ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E5-95-122

P.E.Zhidkov*

ON A PROBLEM WITH TWO-TIME DATA FOR THE VLASOV EQUATION

Submitted to «Nonlinear Analysis: Theory, Methods and Applications»

*E-mail: ZHIDKOV@THEOR.JINRC.DUBNA.SU

1995

1 Introduction

Consider the Vlasov equation

$$\frac{\partial}{\partial t} f(t,x,v) + (v, \nabla_x f) -$$

$$-(\nabla_v f, \nabla_x G(x) + \int \nabla_x W(|x-y|^2) f(t,y,v) dy\ dv) = 0. \quad (1)$$

Here all variables are real, $f = f(t,x,v)$ is the unknown distribution function of particles with values in R defined in the phase space which is the Cartesian product $R^d \otimes R^d$ of the space of coordinates $x \in R^d$ and the space of velocities $v \in R^d$, $G(x)$ is the potential of exterior forces, $W = W(|x-y|^2)$ is the potential of interaction between two particles occupying the points with coordinates x and y. We shall consider $d = 1, 2, 3$.

The physical sense of Eq. (1) is that it describes a large number of interacting particles and any particle at any moment of time is moving according to the Newtonian laws and is subject to forces of the exterior field $\nabla_x G(x)$ and of the mean field created by all other particles (see [1]). In view of the sense of the function f, one should require

$$\int f(t,x,v) dx\ dv = 1 \quad (2)$$

for any t.

In the rigorous mathematical sense, the Vlasov equation (1) was considered in several papers (see, for example, [2-8]). Papers [2,3,5,6] contain results on the well-posedness of the Cauchy problem for Eq. (1) with the initial data

$$f\big|_{t=0} = f_0(x,v) \quad (3)$$

where $f_0 \geq 0$ and $\int f_0(x,v) dx\ dv = 1$. Supplementary, in papers [4,7,8], various approaches to the derivation of the Vlasov equation as a limit equation of the Newtonian dynamics of a large number of particles are presented.

In what follows, we shall not consider the Cauchy problem (1),(3), but we shall study the Vlasov equation (1) completed by a joint distribution of particles in the coordinate space at two moments of time $t_1 < t_2$. In the next section, we shall give careful formulations. Here, we want to discuss our results.

As we noted above, we introduce the joint distribution $g(x, y)$ of our particles in the coordinate space $(x, y) \in R^{2d}$ at two moments of time. Intuitively, the sense of the function g is clear: fixing two small volumes dx and dy in R^d, we accept that the number of particles belonging to dx at $t = t_1$ and to dy at $t = t_2$ is equal to $Ng(x, y)dx\, dy$ up to higher corrections (here N is the number of particles).

We want to make two remarks here. First, if one has densities of particles ρ_1 and ρ_2 in the coordinate space at two moments of time $t_1 << t_2$ (of course, densities are observable experimentally), then, as it was made many times in statistical physics, one may set $g(x, y) = \rho_1(x)\rho_2(y)$. Although this method is not rigorously proved, it may give good results because, for $t_1 << t_2$, one can hope that the partitions of particles at these moments of time are almost "independent" (see [1,9], for example). Also, we would like to note that, if one has solved our problem, then he can find the distribution of particles over velocities. It is clear that this distribution cannot be simply found experimentally.

In the paper, we do not present our results in the most general form, but we want to show, under severe constraints on variables in Eq. (1), only the possibility of our formulation of the problem. However, we consider generalized solutions of this equation of the most general kind. These solutions belong to the space of bounded non-negative Borel measures. In addition, we do not use the probability approach but we exploit the deterministic one to avoid unnecessary complications.

In what follows, by $C, C_1, C_2, C', C'', \ldots$ we denote positive constants.

2 Formulation of the problem. Existence of solutions.

In further considerations, we use the concept of the generalized solution of Eq. (1) which is a normalized Borel measure at any moment of time. We do not discuss this concept in detail but we introduce a definition sufficient for our goals of the solution as a limit in some sense of solutions of finite-dimensional hamiltonian equations when the number of particles tends to infinity and referring readers to the papers [2,4,6-8], for details.

Let $M(R^n)$ be the set of non-negative normalized Borel measures in R^n $(n = 1, 2, 3\ldots)$. The set $M(R^n)$ is equipped by the topology of the weak convergence of sequences of measures (we recall that a sequence w_k from $M(R^n)$ of measures weakly converges to $w \in M(R^n)$ iff

$$\int_{R^n} \phi(x) dw_k(x) \to \int_{R^n} \phi(x) dw(x) \qquad (k \to \infty)$$

for all bounded continuous functions ϕ defined on R^n).

Further, let $w_1, w_2 \in M(R^n)$ and let

$$\nu(w_1, w_2) = \sup_{\phi} \Big| \int_{R^n} \phi(x) dw_1(x) - \int_{R^n} \phi(x) dw_2(x) \Big|$$

where the supremum is taken over the set of all Lipschitz continuous functions ϕ satisfying the condition

$$||\phi||_{\mathrm{Lip}} = \sup_{x \in R^n} |\phi(x)| + \sup_{x,y \in R^n} \frac{|\phi(x) - \phi(y)|}{|x - y|} \leq 1.$$

It is known (see [2]) that this distance generates the topology of the weak convergence in the space $M(R^n)$.

Let $t_1 < t_2$. We denote by $C(I; M(R^n))$ the metric space of continuous in the above sense functions defined on the segment $I = [t_1, t_2]$ with values in $M(R^n)$, with the distance

$$\rho(w_1, w_2) = \sup_{t\in I} \nu(w_1(t), w_2(t))$$

where $w_1, w_2 \in C(I; M(R^n))$.

Consider a hamiltonian system of N interacting mass points (here N is a positive integer), with the hamiltonian

$$H_N(x, v) = \sum_{n=1}^{N}\{\frac{|v_n|^2}{2} + G(x_n) + N^{-1}\sum_{k=n+1}^{N} W(|x_n - x_k|^2)\}.$$

Here $x_n = (x_n^1, ..., x_n^d) \in R^d$ and $v_n = \dot{x}_n = (v_n^1, ..., v_n^d) \in R^d$ are the coordinates and the velocity of the nth particle, $x = (x_1, ..., x_N) \in R^{dN}$, $v = (v_1, ..., v_N) \in R^{dN}$, $G(.) : R^d \to R$ is the potential of exterior forces and $N^{-1}W(|x-y|^2)$ is the potential of interaction between particles occupying the points with the coordinates x and y. This system obeys the following system of equations:

$$\dot{x}_n(t) = v_n(t), \tag{4}$$

$$\dot{v}_n(t) = -\nabla_{x_n} H_N(x; v)\big|_{(x,v)=(x(t),v(t))} \tag{5}$$

where $\nabla_{x_n} H_N = (\frac{\partial H_N(x,v)}{\partial x_n^1}, ..., \frac{\partial H_N(x,v)}{\partial x_n^d})$. Let $(x(t), v(t))$ be a solution of the system (4),(5) and let $w_N(t) \in C(I; M(R^{2d}))$ be the measure for each t with the density $N^{-1}\sum_{n=1}^{N} \delta(x - x_n(t))\delta(v - v_n(t))$ where δ is the Dirac delta-function in R^d. As it is known (see [2,4,6,8]), the function $w_N \in C(I; M(R^{2d}))$ is in a sense a generalized solution of Eq. (1). Using this fact, we introduce the following

Definition 1 *Let μ_N be the measure with the density $N^{-1}\sum_{n=1}^{N} \delta(x - x_n(t_1))\delta(y - x_n(t_2))$ and let a function*

$w(t) \in C(I; M(R^{2d}))$ and $\mu \in M(R^{2d})$ be such that $\rho(w_{N_k}, w) \to 0$ and $\nu(\mu_{N_k}, \mu) \to 0$ as $k \to +\infty$ where N_k is a subsequence of the sequence $1, 2, 3, \ldots$. Then, we call the function w the (generalized) solution of Eq. (1) *with the joint distribution μ in the coordinate space at the moments of time $t = t_1$ and $t = t_2$. Clearly, one has $w(t)(R^{2d}) = 1$ for each t instead of* (2).

Remark 2 We could call this solution physical.

Our first main hypothesis is the following.

(h1) *Let the functions $W(|x|^2), G(x)$ belong to $C^2_{\rm loc}(R^d; R)$ and all their first partial derivatives be bounded as functions in R^d.*

Lemma 3 *Let the hypothesis* (h1) *be satisfied. Consider the following system of equations*

$$\ddot{x}_n = -\nabla_{u_n} G(u_n) - N^{-1}\sum_{k\neq n} \nabla_{u_n} W(|u_n - u_k|^2), \quad n = \overline{1, N}, \tag{6}$$

$$x_n(t_i) = x_n^i, \quad i = 1, 2 \tag{7}$$

where $x_n^i, x_n(.), u_n(.) \in R^d$ and $u_n(t) \in C(I; R^d)$. Then, for any $R > 0$ there exists $D > 0$ such that

$$|x_n(t)| + |\dot{x}_n(t)| + |\ddot{x}_n(t)| < D$$

for an arbitrary solution of this system, all $N = 1, 2, 3, \ldots$, all $t \in [t_1, t_2]$, all continuous functions u_n and for all n for which $|x_n(t_i)| < R$ $(i = 1, 2)$.

Proof. Fix an arbitrary $R > 0$. Taking

$$z_n(t) = x_n(t) - \frac{x_n(t_2) - x_n(t_1)}{t_2 - t_1}(t - t_1) - x_n(t_1),$$

we find that the functions z_n satisfy the following equations:

$$\ddot{z}_n(t) = g_n(u_1, \ldots, u_N, t), \quad n = \overline{1, N},$$

$$z_n(t_i) = 0, \quad i = 1, 2$$

and there exists $P = P(R) > 0$ such that

$$|g_n(z_1, ..., z_N, t)| < P$$

for all numbers N and $n = \overline{1, N}$, which implies:

$$|z_n(t)| + |\dot{z}_n(t)| + |\ddot{z}_n(t)| \leq D'(R)$$

for all N and $n = \overline{1, N}$. Thus, the statement of Lemma 3 follows.

Remark 4 A corollary of the proof of Lemma 3 is that for any N and $x_n(t_i)$ ($n = \overline{1, N}$, $i = 1, 2$) there exists $C_1 > 0$ such that

$$|x_n(t)| < \max_{i=1,2} |x_n(t_i)| + C_1$$

for all functions $u_n(.)$, all $t \in [t_1, t_2]$ and $n = \overline{1, N}$.

Lemma 5 *Consider the system* (4),(5) *completed by the boundary conditions*

$$x(t_i) = x^i \in R^{dN} \quad (i = 1, 2). \tag{8}$$

Then, under the hypothesis (h1) *the problem* (4),(5),(8) *has a solution.*

Sketch of the *Proof.* Fix an arbitrary number $R > \max_{i=1,2} \max_{n=\overline{1,N}} |x_n(t_i)|$. Consider the set

$$P = \{x(.) = (x_1(.), ..., x_N(.)) \in C([t_1, t_2]; R^{dN})|$$

$$x_n(t_i) = x^i_n, \ |x_n(t)| \leq R + C_1\}$$

where $x^i = (x^i_1, ..., x^i_N)$ with $x^i_n \in R^d$ ($i = 1, 2$, $n = \overline{1, N}$) and C_1 is the positive constant from Remark 4. Then, according to Remark 4, to any $u \in P$ there corresponds a unique solution $x(t)$ of the system (6),(7) which belongs to P. In addition, according to Lemma 3 there exists $D > 0$ such that

$$|\dot{x}_n(t)| + |\ddot{x}_n(t)| \leq D$$

for all $n = \overline{1, N}$ and all $u \in P$. Hence, one has a compact mapping of any $u(t) \in P$ in $x(t) \in P$. Therefore, according to the Schauder's theorem, this map has a fixed point in P, and Lemma 5 is proved.

Theorem 1 *Let* $\mu \in M(R^{2d})$. *Then, under the hypothesis* (h1) *for any* $\mu \in M(R^{2d})$ *Eq.* (1) *has a solution* $w(t) \in C(I; M(R^{2d}))$ *with the joint distribution* μ *at moments of time* t_1 *and* t_2, *such that* $w(t)(R^{2d}) = 1$ *for each* $t \in I$.

Proof. We shall use a theorem about the compactness of a family of functions $\sigma_k(.) \in C(I; M(R^{2d}))$ ($k = 1, 2, 3, ...$) (see [2]). According to this result, the above family is relatively compact if for any $\epsilon > 0$ there exists a compact set $Q \subset R^{2d}$ such that

(a) $\sigma_k(t)(R^{2d} \setminus Q) < \epsilon$ for all $t \in I$ and k;

(b) there exists $\delta > 0$ such that

$$\nu(\sigma_k(t), \sigma_k(s)) < \epsilon$$

for all k and for all $t, s \in I$ satisfying $|t - s| < \delta$.

So, let μ_N be a sequence of measures with densities $N^{-1} \sum_{n=1}^{N} \delta(x - x^1_n)\delta(y - x^2_n)$ weakly converging to μ. Fix arbitrary $\epsilon > 0$. Then, according to the Prokhorov's theorem (see [10]), there exists a ball $K_\epsilon \subset R^{2d}$, $K_\epsilon = \{z \in R^{2d} | \ |z| \leq R\}$ where $R > 0$ such that $\mu_N(R^{2d} \setminus K_\epsilon) < \epsilon$ for all N.

Further, let $\{x_N, v_N\}$ be a sequence of solutions of the problem (4),(5),(8) and let w_N be the corresponding sequence of measures with densities $N^{-1} \sum_{n=1}^{N} \delta(x - x_n(t))\delta(y - v_n(t))$. Let $D > 0$ be the constant from Lemma 3 corresponding to our $R > 0$ from the definition of K_ϵ. Take $B_\epsilon = \{z \in R^{2d} | \ |z| \leq R + D\}$. Obviously, by construction, $w_N(t)(R^{2d} \setminus B_\epsilon) < \epsilon$ for all t and $N = 1, 2, 3, \ldots$. Therefore, the condition (a) of the compactness is satisfied.

Let us verify the condition (b). Take an arbitrary function ϕ continuous according to Lipschitz defined on R^{2d} satisfying the condition $||\phi||_{\text{Lip}} \leq 1$. Then, according to Lemma 3, one has

$$\left| \int \phi(z) dw_N(t) - \int \phi(z) dw_N(s) \right| \leq$$

$$\leq N^{-1} \left| \sum_{n:\ (x_n^1, x_n^2) \in K_{\frac{\epsilon}{3}}} [\phi(x_n(t), \dot{x}_n(t)) - \phi(x_n(s), \dot{x}_n(s))] \right| + \frac{2\epsilon}{3} \leq$$

$$\leq C_2|t-s| + \frac{2\epsilon}{3}.$$

Taking $\delta = \epsilon(6C_2)^{-1}$, we obtain that

$$\nu(w_N(t), w_N(s)) < \epsilon$$

if $|t-s| < \delta$. Thus, the condition (b) of the compactness is satisfied, too, and hence, the sequence $\{w_N\}$ is relatively compact in the space $C(I; M(R^{2d}))$. Therefore, it contains a subsequence converging to some $w \in C(I; M(R^{2d}))$. Obviously, $w(t)(R^{2d}) = 1$ for all t and $w(.)$ is a non-negative measure. Thus, Theorem 1 is proved.

3 Uniqueness of solutions

In this section, we assume for simplicity that $d = 1$. Our hypothesis is the following.

(h2) *Let $G \equiv 0$ and $W = W(s^2) = T(s) \in C^2_{\text{loc}}(R^1)$ be a concave function. Let $T'(s)$ and $T''(s)$ be bounded functions ($s \in R$).*

Lemma 6 *Let $d = 1$ and the hypothesis* (h2) *be valid. Consider the following system of equations*

$$\ddot{g}_n = -\sum_{k=1,\ k\neq n}^{M} m_k \frac{\partial}{\partial g_n} T(g_n - g_k) + \phi_n(t), \quad t \in (t_1, t_2), \quad (9)$$

$$g_n(t_i) = g_n^i, \quad i = 1, 2, \ n = \overline{1, M} \quad (10)$$

where $g_n, g_n^i \in R$, M is a positive integer, $m_k \geq 0$ and $\sum_k m_k \leq 1$ and ϕ_n are functions continuous on $[t_1, t_2]$. Let $g_n = z_n$ and $g_n = \bar{z}_n$ be two solutions of this system with $\phi_n = \psi_n$, $g_n^i = z_n^i$ and $\phi_n = \bar{\psi}_n$, $g_n^i = \bar{z}_n^i$, respectively. Then, one has the estimate

$$\max_{t \in [t_1, t_2]} \left\{ \sum_{n=1}^{M} m_n [|z_n - \bar{z}_n| + |\dot{z}_n - \dot{\bar{z}}_n|] \right\} \leq C(\delta_1 + \delta_2)$$

where

$$\delta_1 \geq \max_{n=1,\ldots,M,\ i=1,2} |z_n^i - \bar{z}_n^i|, \quad \delta_2 \geq \max_{n=1,\ldots,M} \max_{t \in [t_1, t_2]} |\psi_n(t) - \bar{\psi}_n(t)|$$

and $C =$ Constant > 0 is independent of t, δ_1, δ_2, M and m_k.

Proof. From (9),(10) one has for the functions $v_n = z_n - \bar{z}_n$:

$$\ddot{v} = A(t)v + R, \quad t \in [t_1, t_2], \quad (11)$$

$$|v_n(t_i)| \leq \delta_1 \quad (12)$$

where $v = (v_1, \ldots, v_M) \in R^M$, $R(t) = (R_1(t), \ldots, R_M(t)) \in C(I; R^M)$ with $|R_n(t)| \leq \delta_2$ for all n and t and $A(t)$ is a continuous in $[t_1, t_2]$ matrix with the elements $a_{n,n}(t) = -\sum_{k\neq n} m_k T''(\theta_{n,k}(t))$ and $a_{n,k}(t) = m_k T''(\theta_{n,k}(t))$ for $k \neq n$ (here $k, n = 1, \ldots, M$ and $\theta_{n,k} \in (0, 1)$). According to the hypothesis (h2), the matrix $A(t)$ is positive semidefinite for all t because $a_{n,n} = \sum_{k\neq n} |a_{n,k}|$ for all n.

Introduce the following norms and spaces (here $z(t) = = (z_1(t), ..., z_M(t)) \in C(I; R^M)$):

$$|z(t)|_p = \left\{ \sum_{n=1}^{M} m_n |z_n(t)|^p \right\}^{\frac{1}{p}},$$

$$||z||_p = \left\{ \int_{t_1}^{t_2} |z(t)|_p^p dt \right\}^{\frac{1}{p}} \quad \text{(with the space } L_p = L_p(I; R^M)).$$

Let $z(t_i) = 0$ $(i = 1, 2)$. Then, one obviously has the following embedding inequalities:

$$||z||_1 \leq C||z||_2, \tag{13}$$

$$||z||_2 \leq C||\dot{z}||_2, \tag{14}$$

$$\max_{t \in [t_1, t_2]} \sum_{n=1}^{M} m_n \{|z_n(t)| + |\dot{z}_n(t)|\} \leq C||\ddot{z}||_1 \tag{15}$$

with a constant $C > 0$ independent of z and M.

Set $u(t) = v(t) - \frac{v(t_2) - v(t_1)}{t_2 - t_1}(t - t_1) - v(t_1)$. Then, one obtains from (11),(12):

$$\ddot{u} = A(t)u + \sigma(t), \quad t \in [t_1, t_2], \tag{16}$$

$$u(t_i) = 0 \tag{17}$$

with $|\sigma_n(t)| < C(\delta_1 + \delta_2)$ $(n = 1, ..., M)$ where $C = \text{Constant} > 0$ depends only on the potential T.

Multiplying the nth equation from (16) by m_n, multiplying the obtained system by u and integrating over $[t_1, t_2]$, one obtains,

$$||\dot{u}||_2^2 \leq C(\delta_1 + \delta_2)||u||_2 \tag{18}$$

due to the matrix $A(t)$ being positive semidefinite and (13). By analogy, multiplying the nth equation from (16) by m_n, summing over n and integrating over $[t_1, t_2]$, we find:

$$||\ddot{u}||_1 \leq C_2||u||_2 + C_3(\delta_1 + \delta_2). \tag{19}$$

Then, (18) implies in view of (14):

$$||u||_2 \leq C_4(\delta_1 + \delta_2).$$

Therefore, by (19)

$$||\ddot{u}||_1 \leq C_5(\delta_1 + \delta_2).$$

Hence, by (15)

$$\max_{t \in [t_1, t_2]} \sum_{n=1}^{M} m_n |\dot{u}_n(t)| \leq C_6(\delta_1 + \delta_2)$$

and

$$\max_{t \in [t_1, t_2]} \sum_{n=1}^{M} m_n |u_n(t)| \leq C_7(\delta_1 + \delta_2).$$

Thus, Lemma 6 is proved.

Lemma 7 *Consider the system* (9),(10) *from Lemma* 6 *with* $N \geq M$ *and* $m_n = N^{-1}$. *Then, one has for any* n *and* m

$$\max_{t \in [t_1, t_2]} \{|g_n(t) - g_m(t)| + |\dot{g}_n(t) - \dot{g}_m(t)|\} \leq C(\delta_1 + \delta_2)$$

where $\delta_1 = \max_{i=1,2} |g_n(t_i) - g_m(t_i)|$, $\delta_2 = \max_{t \in [t_1, t_2]} |\phi_n(t) - \phi_m(t)|$ *and* $C = \text{constant} > 0$ *is independent of* t, δ_1, δ_2, M *and* N.

Proof. One obtains from (9),(10):

$$\frac{d^2}{dt^2}(g_n(t) - g_m(t)) = -pT''(\theta(t))(g_n - g_m) + \phi_n(t) - \phi_m(t), \quad t \in [t_1, t_2],$$

$$|g_n(t_i) - g_m(t_i)| \leq \delta_1$$

where $p, \theta(.) \in (0, 1)$. Set

$$g(t) = g_n(t) - g_m(t) - \frac{g_n(t_2) - g_m(t_2) - g_n(t_1) + g_m(t_1)}{t_2 - t_1} \times$$

$$\times (t - t_1) - g_n(t_1) + g_m(t_1). \tag{20}$$

Then, one has

$$\ddot{g}(t) = -pT''(\theta(t))g(t) + \psi(t), \quad t \in [t_1, t_2], \tag{21}$$

$$g(t_i) = 0, \quad i = 1, 2 \tag{22}$$

where $|\psi(t)| < C_1(\delta_1 + \delta_2)$ for all t. Multiplying (21) by g and integrating over $[t_1, t_2]$, we obtain, using (22),

$$\int_{t_1}^{t_2} (\dot{g}(t))^2 dt \leq C_2(\delta_1 + \delta_2) \int_{t_1}^{t_2} |g(t)| dt.$$

Since by the embedding theorem

$$\int_{t_1}^{t_2} (\dot{g}(t))^2 dt \geq C_3 \left\{ \int_{t_1}^{t_2} |g(t)| dt \right\}^2$$

where $C_3 > 0$ is independent of g, one has

$$\int_{t_1}^{t_2} |g(t)| dt \leq C_4(\delta_1 + \delta_2).$$

Then, it follows from (21) that

$$\int_{t_1}^{t_2} |\ddot{g}(t)| dt \leq C_5 \int_{t_1}^{t_2} |g(t)| dt + C_6(\delta_1 + \delta_2).$$

Therefore,

$$\int_{t_1}^{t_2} |\ddot{g}(t)| dt \leq C_7(\delta_1 + \delta_2).$$

This inequality with (20) implies the statement of Lemma 7.

Theorem 2 *Let the hypothesis* (h2) *be valid. Then, Eq.* (1) *with the joint distribution of particles in the coordinate space* $\mu \in M(R^2)$ *at the moments of time* $t = t_1$ *and* $t = t_2$ *has exactly one solution.*

Proof of Theorem 2. Fix an arbitrary $\delta > 0$. According to assumptions, there exists a ball $B_R = \{z \in R^2 | \, |z| \leq R\}$ where $R > 0$, such that

$$\mu(B_R) > 1 - \delta.$$

Let $h_1 = h_2 = \delta$ and $a = (a_1, a_2)$. Then, the straight lines in R^2 given by the equations $x_i = a_i + n_i h_i$ generate the partition of the space R^2 into open cubes P_{n_1,n_2} where $\bigcup\limits_{n_1,n_2=-\infty}^{\infty} \overline{P}_{n_1,n_2} = R^2$ (here n_i are integers). Obviously, there exist numbers a_1 and a_2 such that the measure μ of the set $\bigcup\limits_{n_1,n_2=-\infty}^{\infty} \{z = (x_1, x_2) \in R^2 | \, x_1 = a_1 + n_1 h_1$ or $x_2 = a_2 + n_2 h_2\}$ is equal to zero. We fix this choice of the vector a. Further, let $\bar{P}_1, ..., \bar{P}_l$ be a reindexing of cubes, the intersections of which with B_R are non-empty. Let $m_i = \mu(P_i)$ $(i = \overline{1, l})$.

Consider an arbitrary sequence $\mu_N \to \mu$ as $N \to \infty$ where μ_N has the density $\sum\limits_{n=1}^{N} \delta(x_n - x_n^1)\delta(y - x_n^2)$ and let $w_N(t)$ be the corresponding sequence of solutions of Eq. (1) and $(x^N(t), v^N(t)) = (x_1^N, ..., x_N^N, v_1^N, ..., v_N^N)$ be the corresponding sequence of solutions of the hamiltonian system (4),(5),(8) (generally, these solutions are not unique). To prove the Theorem, it suffices to prove that the sequence w_N converges in $C(I; M(R^2))$.

Let $P = \bigcup_{i=1}^{l} P_i$. According to [10], $\lim_{N\to\infty} |\mu_N(P_i) - m_i| = 0$ and $\lim_{N\to\infty} |\mu_N(R^2 \setminus \bar{P}) - \mu(R^2 \setminus \bar{P})| = 0$ for $i = \overline{1,l}$. Fix arbitrary positive integers $M < N$. Let N_i and M_i be the numbers of trajectories of the hamiltonian system (4),(5),(8) satisfying the conditions $(x_n^N(t_1), x_n^N(t_2)) \in P_i$ and $(x_n^M(t_1), x_n^M(t_2)) \in P_i$, respectively. Reindex coordinates $x_n^N(t)$ so that $(x_n^N(t_1), x_n^N(t_2)) \in P_i$ for $N_1 + ... + N_{i-1} < n \le N_1 + ... + N_i$ and enumerate other coordinates arbitrary. Repeat this procedure for coordinates $x_n^M(t)$.

Denote $x^N_{N_1+...+N_{i-1}+1}(t)$ by $u_i(t)$ and $x^M_{M_1+...+M_{i-1}+1}(t)$ by $v_i(t)$. Let us estimate the distance between $p_i = (u_i, \dot{u}_i)$ and $q_i = (v_i, \dot{v}_i)$. Using Lemma 7, we have for large enough numbers M and N:

$$\frac{d^2}{dt^2} u_i = - \sum_{j=1,\ j\neq i}^{l} m_j T'(u_i - u_j) + a_i(t)\delta$$

and

$$\frac{d^2}{dt^2} v_i = - \sum_{j=1,\ j\neq i}^{l} m_j T'(v_i - v_j) + b_i(t)\delta$$

with $|a_i(t)| + |b_i(t)| \le C'$ for sufficiently large M and N where C' is independent of N and M. Hence, applying Lemma 6, one finds that

$$\max_{t\in[t_1,t_2]} \sum_{i=1}^{l} m_i \{ |u_i(t) - v_i(t)| + |\dot{u}_i(t) - \dot{v}_i(t)| \} < C'\delta \quad (i = \overline{1,l}). \tag{23}$$

Let us estimate $\rho(w_N, w_M)$. Take an arbitrary function ϕ continuous according to Lipschitz defined on R^2 and satisfying $||\phi||_{\mathrm{Lip}} \le 1$. One has, using (23), for large enough numbers N and M:

$$\left| \int \phi(z) dw_N(z) - \int \phi(z) dw_M(z) \right| \le$$

$$\le \Bigg| N^{-1} \sum_{i=1}^{l} \sum_{k=N_1+...+N_{i-1}+1}^{N_1+...+N_i} \phi(z_k^N) -$$

$$- M^{-1} \sum_{i=1}^{l} \sum_{k=M_1+...+M_{i-1}+1}^{M_1+...+M_i} \phi(z_k^M) \Bigg| + C_1\delta \le$$

$$\le \left| \sum_{i=1}^{l} m_i [\phi(p_i) - \phi(q_i)] \right| + C_2\delta \le$$

$$\le C_3 \sum_{i=1}^{l} m_i |p_i - q_i| + C_2\delta \le C_4\delta$$

(here $z_n^N = (x_n^N, \dot{x}_n^N)$). Since $\delta > 0$ is arbitrary small, it implies that there exists $\lim_{N\to\infty} w_N(t)$, and Theorem 2 is proved.

References

[1] Balescu R. *Equilibrium and Nonequilibrium Statistical Mechanics.* New York: John Wiley and Sons, Inc., (1975).

[2] Arsen'ev A.A. *Lectures on Kinetic Equations.* Moscow: Nauka, (1992) (in Russian).

[3] Arsen'ev A.A. *Global Existence of a Weak Solution of Vlasov's System of Equations.* U.S.S.R. Comp. Math. and Math. Phys. **15**, 136-147 (1975).

[4] Braun W. & Hepp K. *The Vlasov Dynamics and Its Fluctuations in the $\frac{1}{N}$ Limit of Interacting Classical Particles.* Commun. Math. Phys. **56**, 101-113 (1977).

[5] DiPerna R.J. & Lions P.L. *Global Weak Solutions of Vlasov-Maxwell Systems.* Commun. Pure Appl. Math. **42**, 729-757 (1989).

[6] Dobrushin R.L. *Vlasov Equations.* Funk. Anal. i Ego Prilozheniya **13**, 48-58 (1979) (in Russian).

[7] Oelschläger K. *A Martingale Approach to the Law of Large Numbers for Weakly Interacting Stochastic Processes.* The Annals of Probability **12**, 458-479 (1984).

[8] Spohn H. *Large Scale Dynamics of Interacting Particles.* Berlin–Heidelberg: Springer-Verlag, (1991).

[9] Bogoliubov N.N. *Problems of Dynamical Theory in Statistical Physics.* In: Selected papers, Vol. 2. Kiev: Naukova Dumka, (1970) (in Russian).

[10] Prokhorov Yu.V. & Rosanov Yu.A. *Probability Theory.* Moscow: Nauka, (1987) (in Russian).

Received by Publishing Department on March 20, 1995.

Жидков П.Е. E5-95-122

Об одной задаче для уравнения Власова с данными в два момента времени

Исследуется уравнение Власова, дополненное совместным распределением частиц в координатном пространстве в два момента времени. Для гладкого потенциала взаимодействия с ограниченными производными доказаны существование и единственность решения.

Работа выполнена в Лаборатории теоретической физики им.Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1995

Zhidkov P.E. E5-95-122

On a Problem with Two-Time Data for the Vlasov Equation

We investigate the Vlasov equation completed by a joint distribution of particles in the coordinate space at two moments of time. For a smooth potential of interaction with bounded derivatives, we prove the existence and uniqueness of a solution.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1995