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TRANSPORTS ALONG PATHS IN FIBRE BUNDLES.
III. Consistency with Bundle Morphisms

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1. INTRODUCTION

In Ref. [1] the problem has been investigated on consistency, or compatibility, of linear transports along paths in vector bundles and bundle metrics, i.e. when the transports preserve the (scalar products defined by the) metric. The present paper generalizes this problem to and deals with the problem of consistency (compatibility) of arbitrary transports along paths in fibre bundles [2] and acting between these fibre bundles bundle morphisms [3,4]. This task is sufficiently general, to cover from a unified point of view, all analogous problems from the literature available to the author.

The problem for consistency of transports along paths and bundle morphisms is stated in a general form in Sect. 2. Some necessary and sufficient conditions for such consistency are found. It is proved that the introduced concept for consistency is a special case of the one for sections of a fibre bundle transported by means of a transport along paths [2]. In Sect. 3 these concept and results are applied to the special case of a transport along paths and bundle morphisms acting in one and the same fibre bundle. Also some examples are presented. Sect. 4 contains a detailed investigation of the consistency between linear transports along paths in a vector bundle and a Hermitian structure in it. Sect. 5 closes the paper with some concluding remarks.

Below we summarize certain definitions and results from [2] needed for this paper.

By (E, π, B) is denoted an arbitrary (topological) fibre bundle with a base B, bundle space E, projection $\pi : E \to B$, and homeomorphic fibres $\pi^{-1}(x)$, $x \in B$ [4-6].

The set of sections of (E, π, B) is $Sec(E, \pi, B)$, i.e. $\sigma \in Sec(E, \pi, B)$ means $\sigma : B \to E$ and $\pi \circ \sigma = id_B$, where id_X is the identity map of the set X.

By J and $\gamma : J \to B$ are denoted, respectively, an arbitrary real interval and a path in B.

The transport along paths in (E, π, B) is a map $I : \gamma \mapsto I^\gamma$, where $I^\gamma : (s,t) \mapsto I^\gamma_{s\to t}$, $s, t \in J$ in which the maps $I^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t))$ satisfy the equalities

$$I^\gamma_{t\to r} \circ I^\gamma_{s\to t} = I^\gamma_{s\to r}, \quad r, s, t \in J, \tag{1.1}$$

$$I^\gamma_{s\to s} = id_{\pi^{-1}(\gamma(s))}, \quad s \in J, \tag{1.2}$$

and its general form is described by

$$I^{\gamma}_{s\to t} = (F^{\gamma}_{t})^{-1} \circ F^{\gamma}_{s}, \quad s,t \in J, \tag{1.3}$$

$F^{\gamma}_{s} : \pi^{-1}(\gamma(s)) \to Q$, $s \in J$ being one-to-one maps onto one and the same set Q.

In the case of a linear transport along paths in a vector bundle [7] the corresponding to (1.3) general form of the transport matrix is (see [7], proposition 2.4)

$$H(t,s;\gamma) = F^{-1}(t;\gamma)F(s;\gamma), \quad s,t \in J \tag{1.4}$$

in which $F(s;\gamma)$ is a nondegenerate matrix function.

2. GENERAL THEORY

Let there be given two fibre bundles $\xi_h := (E_h, \pi_h, B_h)$, $h = 1,2$ in which defined are, respectively, the transports along paths ${}^1I$ and ${}^2I$. Let (F,f) be a bundle morphism from ξ_1 into ξ_2, i.e. (see [3,4]) $F : E_1 \to E_2$, $f : B_1 \to B_2$ and $\pi_2 \circ F = f \circ \pi_1$. Let $F_x := F \mid \pi_1^{-1}(x)$ for $x \in B_1$ and $\gamma : J \to B_1$ be an arbitrary path in B_1.

Definition 2.1. The bundle morphism (F,f) and the pair $({}^1I, {}^2I)$ of transports, or the transports ${}^1I$ and ${}^2I$, along paths will be called consistent (resp. along the path γ) if they commute in a sense that the equality

$$F_{\gamma(t)} \circ {}^1I^{\gamma}_{s\to t} = {}^2I^{f\circ\gamma}_{s\to t} \circ F_{\gamma(s)}, \quad s,t \in J \tag{2.1}$$

is fulfilled for every (resp. the given) path γ.

This definition contains as an evident special case definition 1.1 from [1]. In fact, to prove this it is sufficient to put in it: $\xi_1 = (E,\pi,B) \times (E,\pi,B)$, where (E,π,B) is a vector bundle; $\xi_2 = (\mathbf{R}, \pi_0, 0)$, where $0 \in \mathbf{R}$ and $\pi_0 : \mathbf{R} \to \{0\}$; $F_x = g_x$, where $g_x : \pi^{-1}(x) \times \pi^{-1}(x) \to \mathbf{R}$, $x \in B$ are nondegenerate symmetric and bilinear maps; $f : B \times B \to \{0\} \subset \mathbf{R}$; ${}^1I^{\gamma}_{s\to t} = I^{\gamma}_{s\to t} \times I^{\gamma}_{s\to t}$, where I^{γ} is a transport along $\gamma : J \to B$ in (E,π,B) and $s,t \in J$; ${}^2I^{f\circ\gamma}_{s\to t} = id_{\mathbf{R}}$.

Analogously to the considerations in [1], Sect. 2 here can be formulated in a general form, for example, the following problems: to be found necessary and/or sufficient conditions for consistency between bundle morphisms and (ordered) pairs of transports along paths; to be found, if any, all bundle morphisms (resp. transports along paths) which are consistent with a given pair of transports along paths (resp. bundle morphism), etc. Further we will consider some results in this field.

Proposition 2.1. The bundle morphism (F,f) and the pair $({}^1I, {}^2I)$ of transports along paths are consistent (resp. along the path γ) iff there exist $s_0 \in J$ and a map

$$C(s_0;\gamma, f\circ\gamma) : \pi_1^{-1}(\gamma(s_0)) \to \pi_2^{-1}((f\circ\gamma)(s_0)), \tag{2.2}$$

such that

$$F_{\gamma(s)} = {}^2I^{f\circ\gamma}_{s_0\to s} \circ C(s_0;\gamma,f\circ\gamma) \circ {}^1I^{\gamma}_{s\to s_0} \tag{2.3}$$

for every (resp. the given) path γ.

Remark. The reason for which as arguments of C are written s_0, γ and $f\circ\gamma$ will be cleared up below in proposition 2.2, where its general structure is described.

Proof. If (F,f) is consistent with $({}^1I, {}^2I)$ (resp. along γ), then, by definition, (resp. along γ) is valid (2.1), which, due to $(I^{\gamma}_{s\to t})^{-1} = I^{\gamma}_{t\to s}$ (see (1.3)), is equivalent to

$$F_{\gamma(t)} = {}^2I^{f\circ\gamma}_{s\to t} \circ F_{\gamma(s)} \circ {}^1I^{\gamma}_{t\to s}, \quad s,t \in J. \tag{2.4}$$

Fixing $s_0 \in J$ and putting $s = s_0$ and $t = s$ in this equality, we get (2.3) (resp. along γ) with $C(s;\gamma,f\circ\gamma) = F_{\gamma(s_0)}$. And vice versa, if (2.3) is true (resp. along γ) for some s_0 and C, then due to (1.1), we have

$$F_{\gamma(t)} \circ {}^1I^{\gamma}_{s\to t} = {}^2I^{f\circ\gamma}_{s_0\to t} \circ C(s_0;\gamma,f\circ\gamma) \circ {}^1I^{\gamma}_{t\to s_0} \circ {}^1I^{\gamma}_{s\to t} =$$
$$= {}^2I^{f\circ\gamma}_{s\to t} \circ {}^2I^{f\circ\gamma}_{s_0\to s} \circ C(s_0;\gamma,f\circ\gamma) \circ {}^1I^{\gamma}_{s\to s_0} = {}^2I^{f\gamma}_{s\to t} \circ F_{\gamma(s)},$$

i.e. (2.1) is identically satisfied (resp. along γ) for $s,t \in J$, so (F,f) and $({}^1I, {}^2I)$ are consistent (resp. along γ).■

Corollary 2.1. The equality (2.3) is valid iff

$$F_{\gamma(s)} = {}^2I^{f\circ\gamma}_{t_0\to s} \circ C(t_0;\gamma,f\circ\gamma) \circ {}^1I^{\gamma}_{s\to t_0}, \tag{2.3'}$$

where

$$C(t_0;\gamma,f\circ\gamma) = {}^2I^{f\circ\gamma}_{s_0\to t_0} \circ C(s_0;\gamma,f\circ\gamma) \circ {}^1I^{\gamma}_{t_0\to s_0} \tag{2.5}$$

for arbitrary $t_0 \in J$, i.e. the existence of $s_0 \in J$ and a map (2.2) for which (2.3) is valid leads to the existence of maps (2.5) for which (2.3′) is true for every $t_0 \in J$ and vice versa.

We have written (2.5) as an equality, but not as a definition of $C(t_0;\gamma,f\circ\gamma)$, because from (1.3) and (2.5), considered as a definition of C, for arbitrary $t_0\in J$ and fixed $s_0\in J$ the validity of (2.5) follows for every $t_0,s_0\in J$.

Proof. If (2.3) is valid, then by proposition 2.1 (F,f) and $({}^1I,{}^2I)$ are consistent, i.e. (2.4) is true, from where (2.3′) with $C(t_0;\gamma,f\circ\gamma)=F_{\gamma(t_0)}$ follows. In the opposite direction the proposition follows from the substitution of (2.5) into (2.3′) and the usage of (1.1). (The same result follows also directly from (2.3) and $(I^{\gamma}_{s\to t})^{-1}=I^{\gamma}_{t\to s}$ (see (1.3))).■

Corollary 2.2. The bundle morphism (F,f) and the pair $({}^1I,{}^2I)$ of transports along paths are consistent (resp. along the path γ) iff there exists a bundle morphism (C,f) from ξ_1 on ξ_2 such that for every (resp. the given) $\gamma:J\to B_1$ is fulfilled: a) $(C,f)\mid\gamma(J):=(C\mid\pi_1^{-1}(\gamma(J)),f\mid\gamma(J))$ is a bundle morphism from $\xi_1\mid_{\gamma(J)}$ on $\xi_2\mid_{(f\circ\gamma)(J)}$, i.e. $C(s;\gamma,f\circ\gamma):=C_{\gamma(s)}:=C\mid\pi_1^{-1}(\gamma(s)):\pi_1^{-1}(\gamma(J))\to\pi_2^{-1}((f\circ\gamma)(J)),\quad s\in J$; b) (C,f) is consistent with $({}^1I,{}^2I)$ and c) the equality (2.3′) is valid.

Proof. If (F,f) and $({}^1I,{}^2I)$ are consistent (resp. along γ), then by proposition 2.1 and corollary 2.1 (2.3′) and (2.5) are fulfilled; so defining the bundle morphism (C,f) through $C_{\gamma(s_0)}:=C\mid\pi_1^{-1}(\gamma(s_0)):=C(s_0;\gamma,f\circ\gamma),\quad s\in J$ from (2.5) and definition 2.1 (see also (2.4)), we see that (C,f) and $({}^1I,{}^2I)$ are consistent (resp. along γ). On the opposite, if there exists a bundle morphism (C,f) with the pointed properties, then there are valid (2.3′) (the condition c)) and (2.5) (follows from the condition b) and (2.4) which is equivalent to definition 2.1) and by proposition 2.1 (F,f) and $({}^1I,{}^2I)$ are consistent (resp. along γ).■

From a functional point of view the general structure of the transports along paths is described by [2], theorem 3.1 and has the form (1.3). The usage of this theorem allows us to clear up the sense of proposition 2.1, as well as to solve (locally, i.e. along a given path) the question for the full description of all bundle morphisms which are (locally) consistent with a given pair of transports along paths.

Let, in accordance with [2], theorem 3.1, be chosen sets Q_1 and Q_2 and one-to-one maps ${}^hF^{\gamma_h}_{s_h}:\pi_h^{-1}(\gamma_h(s_h))\to Q_h$, $h=1,2$, which are associated, respectively, with the paths $\gamma_h:J_h\to B_h,\quad s_h\in J_h,\quad h=1,2$ and are such that (cf. (1.3))

$$ {}^hI^{\gamma_h}_{s_h\to t_h}=\left({}^hF^{\gamma_h}_{t_h}\right)^{-1}\circ{}^hF^{\gamma_h}_{s_h},\quad s_h,t_h\in J_h,\quad h=1,2. \tag{2.6}$$

Proposition 2.2. The bundle morphism (F,f) and the pair $({}^1I,{}^2I)$ of transports along paths, which are given through (2.6) by means of the maps ${}^1F$ and ${}^2F$, are consistent (resp. along a path γ) iff there exists a map

$$C_0(\gamma,f\circ\gamma):Q_1\to Q_2, \tag{2.7}$$

such that

$$F_{\gamma(s)}=\left({}^2F^{f\circ\gamma}_s\right)^{-1}\circ C_0(\gamma,f\circ\gamma)\circ\left({}^1F^{\gamma}_s\right), \tag{2.8}$$

or, equivalently, that

$$F_{\gamma(s)}={}^2I^{f\circ\gamma}_{s_0\to s}\circ C(s_0;\gamma,f\circ\gamma)\circ{}^1I^{\gamma}_{s\to s_0}, \tag{2.9}$$

where $s_0\in J$ is arbitrary and

$$C(s_0;\gamma,f\circ\gamma):=\left({}^2F^{f\circ\gamma}_{s_0}\right)^{-1}\circ C_0(\gamma,f\circ\gamma)\circ\left({}^1F^{\gamma}_{s_0}\right) \tag{2.10}$$

for every (resp. the given) path γ.

Proof. The substitution of (2.6) into (2.1) shows the equivalence of the latter with

$$\left({}^2F^{f\circ\gamma}_t\right)\circ F_{\gamma(t)}\circ\left({}^1F^{\gamma}_t\right)^{-1}=\left({}^2F^{f\circ\gamma}_s\right)\circ F_{\gamma(s)}\circ\left({}^1F^{\gamma}_s\right)^{-1}$$

for any $s,t\in J$. Hence, if (F,f) and $({}^1I,{}^2I)$ are consistent (resp. along γ), i.e. (2.1) is satisfied, then the last expression does not at all depend on $s,t\in J$ and, e.g., fixing arbitrary some $t_0\in J$ and putting $t=t_0$ and $C_0(\gamma,f\circ\gamma):=\left({}^2F^{f\circ\gamma}_{t_0}\right)\circ F_{\gamma(t_0)}\circ\left({}^1F^{\gamma}_{t_0}\right)^{-1}$ from the last equality, we easily obtain (2.8).

The equivalence of (2.8) and (2.9) follows directly from the eq. (2.6) and definition (2.10).

On the opposite, if it is valid (resp. along γ), e.g., (2.9), then by proposition 2.1 the bundle morphism (F,f) and the pair $({}^1I,{}^2I)$ of transports along paths are consistent (resp. along γ). ■

Evidently, the proposition 2.2 is a direct generalization of proposition 2.2 of [1], which is its special case.

The difference between propositions 2.1 and 2.2 is that the latter, through the equality (2.10), establishes the general functional form of the map (2.2).

As by [2], proposition 3.5 the maps ${}^hF^{\gamma_h}_{s_h}$, $h=1,2$ are defined up to a transformation of a form (see [2], eq. (3.11)) ${}^hF^{\gamma_h}_{s_h}\to({}^hD^{\gamma_h})\circ\left({}^hF^{\gamma_h}_{s_h}\right)$, $h=1,2$, where ${}^hD^{\gamma_h}:Q_h\to Q_h^{\circ}$, is a one-to-one map of Q_h onto some set

Q_h°, $h = 1,2$, then there exists also nonuniqueness in the choice of the map (2.8). An elementary check shows the validity of (cf. [1], eq. (2.7))

$$ {}^hF_{s_h}^{\gamma_h} \to \left({}^hD^{\gamma_h}\right) \circ \left({}^hF_{s_h}^{\gamma_h}\right), \quad h = 1,2 $$

$$ \Longleftrightarrow C_0(\gamma, f\circ\gamma) \to \left({}^2D^{f\circ\gamma}\right)^{-1} \circ C_0(\gamma, f\circ\gamma) \circ \left({}^1D^{\gamma}\right). \qquad (2.11) $$

Proposition 2.3. If for a given pair $({}^1I, {}^2I)$ of transports along paths the representation (2.6) is chosen, then all consistent along $\gamma : J \to B_1$ with it bundle morphisms $(F^{\gamma, f\circ\gamma}, f)$ along γ are obtained from the equality

$$ F_{\gamma(s)}^{\gamma, f\circ\gamma} := \left({}^2F_s^{f\circ\gamma}\right)^{-1} \circ C_0(\gamma, f\circ\gamma) \circ \left({}^1F_s^{\gamma}\right) = {}^2I_{s_0\to s}^{f\circ\gamma} \circ C(s_0;\gamma, f\circ\gamma) \circ {}^1I_{s\to s_0}^{\gamma}, \qquad (2.12) $$

where $s_0 \in J$ is arbitrary, C is defined by (2.10) in which $C_0(\gamma, f\circ\gamma)$ is a one-to-one map from Q_1 onto Q_2.

Proof. This proposition is a consequence of the proof of proposition 2.2, as from it is clear that (2.12) is the general solution of the equation (2.1) with respect to $F_{\gamma(t)}$ when ${}^1I$ and ${}^2I$ are given. ■

The definition 1.1 of [1] for consistency between bundle metrics and transports along paths seems rather natural by itself for a difference of the definition 2.1 for consistency between bundle morphisms and a pair of transports along paths, whose introduction needs some explanation. As we saw in the written after definition 2.1 the former definition is a special case of the latter. Now we shall show that in this context the definition 2.1 itself is a special case of definition 2.2 of [2] for a section of a given fibre bundle transported along a path.

Let there be given two fibre bundles $\xi_h = (E_h, \pi_h, B_h)$, $h = 1,2$. We define the fibre bundle $\xi_0 = (E_0, \pi_0, B_1)$ of bundle morphisms from ξ_1 on ξ_2 in the following way:

$$ E_0 := \{(F_{b_1}, f) : F_{b_1} : \pi_1^{-1}(b_1) \to \pi_2^{-1}(f(b_1)),\ b_1 \in B_1,\ f : B_1 \to B_2\}, \qquad (2.13) $$

$$ \pi_0((F_{b_1}, f)) := b_1, \quad (F_{b_1}, f) \in E_0, \quad b_1 \in B_1. \qquad (2.14) $$

It is clear that every section $(F, f) \in Sec\xi_0$ is a bundle morphism from ξ_1 into ξ_2 and vice versa, every bundle morphism from ξ_1 on ξ_2 is a section of ξ_0. (Thus a bundle structure in the set $Morf(\xi_1, \xi_2)$ of bundle morphisms from ξ_1 on ξ_2 is introduced.)

If in ξ_0 given is a transport K along the paths in B_1, then according to [2], definition 2.2 (see therein eq. (2.4)), the bundle morphism $(F, f) \in Sec\xi_0$ is $(K-)$transported along $\gamma : J \to B_1$ if

$$ (F_{\gamma(t)}, f) = K_{s\to t}^{\gamma}(F_{\gamma(s)}, f), \quad s,t \in J. \qquad (2.15) $$

If in ξ_1 and ξ_2 are given, respectively, the transports ${}^1I$ and ${}^2I$ along the paths, respectively, in B_1 and B_2, then they generate in ξ_0 a "natural" transport ${}^0K$ along the paths in B_1. The action of this transport along $\gamma : J \to B_1$ on $(F_{\gamma(s)}, f) \in \pi_0^{-1}(\gamma(s))$ for a fixed $s \in J$ and arbitrary $t \in J$ is defined by

$$ {}^0K_{s\to t}^{\gamma}(F_{\gamma(s)}, f) := \left({}^2I_{s\to t}^{f\circ\gamma} \circ F_{\gamma(s)} \circ {}^1I_{t\to s}^{\gamma},\ f\right) \in \pi_0^{-1}(\gamma(t)). \qquad (2.16) $$

Proposition 2.4. The map ${}^0K_{s\to t}^{\gamma} : \pi_0^{-1}(\gamma(s)) \to \pi_0^{-1}(\gamma(t))$ defined through (2.16) is a transport along γ from s to t, $s,t \in J$ and, consequently, in ξ_0 ${}^0K$ is a transport along paths.

Proof. Using the properties (1.1) and (1.2) of the transports along paths it is easy to check with the help of (2.16) that the maps ${}^0K_{s\to t}^{\gamma}$, $s,t \in J$ satisfy the equalities

$$ {}^0K_{s\to t}^{\gamma} \circ {}^0K_{r\to s}^{\gamma} = {}^0K_{r\to t}^{\gamma}, \quad r,s,t \in J, \qquad (2.17) $$

$$ {}^0K_{s\to s}^{\gamma} = id_{\pi_0^{-1}(\gamma(s))}, \quad s \in J \qquad (2.18) $$

and hence, by [2], definition 2.1, ${}^0K_{s\to t}^{\gamma}$ is a transport along γ from s to t, i.e. in ξ_0 ${}^0K$ really defines a transport along paths.■

Lemma 2.1. If $(F, f) \in Sec\xi_0$, then (2.1) is equivalent to

$$ (F_{\gamma(t)}, f) = {}^0K_{s\to t}^{\gamma}(F_{\gamma(s)}, f), \quad s,t \in J. \qquad (2.19) $$

Proof. At the beginning of the proof of proposition 2.1 we saw that (2.1) is equivalent to (2.4), that is equivalent to (2.19) because of (2.16), i.e. (2.1) and (2.19) are equivalent.■

Proposition 2.5. The bundle morphism (F, f) and the pair $({}^1I, {}^2I)$ of transports along paths are consistent (resp. along the path γ) iff (F, f) is transported along every (resp. the given) path γ with the help of the defined from $({}^1I, {}^2I)$ in ξ_0 transport along paths ${}^0K$.

Proof. The proposition follows directly from lemma 2.1, definition 2.1 and definition 2.2 of [2] (see (2.15) and [2], eq. (2.4)).■

Taking into account the comment after definition 2.1 it is not difficult to verify that proposition 3.1′ of [1] is, in fact, a variant of proposition 2.5 in the special case when one studies the consistency of S-transports and bundle metrics.

3. CONSISTENCY WITH MORPHISMS OF THE FIBRE BUNDLE

In this section we are going to apply the general theory of Sect. 2 to the case of bundle morphisms of a given fibre bundle. The results so-obtained will be illustrated with often used examples.

Let in the fibre bundle (E,π,B) a (bundle) morphism F be given. By definition [4,5] this means that (F,id_B) is a bundle morphism from (E,π,B) into (E,π,B). Hence it is natural (cf. definition 2.1) F and the transport along paths I in (E,π,B) to be called globally (resp. locally) consistent if (F,id_B) and the pair (I,I) are globally (resp. locally) consistent, i.e. in this case definition 2.1 reduces to

Definition 3.1. The transport along paths I in (E,π,B) is globally (resp. locally) consistent with the bundle morphism F of (E,π,B) if

$$F_{\gamma(t)}\circ I^{\gamma}_{s\to t} = I^{\gamma}_{s\to t}\circ F_{\gamma(s)},\quad s,t\in J \tag{3.1}$$

is fulfilled for every (resp. the given) path $\gamma : J\to B$.

This definition formalizes the condition for commutation of a transport along paths and a bundle morphism of the fibre bundle, in which the transport acts, or, in other words, the equality (3.1) is an exact expression of the phrase that "the bundle morphism F and the transport along paths I commute".

Comparing definitions 3.1 and 2.1, we see that from the first of them the second can be obtained if in the former we put

$$(E_h,\pi_h,B_h) = (E,\pi,B),\quad h=1,2,\ f=id_B,\ {}^1I={}^2I=I. \tag{3.2}$$

If we make these substitutions in the whole section 2 and take into account definition 3.1, then the stated therein propositions and definitions, concerning bundle morphisms between fibre bundles, take the following formulations in the case of bundle morphisms of the fibre bundle (E,π,B):

Proposition 3.1. The bundle morphism F and the transport along paths I are consistent (resp. along γ) if and only if there exist $s_0\in J$ and a map

$$C(s_0;\gamma):\pi^{-1}(\gamma(s_0))\to\pi^{-1}(\gamma(s_0)), \tag{3.3}$$

such that

$$F_{\gamma(s)} = I^{\gamma}_{s_0\to s}\circ C(s_0;\gamma)\circ I^{\gamma}_{s\to s_0} \tag{3.4}$$

for every (resp. the given) path γ.

Corollary 3.1. The equality (3.4) is true iff for every $t_0\in J$, we have

$$F_{\gamma(s)} = I^{\gamma}_{t_0\to s}\circ C(t_0;\gamma)\circ I^{\gamma}_{s\to t_0}, \tag{3.4'}$$

where

$$C(t_0;\gamma) = I^{\gamma}_{s_0\to t_0}\circ C(s_0;\gamma)\circ I^{\gamma}_{t_0\to s_0}. \tag{3.5}$$

Corollary 3.2. The bundle morphism F and the transport along paths I are consistent (resp. along γ) iff there exists a bundle morphism C of (E,π,B) such that for every (resp. the given) path $\gamma : J\to B$ we have: a) $C\mid\pi^{-1}(\gamma(J))$ is a morphism of $(E,\pi,B)\mid_{\gamma(J)} = (\pi^{-1}(\gamma(J)),\pi\mid\gamma(J),\gamma(J))$; b) C and I are consistent (resp. along γ); and c) (3.4′) is true.

Proposition 3.2. The bundle morphism F and the transport along paths I defined by the equality $I^{\gamma}_{s\to t} = (F^{\gamma}_t)^{-1}\circ F^{\gamma}_s,\quad t,s\in J$ (see (1.3)) are consistent (resp. along γ) if and only if there exists a map $C_0(\gamma):Q\to Q$, such that

$$F_{\gamma(s)} = (F^{\gamma}_s)^{-1}\circ C_0(\gamma)\circ F^{\gamma}_s = I^{\gamma}_{t\to s}\circ C(s;\gamma)\circ I^{\gamma}_{s\to t},\quad s,t\in J \tag{3.6}$$

where

$$C(s;\gamma) := (F^{\gamma}_s)^{-1}\circ C_0(\gamma)\circ F^{\gamma}_s. \tag{3.7}$$

Proposition 3.3. If a transport along paths I with a representation $I^{\gamma}_{s\to t} = (F^{\gamma}_t)^{-1}\circ F^{\gamma}_s,\quad s\in J$ (see (1.3)) is fixed, then any bundle morphism ${}^{\gamma}F$ along γ consistent along $\gamma : J\to B$ with it is obtained from the equality

$${}^{\gamma}F_{\gamma(s)} := (F^{\gamma}_s)^{-1}\circ C_0(\gamma)\circ F^{\gamma}_s = I^{\gamma}_{s_0\to s}\circ C(s_0;\gamma)\circ I^{\gamma}_{s\to s_0}, \tag{3.8}$$

where $s_0\in J$ is arbitrary, $C_0 : Q\to Q$ is one-to-one and C is defined by (3.7).

According to (2.13) and (2.15) the fibre bundle (E_0,π_0,B) of the bundle morphisms of (E,π,B) is defined through the equalities

$$E_0 := \{F_b :\quad F_b:\pi^{-1}(b)\to\pi^{-1}(b),\quad b\in B\}, \tag{3.9}$$

$$\pi_0(F_b) := b,\quad F_b\in E_0,\quad b\in B. \tag{3.10}$$

Evidently, if F is a morphism of (E,π,B), then $F\in Sec(E_0,\pi_0,B)$ and vice versa.

According to [2], definition 2.2 if in (E_0,π_0,B) is given a transport along paths K, then $F\in Sec(E_0,\pi_0,B)$ is K-transported along $\gamma : J\to B$ if

$$F_{\gamma(t)} = K^{\gamma}_{s\to t}(F_{\gamma(s)}). \tag{3.11}$$

If I is a transport along paths in (E, π, B), then in (E_0, π_0, B) it induces in a "natural" transport along paths ${}^0K$ whose action along $\gamma : J \to B$ on $F_{\gamma(s)} \in \pi_0^{-1}(\gamma(s))$ is

$$ {}^0K^{\gamma}_{s\to t}(F_{\gamma(s)}) := I^{\gamma}_{s\to t} \circ F_{\gamma(s)} \circ I^{\gamma}_{t\to s} \in \pi_0^{-1}(\gamma(t)), \quad s,t \in J. \qquad (3.12) $$

Proposition 3.4. The mapping ${}^0K^{\gamma}_{s\to t} : \pi_0^{-1}(\gamma(s)) \to \pi_0^{-1}(\gamma(t))$, defined by (3.12), is a transport along γ from s to t, $s,t \in J$ and, consequently, ${}^0K$ defines a transport along paths in (E_0, π_0, B).

Lemma 3.1. The equality (3.1) is equivalent to

$$ F_{\gamma(t)} = {}^0K^{\gamma}_{s\to t}(F_{\gamma(s)}), \quad s,t \in J. \qquad (3.13) $$

Proposition 3.5. In (E_0, π_0, B) the bundle morphism F and the transport along paths I are consistent (resp. along γ) iff F is transported along every (resp. the given) path γ with the help of the transport along paths ${}^0K$ defined by I in (E_0, π_0, B).

Now we shall consider examples for consistency of concrete bundle morphisms in vector bundles with transport along paths in them.

Example 3.1. *Consistency with an almost complex structure*

Let the bundle morphism $\mathbf{J}$ of the real vector bundle (E, π, B) define an almost complex structure in it [8-10], i.e. $\mathbf{J}_x := \mathbf{J} \mid \pi^{-1}(x)$, $x \in B$ to be $\mathbf{R}$-linear isomorphisms defining complex structure in the fibres $\pi^{-1}(x)$, which means that

$$ \mathbf{J}_x \circ \mathbf{J}_x := -id_{\pi^{-1}(x)}. \qquad (3.14) $$

Evidently, if (E, π, B) is the tangent bundle to some manifold and $\mathbf{J}$ is a linear endomorphism, then $\mathbf{J}$ defines an almost complex structure on that manifold [3,8]. In this case, following the accepted terminology, a transport along paths consistent with $\mathbf{J}$ may be called almost complex.

Proposition 3.6. A bundle morphism $\mathbf{J}$ consistent with a transport along paths I of the vector fibre bundle (E, π, B) defines an almost complex structure in it if and only if the involved in (3.4)-(3.7) (with $\mathbf{J}_{\gamma(s)}$ instead of $F_{\gamma(s)}$) map $C(s;\gamma)$ or map $C_0(\gamma)$ define a complex structure in $\pi^{-1}(\gamma(s))$ or Q respectively, i.e. when they satisfy the following equalities:

$$ C(s;\gamma) \circ C(s;\gamma) = -id_{\pi^{-1}(\gamma(s))}, \qquad (3.15) $$

$$ C_0(\gamma) \circ C_0(\gamma) = -id_Q. \qquad (3.15') $$

Or, in other words, the almost complex structure $\mathbf{J}$ and the transport along paths I are globally (resp. locally) consistent, i.e. they commute globally (resp. locally), if and only if there are fulfilled (3.4) (or (3.6)), (3.15) and (3.15′) for every (resp. the given) path γ; besides, eqs. (3.15) and (3.15′) are equivalent.

Proof. According to proposition 3.1 $\mathbf{J}$ and I are consistent (resp. along γ) iff (3.4) holds for $F = \mathbf{J}$ and every (resp. the given) path γ, so we get $\mathbf{J}_{\gamma(s)} \circ \mathbf{J}_{\gamma(s)} = I^{\gamma}_{s_0\to s} \circ C(s_0;\gamma) \circ C(s_0;\gamma) \circ I^{\gamma}_{s\to s_0}$, whence it follows that (resp. along γ) (3.13) is equivalent to (3.15) for $s = s_0$.

Analogously, based on the considerations on (3.6), one proves (resp. along γ) the equivalence of (3.13) with (3.14) (for every s) and (3.15′).

The equivalence of (3.15) and (3.15′) is a consequence of (3.7).■

Example 3.2. *Consistency with a multiplication with numbers*

Let (E, π, B) be a real (resp. complex) vector bundle, $\lambda \in \mathbf{R}$ (resp. $\lambda \in \mathbf{C}$) and the bundle morphism ${}^{\lambda}F$ of (E, π, B) be defined by ${}^{\lambda}F(u) := \lambda \cdot u = {}^{\lambda}F_{\pi(u)}(u)$ for every $u \in$E.

Definition 3.2. The transport along paths I in the real (resp. complex) vector bundle (E, π, B) is called consistent with the operation multiplication with real (resp. complex) numbers if it is consistent with the bundle morphisms ${}^{\lambda}F$ for every $\lambda \in \mathbf{R}$ (resp. $\lambda \in \mathbf{C}$).

Proposition 3.7. The transport along paths I is globally (resp. locally) consistent with the multiplication with, respectively, real or complex numbers if and only if

$$ I^{\gamma}_{s\to t}(\lambda u) = \lambda (I^{\gamma}_{s\to t}(u)), \quad u \in \pi^{-1}(\gamma(s)) \qquad (3.16) $$

for every, respectively, $\lambda \in \mathbf{R}$ or $\lambda \in \mathbf{C}$ and every (resp. the given) path $\gamma : J \to B$.

Proof. The proposition is a simple corollary of definitions 3.1 and 3.2, proposition 3.5, lemma 3.1 and (3.12).■

In other words we may say that the consistency with multiplication with numbers means simply the validity of the condition for homogeneity (3.16) or, which is the same, the operations of I-transportation along paths and multiplication with numbers in the fibres to commute.

Example 3.3. *Consistency with the operation addition*

Let (E, π, B) be a vector fibre bundle, $A \in Sec(E, \pi, B)$ and ${}^AF$ be a bundle morphism of (E, π, B) defined by ${}^AF(u) := u + A(\pi(u)) = {}^A F_{\pi(u)}(u)$ for every $u \in E$.

Definition 3.3. The transport along paths I in the vector fibre bundle (E, π, B) is called globally (resp. locally) consistent with the operation addition if it is consistent with the bundle morphisms ${}^{A}F$ for every section A I-transported along every (resp. the given) path.

Proposition 3.8. The transport I is globally (resp. locally) consistent with the operation addition iff these operations commute, i.e. iff

$$I^{\gamma}_{s\to t}(u+v) = I^{\gamma}_{s\to t}(u) + I^{\gamma}_{s\to t}(v), \quad u, v \in \pi^{-1}(\gamma(s)), \tag{3.17}$$

which means that the usual additivity condition in the fibres is to be fulfilled for every (resp. the given) path $\gamma : J \to$B.

Proof. According to definition 3.1 the transport I and the above defined bundle morphisms ${}^{A}F$ are globally (resp. locally) consistent iff the equality

$$A(\gamma(t)) + I^{\gamma}_{s\to t}(u) = I^{\gamma}_{s\to t}[A(\gamma(s)) + u], \quad u \in \pi^{-1}(\gamma(s)) \tag{3.18}$$

is valid for every (resp. the given) path γ.

The condition A to be I-transported along γ section means (see [2], definition 2.2 and proposition 2.1) that $A(\gamma(t)) = I^{\gamma}_{s\to t}A(\gamma(s))$, $s, t \in J$, which, when substituted into the previous equality, due to the arbitrariness of A (i.e. of $A(\gamma(s))$) gives (3.17) (see [2], proposition 2.2) in which $A(\gamma(s))$ is denoted with v. On the opposite, if (3.17) is valid and A is a section I-transported along γ section, then (3.18) is identically satisfied and, consequently, ${}^{A}F$ and I are consistent.■

Example 3.4. *Consistency between transport along paths and Finslerian metrics*

Let in a manifold M a Finslerian metric be given [11] by means of a Finslerian metric function $F : T(M) \to \mathbf{R}_0 := \{\lambda : \lambda \in \mathbf{R}, \lambda \geq 0\}$ having the property $F(x, \lambda A) = \lambda F(x, A)$ for $\lambda \in \mathbf{R}_+ := \{\lambda : \lambda \in \mathbf{R}, \lambda > 0\}$, $x \in M$, $A \in T_x(M)$ and satisfying the conditions described in the above references. Here $T(M) := \cup_{x\in M} T_x(M)$, where $T_x(M)$ is the tangent to M space at $x \in$M.

Definition 3.4. The Finslerian metric and the transport along paths I are consistent (resp. along $\gamma : J \to M$) if the equality

$$F(\gamma(s), A) = F(\gamma(t), I^{\gamma}_{s\to t}A), \quad s, t \in J, \quad A \in T_{\gamma(s)}(M) \tag{3.19}$$

is fulfilled for every (resp. the given) path γ.

This definition is a special case of definition 2.1 and it is obtained from it for: $\xi_1 = (T(M), \pi, M)$; $\xi_2 = (\mathbf{R}_0, \pi_0, 0)$, where $0 \in \mathbf{R}$ and $\pi_0 : \mathbf{R}_0 \to \{0\}$; in the bundle morphism (F, f) $F : T(M) \to \mathbf{R}_0$ is the Finslerian metric function, $F_x(A) = F(x, A)$, $A \in T_x(M)$ and $f : M \to \{0\}$; ${}^{1}I^{\gamma} = I^{\gamma}$; ${}^{2}I^{f\circ\gamma} = id_{\mathbf{R}_0}$.

Example 3.5. *Consistency between transports along paths and symplectic metrics*

A real symplectic metric $a : x \mapsto a_x$, $a_x : \pi^{-1}(x) \times \pi^{-1}(x) \to \mathbf{R}$ in a fibre bundle (E, π, B) differs from a real symmetric metric g (cf. [1]) only in that it is antisymmetric, i.e. $a_x(u, v) = -a_x(v, u)$ for $u, v \in \pi^{-1}(x)$. Hence, modifying definition 1.1 of [1], we can say that the transport along paths I and a are consistent (resp. along γ) if

$$a_{\gamma(s)} = a_{\gamma(t)} \circ (I^{\gamma}_{s\to t} \times I^{\gamma}_{s\to t}), \quad s, t \in J \tag{3.20}$$

is fulfilled for every (resp. the given) path $\gamma : J \to$B.

4. CONSISTENCY WITH A HERMITIAN STRUCTURE

By a Hermitian structure in the real vector bundle (E, π, B) we understand (cf. [10]) a pair $(\mathbf{J}, g)$ of almost complex structure $\mathbf{J}$, i.e. a bundle morphism $\mathbf{J} : E \to E$ with the property $\mathbf{J} \circ \mathbf{J} = -id_E$, and a consistent with it bundle symmetric metric g ($g : x \to g_x$, such that g_x is bilinear, symmetric and $g_x = g_x \circ (\mathbf{J}_x \times \mathbf{J}_x)$) in that bundle (called a Hermitian metric; see [10], ch. IX, §1).

Definition 4.1. If I is a transport along paths in (E, π, B), then the Hermitian structure $(\mathbf{J}, g)$ and I are consistent (resp. along the path γ) if the pairs $\mathbf{J}$ and I and g and I are consistent separately, i.e.

$$\mathbf{J}_{\gamma(t)} \circ I^{\gamma}_{s\to t} = I^{\gamma}_{s\to t} \circ \mathbf{J}_{\gamma(s)}, \tag{4.1}$$

$$g_{\gamma(s)} = g_{\gamma(t)} \circ (I^{\gamma}_{s\to t} \times I^{\gamma}_{s\to t}), \tag{4.2}$$

for every (resp. the given) path γ.

Remark. The condition (4.2) for consistency between g and I was introduced and investigated in [1].

In particular, if $(E, \pi, B) = (T(M), \pi, M)$ is the tangent bundle to the manifold M, then (4.1) and (4.2) define (an "almost Hermitian") transport consistent with the Hermitian structure $(\mathbf{J}, g)$ of the almost Hermitian manifold $(M, \mathbf{J}, g)$ (cf. [10], ch. IX, §2).

Further we shall dwell on the question for consistency between linear transports (L-transports) along paths L in a vector bundle (E, π, B) [7] and Hermitian structures $(\mathbf{J}, g)$ in it, such that $\mathbf{J}$ is a linear endomorphism.

Let along the path $\gamma : J \to B$ there be fixed bases $\{e_i(s), \quad i = 1, \dots, \dim(\pi^{-1}(x)), x \in B\}$ in $\pi^{-1}(\gamma(s)), \quad s \in J$, in which $\mathbf{J}$ is defined by the matrices $\mathbf{J}(\gamma(s)) = \|\mathbf{J}^i_{.j}(\gamma(s))\|$, L is defined through the matrices $F(s;\gamma)$ by (1.4) and g - through the matrix $G(\gamma(s)) = \|g_{\gamma(s)}(e_i(s), e_j(s))\|$. Then the condition $g = g \circ (\mathbf{J} \times \mathbf{J})$ for consistency along γ between $\mathbf{J}$ and g takes the form

$$\mathbf{J}^\top(\gamma(s))G(\gamma(s))\mathbf{J}(\gamma(s)) = G(\gamma(s)), \tag{4.3}$$

where $\top$ means transposition of matrices, and the the condition for consistency between $\mathbf{J}$ and I, due to propositions 3.2 and 3.6, looks like

$$\mathbf{J}(\gamma(s)) = F^{-1}(s;\gamma)C_0(\gamma)F(s;\gamma), \qquad C_0(\gamma)C_0(\gamma) = -\mathbf{I}, \tag{4.4}$$

where $C_0(\gamma)$ is a nondegenerate matrix and $\mathbf{I}$ is the unit matrix.

Let in (E, π, B) an L-transport along paths L be given [7]. The following two propositions solve the problems for the existence and a full (local and global) description of all consistent with L Hermitian structures.

Proposition 4.1. Along the path $\gamma : J \to B$ the class of all Hermitian structures (along γ) which are locally consistent with the L-transport along paths L is given in the above pointed bases through the equalities

$$\mathbf{J}(\gamma(s);\gamma) = F^{-1}(s;\gamma)C_0(\gamma)F(s;\gamma), \tag{4.5a}$$

$$G(\gamma(s);\gamma) = F^\top(s;\gamma)C(\gamma)F(s;\gamma) \tag{4.5b}$$

in which the matrix functions C_0 and C satisfy the equations

$$\mathbf{I} = C^\top C^{-1} = -C_0C_0 = C_0^\top CC_0C^{-1}. \tag{4.6}$$

Proof. The equalities (4.5a) and (4.5b), and also the first two from (4.6), follow, respectively, from (4.4) and [1], proposition 2.3 (see therein eq. (2.8)). The last equality from (4.6) is obtained by the substitution of (4.5) into the condition for consistency between $\mathbf{J}$ and g expressed now by (4.3).■

Proposition 4.2. Let there be given an L-transport along paths L defined along $\gamma : J \to B$ through (1.4) by the matrices

$$F(s;\gamma) = Y(\gamma)Z(s;\gamma)D^{-1}(\gamma(s)), \quad s \in J, \tag{4.7}$$

where Y and D are nondegenerate matrix functions and Z is a pseudoorthogonal of some type (p,q), $p + q = \dim(\pi^{-1}(\gamma(s)))$ matrix function, i.e.

$$Z^\top(s;\gamma)G_{p,q}Z(s;\gamma) = G_{p,q} := diag(\underbrace{1,\dots,1}_{p-\text{times}}, \underbrace{-1,\dots,-1}_{q-\text{times}}). \tag{4.8}$$

Then every Hermitian structure $(\mathbf{J}, g)$ globally consistent with L, if any, is given by the equalities

$$\mathbf{J}(\gamma(s)) = D(\gamma(s))Z^{-1}(s;\gamma)PZ(s;\gamma)D^{-1}(\gamma(s)), \tag{4.9a}$$

$$G(\gamma(s)) = (D^{-1}(\gamma(s)))^\top G_{p,q}D^{-1}(\gamma(s)), \tag{4.9b}$$

where P is a constant with respect to s and γ matrix, which may depend on p and q and is explicitly constructed below (see (4.11)), and Z besides (4.8) satisfies the condition that $Z^{-1}(s;\gamma)PZ(s;\gamma)$ depends only on $\gamma(s)$, but not on s and γ separately.

Remark. The necessity of the representation (4.7) for $F(s;\gamma)$ is a consequence of that we want a globally consistent with L metric g to exist (see [1], proposition 2.6).

Proof. The equality (4.9b) is a corollary from proposition 2.6 from [1]. The consistency of L and $\mathbf{J}$ is equivalent to (4.4), due to which substituting (4.7) into (4.4), we get (4.9b) with

$$P = Y^{-1}(\gamma)C_0(\gamma)Y(\gamma), \quad C_0(\gamma)C_0(\gamma) = -\mathbf{I}. \tag{4.10}$$

As $\mathbf{J}$ and g must form a Hermitian structure, they have to be consistent, i.e. (4.3) must be true which, as we shall now prove, is a consequence of the independence of P of γ. In fact, substituting (4.9) into (4.3) and using (4.8), we find

$$P^\top G_{p,q}P = G_{p,q}, \tag{4.11a}$$

i.e. P is a pseudoorthogonal matrix of type (p,q) which, as a result of (4.10), satisfies

$$PP = -\mathbf{I}. \tag{4.11b}$$

If we consider (4.11) as a system of equations with respect to P, then, its solution, if any (see below), is independent of any parameters as it is a function only of $\mathbf{I}$ and $G_{p,q}$, i.e. the elements of P are independent of γ numbers. This conclusion is a consequence of the observation that (4.11a) does not change when it is transposed, i.e. $P^\top G_{p,q}P$ is a symmetric, and (4.11b), due to (4.11a), is equivalent to $(G_{p,q}P)^\top \equiv P^\top G_{p,q} = -G_{p,q}P$, i.e. $G_{p,q}P$ is antisymmetric. Hence (4.11a) and (4.11b) contain respectively $n(n+1)/2$ and $n(n-1)/2$, $n = \dim(\pi^{-1}(x))$, $x \in P$, or commonly n^2, a number of independent scalar equations for the n^2 components of P. So, if P exists, it is constant (along γ).

The condition for that the matrix $Z^{-1}(s;\gamma)PZ(s;\gamma)$ to depend only on $\gamma(s)$ is a result from that $\mathbf{J}$ must be globally defined, i.e. $\mathbf{J}(\gamma(s))$ must depend only on the point $\gamma(s)$, but not on the path $\gamma : J \to B$, so from (4.9a) the above pointed condition follows.■

Remark. It may be proved that for an even n, i.e. for $n = 2k+1$, $k = 1, 2, \ldots$, the equations (4.11) have no solutions with respect to P, which is in accordance with the fact that in this case in (E, π, B) a complex structure cannot be introduced (see [10], ch. IV, §1). For an odd n, i.e. for $n = 2k$, the equations (4.11) have different solutions with respect to P. For instance, for $n = 2$ these solutions for $p = 2 = 2 - q$, $p = q = 1$ and $2 - p = 2 = q$, respectively, are:

$$P^{\pm}_{2,0} = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P^{\pm}_{1,1} = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P^{\pm}_{0,2} = -P^{\pm}_{2,0}.$$

For $p = n = 2k$, $q = 0$, we have $P = diag(P_1, \ldots, P_k)$, $P_1, \ldots, P_k \in \{P^-_{2,0}, P^+_{2,0}\}$.

In the general case, the answer to the problem for the existence of L-transports consistent with a given Hermitian structure $(\mathbf{J}, g)$ is negative. Below we shall analyze the reasons for this.

Let in a fibre bundle a Hermitian structure $(\mathbf{J}, g)$ be given. We want to see whether there exist L-transports along paths consistent with it and, possibly, to describe them.

First of all, for the existence of L consistent with g the signature (and consequently the number of positive eigenvalues) of g must not depend on the point at which it is (they are) calculated (see [1], proposition 2.4).

By proposition 2.5 of [1], from the consistency between L and g it follows that the matrix function F describing L through (1.4) has the form (4.7) in which Y is arbitrary, Z satisfies (4.8) and p, q and D are define by

$$D^\top(x)G(x)D(x) = G_{p,q} \qquad (4.12)$$

for any point x from the base of the bundle.

The consistency between $\mathbf{J}$ and L shows that for some C_0 the function F must satisfy (4.4) and, due to (4.7), the matrix function Z is a solution of the equation

$$Z^{-1}(s;\gamma)PZ(s;\gamma) = A(\gamma(s)) := D^{-1}(\gamma(s))\mathbf{J}(\gamma(s))D(\gamma(s)), \qquad (4.13)$$

where P is given by (4.10) for some C_0 and, in accordance with the consistency between $\mathbf{J}$ and g (see (4.3)), does not depend on γ (see the proof of proposition 4.2).

So, the matrix function F defining L has the form (4.7) in which Z is a solution of the system (see (4.8) and (4.13))

$$Z^\top(s;\gamma)G_{p,q}Z(s;\gamma) = G_{p,q}, \qquad (4.14a)$$

$$PZ(s;\gamma) - Z(s;\gamma)A(\gamma(s)) = 0. \qquad (4.14b)$$

The equalities (4.14) form a system of $n(n+1)/2 + n^2$ scalar equations for n^2 elements of $Z(s;\gamma)$, as a consequence of which, generally, it has no a solution with respect to Z (see, in particular, the analysis made in [12] for the existence of solutions for the equation $AX + XB = C$ with respect to X).

The above consideration prove the following

Proposition 4.3. Let in a fibre bundle there be given a Hermitian structure $(\mathbf{J}, g)$ and the signature of g be independent of the point at which it is calculated. Let D, p and q be defined by (4.12). Then if for every (resp. a given) path γ and some $Y(\gamma)$ and $C_0(\gamma)$ there exists a (constant) matrix P satisfying (4.10) and (4.11), for which the system (4.14) (with A defined from (4.13)) has a solution with respect to $Z(s;\gamma)$, then the L-transport along paths (resp. the given path γ) defined by the matrices (4.7) is globally (resp. locally along γ) consistent with $(\mathbf{J}, g)$. The L-transports along paths (resp. along γ) obtained in this way form the class of all globally (resp. locally along γ) consistent with $(\mathbf{J}, g)$ L-transports along paths (resp. along γ).

5. CONCLUSION

In this work we have considered the problem for consistency (or compatibility) of transports along paths in (different or coinciding) fibre bundles and bundle morphisms between them. Our approach to this problem is sufficiently general and as its special cases includes all known to the author analogous problems posed in the literature. In particular, one most often comes to the question for consistency of a connection and some other mathematical structure, like a metric, complex or almost complex structure. It can equivalently be formulated as a special case of the above problem in the following way. On one hand, the connection can equivalently be expressed in terms of a corresponding parallel transport, a kind of transport along paths [13]. On the other hand, the mentioned mathematical structures, at least, in the known to the author analogous problems in the available to him literature, can equivalently be put in a form of bundle morphisms of the fibre bundle in which the parallel transport acts. So, the consistency between a connection and a mathematical structure is equivalent to the consistency of a corresponding parallel transport and a bundle morphism. A typical example of this kind is the consistency between a symmetric (Riemannian) metric and a linear connection (in the tangent bundle to a manifold), which in other terms is treated by proposition 3.2 of [1] (see also the comment after definition 4.1 of the present paper).

In connection with proposition 2.3 there arise two problems. First, to describe, if any, all pairs of transports (locally) consistent along a fixed path with a given bundle morphism. Second, to describe, if any, all bundle morphisms (resp. pairs of transports along paths) globally, i.e. along every path, consistent with a given pair of transports along paths (resp. bundle morphism). These problems will be investigated elsewhere.

At the end, we want to note that in the very special case when the bundle morphism (F, f) is such that there exists the inverse map F^{-1} (and hence also the map f^{-1}), then all pairs of transports along paths consistent with (F, f) are $({}^1I, {}^2I)$, where ${}^2I$ is arbitrary and ${}^1I$ is given by ${}^1I^{\gamma}_{s\to t} = F^{-1}_{\gamma(t)} \circ {}^2I^{f\circ\gamma}_{s\to t} \circ F_{\gamma(s)}$. This result is an evident corollary by eq. (2.1).

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