated by flat linear connections in these bundles. On this basis our work is aimed to analyze the concept "centre of mass" of a physical system described with its energy-momentum tensor and to propose an adequate definition of that concept. The generalizations of the presented here results to the case of more general space-times with arbitrary curvature and torsion will be published elsewhere.

In section 2, by means of flat linear transports over a given space-time, we introduce the needed for us dynamical quantities and present a part of their properties. These quantities are similar to the classical ones and coincide with them in the corresponding special case.

In section 3 we define the mass centre of a discrete physical system and consider its connection with some dynamical quantities depending on the energy-momentum tensor of this system.

Section 4 contains analysis of the mass centre of a physical system described by its energy-momentum tensor. As a ground are taken two conditions: (a) in the discrete case one must obtain the results of section 3 and (b) some linear conditions (see (4.3)) are assumed to hold. It turns out that they define the mass centre up to an arbitrary 1-form (covector) which, when, as usual, the space-time is endowed with a metric, is naturally to be assumed to be the covector corresponding with respect to the metric to the energy-momentum vector of the system.

In Section 5 are presented certain concluding remarks.

2. SOME MECHANICAL QUANTITIES DEFINED BY MEANS OF FLAT LINEAR TRANSPORTS

In this section certain necessary for our investigation quantities are defined and some their properties are established.

Let M be a differentiable manifold [7] endowed with a flat linear transport L [12], which can equivalently be thought as a parallel transport generated by a flat linear connection ∇^L on M [12]. Physically M will be interpreted as a space-time of dimension $n:=\dim(M)=4$ and its properties will be specified, when needed, below.

The Latin and Greek indices are referring to M and will run, respectively, from 0 to $n-1=3$ and from 1 to $n-1=3$. The usual summation rule over repeated on different levels Latin (resp. Greek) indices from 1 to n (resp. $n-1$) will be assumed.

The (flat linear) transport from x to y, $x,y\in M$ will be denoted by $L_{x\longrightarrow y}$ and $H^i_{.j}(y,x)$ will mean the components of the matrix representing it (in some local coordinates), which are components of a bivector (vector at y and covector at x) [12]. For details concerning flat linear transports the reader is referred to [12].

Definition 2.1. Let the C^1 path $\gamma:J\longrightarrow M$, J being an $\mathbb{R}$ interval, joins the points $x,y\in M$, i.e., $\gamma(s)=x$ and $\gamma(t)=y$ for some $s,t\in J$. The displacement vector of y with respect to x (as it is defined by the transport L) is the vector

$$h(x,y):=\int_s^t (L_{\gamma(r)\longrightarrow\gamma(t)}\dot\gamma(r))dr, \tag{2.1}$$

where $\dot\gamma$ is the tangent to γ vector field.

In the general case $h(x,y)$ depends on γ. We didn't denote this because hereafter in this work we shall be interested only in the case when $h(x,y)$ doesn't depend on γ. This assumption puts a restriction on the used transport L which is expressed by

Proposition 2.1. If the points x and y belong to some coordinate neighborhood, then the displacement vector (2.1) doesn't depend on the path γ if and only if the torsion of the flat linear connection, for which L is a parallel transport, is zero.

Proof. In a coordinate basis the components of (2.1) are

$$h^i(x,y)=\int_s^t H^i_{.j}(x,\gamma(r))\dot\gamma^j(r)dr=\int_x^y H^i_{.j}(x,z)dz^j, \tag{2.2}$$

where we have made the substitution $z^j=\gamma^j(r)$ and the last integral is along γ. As is well known [6], this last integral is locally independent from γ iff the integrand in it is a full differential (with respect to z), i.e. iff locally it is a closed 1-form which is expressed by eq. (4.3′) of [12]. By its turn this equation, due to proposition 4.3 and the remark after proposition 4.2 from [12] is satisfied iff the mentioned torsion vanishes.■

Remark. If there does not exist a coordinate neighborhood containing x and y, then the vector (2.1) depends on the path γ (see below the remark after (2.6)). That is why further is supposed the point defining some displacement vector to belong to some coordinate neighborhood.

Proposition 2.2. If x,y and z belong to one and the same coordinate neighborhood, it is valid the implication

$$h(x,y)=h(x,z) \Leftrightarrow y=z \tag{2.3}$$

which is equivalent to

$$h(x,y)=0 \Leftrightarrow y=x. \tag{2.3′}$$

Proof. By propositions 4.1 and 4.2 from [12] there exists holonomic coordinates $\{x^{i'}\}$ in the neighborhood containing x, y and z such that $H^{i'}_{..j'}(x,y)=\delta^{i'}_{j'}$. So, in it, we have

$$h^{i'}(x,y)=\int_x^y H^{i'}_{..j'}(x,u)du^{j'}=\int_x^y du^{i'}=y^{i'}-x^{i'}, \tag{2.4}$$

from where immediately follow (2.3) and (2.3′).■

Remark. From (2.4) we infer that in the considered case $h(x,y)$ is a straightforward generalization of the Euclidean (difference of two) radius-vector(s).

From this proposition, evidently, it can be concluded that if $x\in M$ and a basis $\{E_i\}$ in the tangent to M bundle (i.e. if $\{E_i(z)\}$ is a basis in $T_z(M)$) are fixed, then the components $h^i(x,y)$ of $h(x,y)=:h^i(x,y)E_i(x)$ are local coordinates of every y, i.e. the map $y\longmapsto(h^1(x,y),\dots,h^n(x,y)\in\mathbb{R}^n$ is a local coordinate system on M. In this sense $h(x,y)$ may be called a vector coordinate of y.

As a simple corollaries of (2.1), we find

$$h(x,y)=h(x,z)+L_{z\longrightarrow x}h(z,y), \qquad (2.5)$$

$$h(x,y)=-L_{y\longrightarrow x}h(y,x). \qquad (2.6)$$

Remark. If there is not a single coordinate neighborhood containing x and y, then the displacement vector depends on the path γ. For instance, if this is the case and there exist neighborhoods $U'\ni x$, $U''\ni y$ and $U'\cap U''\neq\emptyset$, then using in U' and U'' coordinates like those in (2.4), we, writing explicitly the dependence on γ, find

$$h^{i'}(x,y;\gamma)=h^{i'}(x,z)+\delta^{i'}_{j'}\frac{\partial z^{j'}}{\partial z^{j''}}h^{j''}(z,y)=$$

$$=z^{i'}-x^{i'}+\delta^{i'}_{j'}\frac{\partial z^{j'}}{\partial z^{j''}}(y^{j''}-x^{j''}),$$

where $z\in\gamma(J)\cap U'\cap U''$. From here is evident the explicit dependence of the displacement vector on the path entering in its definition (2.1) in the considered concrete case.

Let M be a 4-dimensional space-time. Let us consider a physical system with a (contravariant) energy-momentum tensor T^{ij}. (The concrete structure of T^{ij} or its dependence on other quantities, physical or geometrical fields, is insignificant.) Let Σ be a (time-like, if there is a metric) hypersurface with a measure $d\Sigma_k=$ $=\varepsilon_{kijl}dx^i_1dx^j_2dx^l_3$, ε_{kijl} being the 4-dimensional antisymmetric ε-symbols and dx^i_1, dx^j_2 and dx^l_3 being three linearly independent displacements on Σ.

We define the (4-)vector of energy-momentum of the system as

$$p^i(x):=\frac{1}{c}\int_\Sigma H^i_{.j}(x,y)T^{jk}(y)d\Sigma_k(y) \qquad (2.7)$$

in which c=const is the light velocity in vacuum.

As a corollary of this definition (see also eq. (2.5) from [12]), we get

$$p(z)=L_{x\longrightarrow z}p(x). \qquad (2.8)$$

Let us define the tensor P by

$$P^{ij}(x):=\frac{1}{c}\int_\Sigma h^i(x,y)H^j_{.k}(x,y)T^{kl}(y)d\Sigma_l(y), \qquad (2.9)$$

the antisymmetric part of which,

$$L^{ij}(x):=2P^{[ij]}(x):=P^{ij}(x)-P^{ji}(x), \qquad (2.10)$$

is the orbital angular momentum tensor [2,3,10] of the investigated physical system. The fact that L isn't a conserved quantity [3] is not significant for the following. (A conserved quantity is the total angular momentum, which is a sum of L and the spin angular momentum tensor [2,3,10].)

Substituting (2.5) into (2.9), we get

$$P(x)=h(x,z)\otimes p(x)+L_{z\longrightarrow x}P(z), \qquad (2.11)$$

which in a case of orbital angular momentum reduces to

$$L(x)=h(x,z)\wedge p(x)+L_{z\longrightarrow x}L(z), \qquad (2.12)$$

where $\otimes$ is the tensor product sign and $\wedge$ is the antisymmetric external (wedge) product sign.

At the end of this section we shall write the expressions for the components of h, p and P in some special bases.

By propositions 4.1 and 4.2 from [12] (see also proposition 2.1 and the assumption before it) there is a local holonomic basis (coordinate system) in which the components of the bivector $H(x,y)$, representing L in it, are Kronecker's deltas, i.e. $H^i_{.j}(x,y)=\delta^i_j$. In this basis, from the definitions of h, p and P, we find:

$$h^i(x,y)=y^i-x^i, \qquad (2.13)$$

$$p^i(x)=\frac{1}{c}\int_\Sigma T^{ik}(y)d\Sigma_k(y)=\text{const}, \qquad (2.14)$$

$$P^{ij}(x)=\frac{1}{c}\int_\Sigma(y^i-x^i)T^{jk}(y)d\Sigma_k(y)=P^{ij}(\underline{0})-x^ip^j(x), \qquad (2.15)$$

with $\underline{0}$ being the point with zero coordinates in the used basis.

Because of $p^i(x)$=const, from the used basis by linear transformation with constant coefficients can be obtained a local holonomic basis with the above-pointed property (see proposition 4.1 of [12]) in which

$$p^i=cM\delta^i_0, \qquad (2.16)$$

where

$$M:=\frac{1}{c^2}\int_\Sigma T^{0k}(y)d\Sigma_k(y)=\text{const}, \qquad (2.17)$$

is the total mass of the investigated physical system. Let's note that if the space is endowed with a metric and x^0 is interpreted as a time (coordinate), then (2.16) expresses the simple fact that $p(x)$ is a time-like vector.

So, in this basis

$$P^{ij}(x)=P^{ij}(\underline{0})-cMx^{i}\delta^{j}_{0}. \tag{2.18}$$

And, at last, if we choose the hypersurface Σ as $y^0=z^0=\text{const}$, then $d\Sigma_k(y)=\delta^0_k d^3y$ and $P^{0j}(\underline{0})=cMz^0\delta^j_0$. Hence, we have

$$p^0(x)=cM=\frac{1}{c}\int_{y^0=z^0} T^{00}(y)d^3y,\quad p^{\alpha}(x)=\frac{1}{c}\int_{y^0=z^0} T^{\alpha 0}(y)d^3y=0, \tag{2.19}$$

$$P^{0j}(x)=cM(z^0-x^0)\delta^j_0,\quad P^{\alpha j}(x)=P^{\alpha j}(\underline{0})-cMx^{\alpha}\delta^j_0. \tag{2.20}$$

3. CENTRE OF MASS IN A DISCRETE CASE

Let us have particles with masses m_a situated at some moment t at the points x_a, where $a=1,\dots,N$ numbers the particles. Below we suppose the particles total mass to be nonzero, i.e. $\sum_a m_a\neq 0$. Let x be a fixed space-time point and the displacement vector $h(x,x_a)$ be well defined (see the previous section).

Definition 3.1. The mass centre of the masses m_a with respect to the reference point x at the moment t is the point x_M such that

$$h(x,x_M):=\Big(\sum_a m_a h(x,x_a)\Big)\Big(\sum_a m_a\Big)^{-1}. \tag{3.1}$$

Remark. With the change of time t the point x_M describes a world line, the world line of the system's mass centre.

As a consequence of (2.5), the mass centres x_M and y_M with respect to the reference points x and y respectively are connected by

$$h(y,y_M)=h(y,x)+L_{x\longrightarrow y}h(x,x_M). \tag{3.2}$$

In a local holonomic basis in which $H^i_{\ .j}(x,y)=\delta^i_j$, from (3.1), we easily get

$$x^i_M=\Big(\sum_a m_a x^i_a\Big)\Big(\sum_a m_a\Big)^{-1}. \tag{3.3}$$

Example 3.1 (Special relativity; cf. [5]). Let us have a Minkowski's space-time M^4 referred to Minkowskian coordinates. As a concrete realization of the transport L we shall use the (pseudo-) Euclidean transport defined by $H^i_{\ .j}(x,y)=\delta^i_j$, $i,j=0,1,2,3$. The coordinates of any event $x\in M^4$ are of the form $(ct,\mathbf{x})$, where c is the velocity of light, t is the time in the used frame and $\mathbf{x}:=(x^1,x^2,x^3)$,

which may depend on t, is the special coordinate of x.

So, in this case (2.13)-(2.15) and (3.3) are valid. The last of these equality, due to $x^0_a=ct$ for every event, reduces to

$$x^0_M=ct,\quad \mathbf{x}_M=\Big(\sum_a m_a\mathbf{x}_a\Big)\Big(\sum_a m_a\Big)^{-1}. \tag{3.4}$$

If we define m_a as $m_a=\mathcal{E}_a/c^2$, where $\mathcal{E}_a=c^2m_a=c^2m^0_a\times$ $\times(1-(d\mathbf{x}/dt)^2)^{-1/2}$, m^0_a being the rest mass of the a-th particle, is the energy of the a-th particle, we find $\mathbf{x}_M=\Big(\sum_a \mathcal{E}_a\mathbf{x}_a\Big)\Big(\sum_a \mathcal{E}_a\Big)^{-1}$. Thus we can make the inference that in the discrete case in special relativity our definition 3.1 of mass centre reduces to the known classical one (see, e.g., [5], ch.2, §14). Let's note that the so-obtained mass centre depends on the used basis (frame of reference), as $\mathcal{E}_a$ are such quantities. If we wish to get an invariant definition of x_M, then instead of $m_a=\mathcal{E}_a/c^2$ we have to take $m_a=m^0_a$.

Now we want to show that in the discrete case there exists a very important for the following section connection between the mass centre x_M and the tensor P with local components (2.9).

To begin with, let us remember that the component $T^{00}(z)$ of an energy-momentum tensor is regarded as an energy density at z [5]. Hence it can be written as $T^{00}(z)=c^2\rho(z)$, $\rho(z)$ being the mass density at z, which in the discrete case is

$$\rho(z)=\sum_a m_a\delta^3(\mathbf{x}_a-\mathbf{z}_a), \tag{3.5}$$

where δ^3 is the 3-dimensional Dirac's delta function.

If a local holonomic basis in which $H^i_{\ .j}(x,y)=\delta^i_j$ is used and Σ is defined by $y^0=z^0=\text{const}$, then $d\Sigma_k(y)=\delta^0_k d^3y$ and from (2.9), we obtain

$$P^{i0}(x)=\frac{1}{c}\int_{y^0=z^0} h^i(x,y)T^{00}(y)d^3y=\frac{1}{c}\int_{y^0=z^0} h^i(x,y)\sum_a m_a\delta^3(\mathbf{x}_a-\mathbf{y})d^3y=$$

$$=c\sum_a m_a h^i(x,x_a)\Big|_{x^0_a=z^0},$$ which may also be written as

$$P^{00}(x)=cM(z^0-x^0),\quad P^{\alpha 0}(x)=cMh^{\alpha}(x,x_M)\Big|_{x^0_M=z^0}, \tag{3.6}$$

where (cf. (2.17)) the total mass of the system is

$$M:=\int_{y^0=z^0}\rho(y)d^3y=\sum_a m_a. \tag{3.7}$$

Analogous calculations (see (2.14) and (2.15)) show that:

$$p^0(x)=\frac{1}{c}\int_{y^0=z^0} T^{00}(y)d^3y=cM,\ \ p^\alpha(x)=\frac{1}{c}\int_{y^0=z^0} T^{\alpha 0}(y)d^3y, \tag{3.8}$$

$$P^{0\alpha}(x)=(z^0-x^0)p^\alpha(x),\ \ P^{\alpha\beta}(x)=\frac{1}{c}\int_{y^0=z^0} y^\alpha T^{\beta 0}(y)d^3y-cx^\alpha p^\beta(x). \tag{3.9}$$

The most important for the following result here is the connection between $h(x,x_M)$ and P expressed explicitly by (3.6).

4. CENTRE OF MASS: GENERAL CASE

The conclusion from the previous section is that the mass centre of a physical system (if it exists!) must be connected with the tensor P and in the discrete case this connection must reduce to the already established one.

So, we state the problem for expressing in a covariant way $h(x,x_M)$ through $P(x_M)$. Due to (3.15) these quantities are connected by the relation

$$P^{ij}(x_M)=H^i_{.k}(x_M,x)H^j_{.l}(x_M,x)P^{kl}(x)+h^i(x_M,x)p^j(x_M), \tag{4.1}$$

which is a simple corollary from the corresponding definitions and directly can't serve as an equation for determination of

$$h(x_M,x)=-L_{x\longrightarrow x_M}h(x,x_M). \tag{4.2}$$

Hence, to express $h(x_M,x)$ through $P(x_M)$ we must impose on the latter a certain number of independent conditions such that by the usage of (4.1) they must be solvable with respect to (some of) the components of $h(x_M,x)$ and such that the so-obtained dependence in a discrete case must coincide with the one established in section 3. The type of these conditions is sufficiently arbitrary and this is the cause for the possible existence of different nonequivalent definitions of the mass centre on the basis of P or the orbital and/or spin angular momentum, all of which in the corresponding special cases reduce to its classical definition. Below we analyze only the linear conditions that can be imposed on P which most of all fit to the general spirit of tensor calculus and general relativity.

The general form of the mentioned linear conditions is $B^i_{jk}(x_M)P^{jk}(x_M)=b^i(x_M)$ for some tensors B^i_{jk} and b^i, i.e.,

$$B^i_{jk}(x_M)H^j_{.l}(x_M,x)H^k_{.n}(x_M,x)P^{ln}(x)\ +\ B^i_{jk}(x_M)h^j(x_M,x)p^k(x_M)=b^i(x_M). \tag{4.3}$$

In section 3 we saw that only h^α, $\alpha=1,\dots,n-1=3$ are connected

with P, the component h^0 being independent of it. Hence only $n-1=3$ of these $n=4$ conditions must be independent, i.e.

$$\det\|B^i_{jk}(x)p^k(x)\|=0,\ \ i,j,k=0,1,\dots,n-1=3, \tag{4.4}$$

$$\det\|B^\alpha_{\beta k}(x)p^k(x)\|\neq 0,\ \ \alpha,\beta,\gamma=1,\dots,n-1=3. \tag{4.5}$$

(The last condition may always be fulfilled with an appropriate renumbering of $B^i_{jk}(x)$.)

The condition (4.4) is equivalent to the existence of nonvanishing covector field q such that

$$B^i_{jk}(x)q_i(x)p^k(x)=0\ \ (\sum_i(q_i(x))^2\neq 0). \tag{4.6}$$

On the opposite, if we fix a covector field $q\neq 0$ and define $B^i_{jk}(x)$ as any solution of (4.5)-(4.6), we shall obtain some relation (4.3) satisfying the needed conditions.

Let there be given a nonvanishing covector field q. It is easily verified that the quantities ${}^0B^i_{jk}(x):=2\delta^{[i}_j\delta^{l]}_k q_l(x)=(\delta^i_j\delta^l_k-\delta^l_j\delta^i_k)q_l(x)$ satisfy all of the above conditions. So, putting $B^i_{jk}(x)=:$ $=:{}^0B^i_{jk}(x)+{}'B^i_{jk}(x)$ and $Q^i(x):=-{}'B^i_{jk}(x)P^{jk}(x)+b^i(x)$ into (4.3), we see that $h(x_M,x)$ must be a solution of $P^{[jk]}(x_M)q_k(x_M)=Q^j(x_M)$, or

$$2q_k(x_M)H^j_{.l}(x_M,x)H^k_{.n}(x_M,x)P^{[ln]}(x)+q_k(x_M)h^{[j}(x_M,x)p^{k]}(x_M)\ =\ Q^j(x_M), \tag{4.7}$$

where q, Q and p must satisfy the conditions

$$Q^i(x)q_i(x)=0,\ \ p^i(x)q_i(x)\neq 0. \tag{4.8}$$

The former of them is a corollary from (4.6) and the latter one ensures the solvability of (4.7) with respect to $h(x_M,x)$ in space-times with dimension greater than one. (Evidently, if $p^i(x)q_i(x)=0$, from (4.7) can be obtained no more than the linear combination $q_i(x_M)h^i(x_M,x)$, but not $h^i(x_M,x)$ itself.)

From (4.7), we get

$$h^i(x_M,x)=\frac{1}{q_k(x_M)p^k(x_M)}[Q^i(x_M)\ +\ (q_k(x_M)h^k(x_M,x))p^i(x_M)\ - \\ -\ 2H^i_{.l}(x_M,x)H^k_{.n}(x_M,x)P^{[ln]}(x)q_k(x_M)]. \tag{4.9}$$

Let us investigate this expression.

Firstly, (4.9) defines only the spacial components $h^\alpha(x_M,x)$, leaving the time component $h^0(x_M,x)$ arbitrary. To prove this, let's take a basis $\{E_i\}$ such that $H^i_{.j}(x,y)=\delta^i_j$. In it (4.9) reduces to

$$h^\alpha(x_M,x)=\frac{1}{q_k(x_M)p^k(x_M)}[Q^\alpha(x_M) + (q_k(x_M)h^k(x_M,x))p^\alpha(x_M) - 2P^{[\alpha k]}(x)q_k(x_M)], \tag{4.10a}$$

$$h^0(x_M,x)=\frac{1}{q_k(x_M)p^k(x_M)}[Q^0(x_M) + (q_k(x_M)h^k(x_M,x))p^0(x_M) - 2P^{[0\beta]}(x)q_\beta(x_M)]. \tag{4.10b}$$

As $q\neq0$, for some i we must have $q_i(x_M)\neq0$. Let, e.g., $q_0(x_M)\neq0$.

Substituting (4.10a) and $Q^\alpha(x_M)q_\alpha(x_M)=-Q^0(x_M)q_0(x_M)$ (see (4.8)) into $h^0(x_M,x)\equiv(q_0(x_M))^{-1}[q_k(x_M)h^k(x_M,x)-h^\alpha(x_M)q_\alpha(x_M)]$, we obtain (4.10b). So, (4.10b) is a consequence of (4.10a). Evidently, the same result is true if $q_i(x_M)\neq0$ for some other fixed value of i.

Now we shall study what conditions must satisfy q and Q if in the discrete case the right-hand side of (4.10a) reproduces the same result as (3.1).

For simplicity and brevity a basis $\{E_i\}$ in which

$$H^i_{.j}(x,y)=\delta^i_j,\ \ p^i(x)=\frac{1}{c}\int_{y^0=z^0} T^{i0}(y)d^3y=cM\delta^i_0,\ \ M=\text{const}\neq0 \tag{4.11}$$

will be used. In it (4.10a) gives

$$h^\alpha(x_M,x)=\frac{1}{cMq_0(x_M)}\ [Q^\alpha(x_M)-2P^{[\alpha0]}(x)q_0(x_M)-2P^{[\alpha\beta]}(x)q_\beta(x_M)]. \tag{4.12}$$

(In this basis $q_0(x_M)\neq0$ because of (4.8) and (4.11).)

Substituting, in accordance with (4.11) and (3.6)-(3.9), here $P^{\alpha0}(x)=cMh^\alpha_0(x,x_M)=-cMh^\alpha_0(x_M,x)$, $P^{0\alpha}(x)=0$ and $P^{\alpha\beta}(x)=P^{\alpha\beta}(\underline{0})$, where h_0 is defined by the right hand side of (3.1), we find

$$h^\alpha(x_M,x)=h^\alpha_0(x_M,x)+\frac{1}{cMq_0(x_M)}\ [Q^\alpha(x_M)-2P^{[\alpha\beta]}(\underline{0})q_\beta(x_M)]. \tag{4.13}$$

Therefore

$$h^\alpha(x_M,x)=h^\alpha_0(x_M,x), \tag{4.14}$$

as we must have, if and only if $Q^\alpha(x_M)=2P^{[\alpha\beta]}(\underline{0})q_\beta(x_M)]$, from which, due to (4.8) and $q_0(x)\neq0$, follows $Q^0(x)=0$, i.e. (4.14) is equivalent to

$$Q^i(x_M)=2\delta^i_\alpha H^\alpha_{.j}(x_M,\underline{0})H^\beta_{.l}(x_M,\underline{0})P^{[jl]}(\underline{0})q_\beta(x_M). \tag{4.15}$$

Hence Q must depend linearly upon q and the antisymmetric part

of P. But, because the Greek indices don't take the value zero, it depends also on the used basis $\{E_i\}$ which isn't uniquely defined by the conditions $H^i_{.j}(x,y)=\delta^i_j$ and $p^i(x)=cM\delta^i_0$. (These conditions fix $\{E_i\}$ up to a transformation with a constant nondegenerate diagonal matrix.) The only way to be skipped that last dependence is to admit that in $\{E_i\}$ is fulfilled $q_\alpha(x_M)=0$, or

$$q_i(x_M)=q_0(x_M)\delta^0_i,\ \ q_0(x_M)\neq0, \tag{4.16}$$

which implies (see (4.15))

$$Q^i(x_M)=0. \tag{4.17}$$

The above discussion and its results can be summarized into

Proposition 4.1. Let $h(x_M,x)$ depend linearly on $P(x_M)$ and in the discrete case reduce to (3.1). Let there be chosen a nonvanishing covector field q. Let there exist a local holonomic basis such that in it:

$$H^i_{.j}(x,y)=\delta^i_j, \tag{4.18a}$$

$$p^i(x)=cM\delta^i_0,\ \ M=\text{const}\neq0, \tag{4.18b}$$

$$q_i(x)=q_0(x)\delta^0_i,\ \ q_0(x)\neq0. \tag{4.18c}$$

Then in any basis the spacial coordinates x^α_M of x_M are uniquely defined by the equation

$$P^{[ik]}(x_M)q_k(x_M)=0, \tag{4.19}$$

or, equivalently, by

$$h^i(x_M,x)=\frac{1}{(q(p))(x_M)}\ [(q(x_M)(h(x_M,x)))p^i(x_M)- -2H^i_{.k}(x_M,x)H^j_{.l}(x_M,x)P^{[kl]}(x)q_j(x_M)], \tag{4.20}$$

which leaves the time component x^0 of x_M in the above special basis arbitrary.

Let us turn now our attention on the covector field q, which must satisfy only the condition (4.18c). In this connection are important the following two observations. Firstly, the defined by (4.19) mass centre x_M, generally, depends on the choice of q which is in a great extend arbitrary and until now hasn't any physical meaning. Secondly, the equations (4.19), as well as the results leading to proposition 4.1, imply the existence of some dependence of q on p. These two facts, the above-considered discrete case and the investigations in [2,3] are a hint for us to propose the following general definition of mass centre.

Let the space-time be endowed with a linear transport L and independently with a metric g with covariant components $g_{ij}=g_{ji}$ and signature (+---). Then, roughly speaking, the mass centre x_M is defined by proposition 4.1 with $q_i=g_{ij}p^j$. More precisely, we give

Definition 4.1. The mass centre of a system described by an energy-momentum tensor is the unique point x_M satisfying the following three conditions:

1. At the point x_M in any local basis is valid the equation

$$p^{[ik]}(x_M)g_{kl}(x_M)p^l(x_M)=0. \tag{4.21}$$

2. In a neighborhood of x_M there exist local coordinates $\{x^i\}$ such that in the associated to them basis $\{\partial/\partial x^i\}$ to be fulfilled:

$$H^i_{\cdot j}(x_M,y)=\delta^i_j, \tag{4.22a}$$

$$p^i(x_M)=cM\delta^i_0,\ M=\text{const}\neq0, \tag{4.22b}$$

$$g_{i0}(x_M)=g_{00}(x_M)\delta^0_i,\ g_{00}(x_M)\neq0. \tag{4.22c}$$

3. In the coordinates in which (4.22) hold the time component of x_M is $x^0_M=ct$, t being the time in these coordinates.

5. Comments

Now we shall make some remarks concerning definition 4.1.

Firstly, the equation (4.21) is a special case of eq. (4.19) when the choice $q_i=g_{ij}p^j$ is made. Our opinion is that this connection between q and p (in a metric space-time) is the only "reasonable" one which prevents the dependence of x_M on a sufficiently arbitrary quantity q. Moreover, in this way is given a physical meaning to q as the covector (1-form) corresponding by means of the metric to the momentum p. This is important because by its meaning the mass centre must depend only on the mass distribution of the matter and it shouldn't depend on arbitrary quantities of unclear physical meaning.

Secondly, the conditions (4.22) ensure the solvability of eq. (4.21) with respect to x_M and the coincidence in the discrete case of the so-obtained value of x_M with the one obtained independently by definition 3.1. Let's also note that the condition (4.22c) is a simple corollary of $q_i=g_{ij}p^j$ and (4.18c).

Thirdly, the three conditions (4.22) have a different meaning and in the general case of arbitrary metric they can't be satisfied simultaneously. The first of them, (4.22a), expresses the fact that the associated to the used transport connection is torsion free (in addition to its zero curvature). The second one, (4.22b), shows that the (linear) momentum p is (by definition) a time-like vector and that its direction is taken as a direction of the time (zeroth) coordinate axes, which is possible because of M≠0. (If M=0, then x is left completely arbitrary by (4.21) and (4.22), i.e. for massless systems any space-time point can serve as their mass centre.) These two conditions are always compatible in accordance with propositions 4.1–4.3 of [12] as in a basis in which (4.22a) is valid it is fulfilled (2.14). The last condition, (4.22c), enables us to interpret $x^0_M/c=t$ as a time in the described frame of reference (if it exists). This condition is very restrictive one. In fact, if x_M was a fixed point, then with a linear transformation with constant coefficients it is possible (see [12], proposition 4.1) to transform the basis in which (4.22a) holds into a basis in which (4.22a) and (4.22c) are valid simultaneously. But, generally, in such a basis (4.22b) will not be satisfied. Moreover, as x_M describes with the change of time a whole world line, the mass centre's world line, in the general case one needs a linear transformation with nonconstant coefficients to satisfy (4.22c) and if this is the real situation, then, by [12], proposition 4.1, in the new basis the property (4.22a) will be lost. The conclusion from these considerations is that (4.22c) puts a significant restriction on the possible metrics which are admitable if we want to be well defined the mass centre (world line) of an arbitrary material system. In short, in a given space-time the equation (4.21) defines a mass centre (world line(s)) if and only if all of the conditions (4.22) can be satisfied in some local holonomic basis.

Fourthly, as it was proved above, the equation (4.21) and the conditions (4.22) define, in a basis in which (4.22) are satisfied, only the spacial coordinates x^α_M of the mass centre x_M, but its time coordinate x^0_M is left by them completely arbitrary. This last component is fixed by the third condition of definition 4.1 in such a way as to give its appropriate value in the discrete and classical cases.

Fifthly, by (4.12) with Q=0 and $q_i=g_{ij}p^j$, in the basis described by (4.22) the mass centre has the following coordinates

$$x_M^0=ct,\ x_M^\alpha=x+\frac{1}{cM}P^{[\alpha 0]}(x). \tag{5.1}$$

In any other basis the coordinates of x_M can be obtained from the components of the displacement vector $h(x,x_M)$ in this basis.

Sixthly, in [2-4] to define the mass centre a "similar" to (4.21) equation is proposed in which the orbital angular momentum $L^{ij}:=P^{[ij]}$ is replace with the total angular momentum $J^{ij}=L^{ij}+S^{ij}$ which includes the spin angular momentum S^{ij}. (In [3,4] the bitensor $H(x,y)$ is replaced with another bitensor and the case of general relativity is considered, but this circumstances are insignificant now.) The only reason for this being that J^{ij} is a conserved quantity. We consider this definition of mass centre as irrelevant by three reasons: (a) the above made analysis leads directly to our definition 4.1, (b) the tensor S^{ij} describes the "pure angular momentum properties" of the matter and has nothing common with the simple mass distribution which in its turn must define uniquely the mass centre and (c) in the mentioned works nothing is said about conditions like (4.22), which insure the solvability of (4.21) with respect to x_M, instead of which there is expressed simple consideration that the corresponding equation is likely to have a solution with the needed properties as it has it in some special cases.

Seventhly, in [5], ch. II, §14 is pointed that the presented therein definition of mass centre gives different points for it in different frames (bases), i.e. it depends explicitly on the used local coordinates, even in the simple case of special relativity (cf. our example 3.1). Evidently, our definition 4.1 is free of this deficiency the cause for this being the condition (4.22c) (see also (4.18c) and (4.17)) and the general usage of the displacement vector $h(x,x_M)$ for the definition of x_M (see also eq. (4.21)).

At the end, the above discussion can be summarize as follows. If in a space-times endowed with a (flat) linear transport (connection) and a metric we admit a linear relationship between $P(x_M)$ and $h(x,x_M)$, then the mass centre (mass centre's world line) is well defined by definition 4.1 and it exists if the conditions (4.22) can be satisfied in some local coordinates. It is important to be noted that just this is the classical case of special relativity.

References

1. Bailey I., W. Israel, Relativistic dynamics of extended bodies and polarized media: An eccentric approach, Annals of Physics. vol. **130**, pp. 188-214, 1980.
2. Dixon W.G., A covariant multipole formalism for extended test bodies in general relativity, Nuovo cimento, vol. **34**, No. 2, pp. 317-339, 1964.
3. Dixon W. G., Dynamics of extended bodies in general relativity. I. Momentum and angular momentum, Proc. Roy. Soc. London, vol. **A 314**, pp. 499-527., 1970.
4. Dixon W.G., Extended bodies in general relativity: Their description and motion, in: Isolated gravitating systems in general relativity, North-Holland Publ. Co., Amsterdam, 1979.
5. Landau L. D., E. M. Lifshiz, The Classical Theory of Fields, Addison-Wesley, Reading, Mass., 1959.
6. Lovelock D., H. Rund, Tensors, Differential Forms, and Variational Principals, Wiley-Interscience Publication, John Wiley & Sons, New York-London-Sydney-Toronto, 1975.
7. Kobayashi S., K. Nomizu, Foundations of differential geometry, vol. 1, Interscience Publishers, New York-London, 1963.
8. Ruse H. S., Proc. London Math. Soc., vol. **32**, p. 87, 1931 (Quart. J. Math., vol. 2, p. 190, 1931).
9. Synge J. L., Proc. London Math. Soc., vol. **32**, p. 241, 1931.
10. Synge J. L., Relativity: The General Theory, North-Holland Publ. Co., Amsterdam, 1960.
11. de Witt B. S., R. Brehme, Radiation dumping in a gravitational fields, Annals of Physics, vol. 9, pp. 220-259, 1960.
12. Iliev B. Z., Flat linear connections in terms of flat linear transports in tensor bundles, Communication JINR, E5-92-544, Dubna, 1992.
13. Iliev B.Z., Linear transports along paths in vector bundles. I. General theory, Communication JINR, E5-93-239, Dubna, 1993; Transports along paths in fibre bundles. General theory, Communication JINR, E5-93-299, Dubna, 1993.

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