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V.P.Gerdt, V.N.Robuk, V.M.Severyanov

ON CONSTRUCTION OF FINITELY PRESENTED LIE ALGEBRAS

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1 Introduction

In recent years one can observe the keen interest to computer algebraic aspects of combinatorial algebra [1]. Under the latter one usually understands analysis of algebraic objects given by generators and defining relations of the polynomial form. In the commutative case quite a number of universal computer algebra methods and tools have been designed to deal with multivariate polynomial systems, first of all, those based on Gröbner bases techniques [2].

Though the concept of a Gröbner basis has been generalized in different extent to non-commutative algebras [3] (see also review paper [1] and references therein) the area of its practical use is still quite restrictive. As it shown in [4], the use of non-commutative Gröbner bases method proves out in a class of algebras, called in [4] algebras of solvable type, which can be considered as intermediate between commutative algebras and general non-commutative ones [5].

Unfortunately, analysis of Lie algebras cannot generally be reduced to algebras of the solvable type, except finitely dimensional Lie algebras whose enveloping algebras are just of that type. On the other side, the problem of construction of finitely presented Lie algebras, which given by a finite set of generators and defining relations, is of great practical importance in the context of investigating the algebraic structure of non-linear partial differential equations in the framework of Wahlquist-Estabrook prolongation method [6]. Different computational aspects of this particular problem and a number of effective algorithmic procedures have been implemented in Reduce considered in [7],[8].

In present paper we describe an iterative algorithm for construction of basis elements of a finitely presented Lie algebra and computation of its commutator table. This algorithm can be considered as the further development of algorithmic ideas of paper [9]. In addition to the given set of generators and relators we introduce a grading of (non-associative) words by their weights. As a first step of the algorithm the initial data are transformed to the special form called *genetic code* of a Lie algebra to be computed. Then the algorithm provides an iterative procedure for computation of all the basis elements of a given weight in terms of those of less weights modulo the Jacobi identities and relators. As an illustration, the defining relations are produced and the prolongation algebras are constructed for the Korteveg-de Vries equation and for the system of equations describing one-dimensional Langmuir turbulence. The proposed algorithm has been implemented in the Rlisp language of computer algebra system Reduce [10].

2 Bases of Free Lie Algebras

2.1 Definition of a Free Lie Algebra

Let K be a field, $X = \{x_1, x_2, \dots, x_n\}$ be a finite alphabet and e be an empty word. The elements of X are called *free generators*).

Following to [11] put $\Gamma_0 = e$, $\Gamma_1(X) = X$ and define $\Gamma_n(X)$ $(n > 1)$ inductively as the set of all words (*monomials*) of the form

$$(u)(v), \quad u \in \Gamma_s,\ v \in \Gamma_r, \quad s + r = n\,.$$

If, say, $s = 1$, we write simply $u(v)$, or even uv if also $r = 1$.

Put

$$\Gamma(X) = \cup_{n=1}^{\infty} \Gamma_n(X)$$

and turn $\Gamma(x)$ into a (non-associative) groupoid subject to the operation

$$u \cdot v = (u)(v)\,.$$

Definition. An element $v \in \Gamma_s(X)$ is said to have *degree* s, i.e. $deg(v) = s$.

Definition. $F(X)$ is said to be a *free algebra* on X if it is a K-algebra of $\Gamma(X)$.

It means that $p \in F(x)$ is a finite sum

$$p = \sum_{u \in \Gamma} \alpha_u u, \quad \alpha_u \in K\,,$$

and the multiplication in $F(X)$ extends the multiplication in $\Gamma(X)$ as follows

$$\left(\sum_{u\in\Gamma} \alpha_u u\right)\left(\sum_{v\in\Gamma} \beta_v v\right) = \sum_{u,v\in\Gamma} \alpha_u \beta_v u \cdot v\,.$$

Remark. $F(X)$ is a graded algebra with the *homogeneous elements* of degree n being those which are linear combinations of words of length n.

This grading can be sharpened by introducing a monoid homomorphism

$$W : (T(X), \cdot, e) \longrightarrow (\mathrm{N}, +, 0)\,,$$

where $T(X)$ being the semigroup of words in the alphabet X with unity e, induced by the groupoid homomorphism $\phi :\ \Gamma(X) \longrightarrow T(X)$ with the identity map $\phi :\ X \longrightarrow X$.

Definition. If W is a grading on $\Gamma(X)$, we refer to the value $w_i = W(x_i)$ as a *weight* of x_i.

We assume that a polynomial is graded by its monomial of the maximal weight.

Let I be the two-sided ideal of $F(X)$ generated by the elements of the form (with $u \cdot v$ denoted as $[u, v]$)

$$\begin{cases} [u, u], \\ J(u, v, w) = [[u, v], w] + [[v, w], u] + [[w, u], v] \end{cases} \quad u, v, w \in \Gamma(X)\,.$$

Definition. The quotient algebra

$$L(X) = F(X)/I$$

is called the *free Lie algebra* on X[1].

2.2 Basis Family and Hall Basis

Definition [11] A linearly ordered (w.r.t. some order $\le$)) set $R = R(X) \subseteq \Gamma(X) \subseteq L(X)$ is called a *basis family* of $L(X)$ if

1. $X \subseteq R$
2. $w = [u, v] \in R$ iff
   (a) $u, v \in R$
   (b) $u < v$ % skew-symmetry
   (c) if $v = [v_1, v_2]$ then $u \ge v_1$ % Jacobi identity
3. $w = [u, v] > u$

The further specification of the above basis family is based on the choice of the monomial order providing the condition 3.

In this paper we consider one of possible orders, and the corresponding basis called a Hall basis [13]. For the compactness of writing we shall often omit the Lie brackets assuming their *right-normed arrangement*, for instance

$$x^2yx \equiv [x, [x, [y, x]]]$$
$$(xy)x^2y \equiv [[x, y], [x, [x, y]]]$$

Let $x_1 > x_2 > \dots x_n > e$ and $u_i, v_j \in \Gamma(X)$

Definition (Lexicographical order).

$$u_1u_2 \dots u_s >_{lex} v_1v_2 \dots v_r \quad iff \quad \exists i :\ (u_j = v_j,\ j \le i\ \wedge u_{i+1} > v_{i+1})$$

Definition (Graded Lexicographical Order).

$$u = u_1u_2 \dots u_s >_{glex} v = v_1v_2 \dots v_r \quad iff$$
$$deg(u) > deg(v)\ \vee\ (deg(u) = deg(v)\ \wedge\ u >_{lex} v)$$

Remark. This order provides, obviously, the condition 3 ($w = uv > u$) in the basis family definition. The corresponding specification of the basis family called *a Hall basis* [13]. Below we use a slightly more general concept of a Hall basis[2] when the words are graded rather by weight than by length.

[1] For more details on free Lie algebras see the recent monograph [12].
[2] We call it Hall basis as well.

2.3 Example

Let $L(x, y, z)$ be a free Lie algebra with three free generators

$$x < y < z$$

Then the Hall basis of $L(x, y, z)$ is

Degree	Basis Elements
1	$x < y < z <$
2	$< xy < xz < yz <$
3	$< x^2y < x^2z < yxy < yxz < y^2z < zxy < zxz < zyz <$
4	$< x^3y < x^3z < yx^2y < yx^2z < y^2xy < y^2xz < y^3z < zx^2y <$ $< zx^2z < zyxy < zyxz < zy^2z < z^2xy < z^2xz < z^2yz <$ $< (xy)xz < (xy)yz < (xz)yz <$
.....	....

The number of elements of a Hall basis of degree m for n free generators ($n = card(X)$) is given by the following expression (*Witt's formula*) [13]

$$N_m = \frac{1}{m} \sum_{d|m} \mu(d) n^{m/d}$$

where d runs through all divisors of n and $\mu(d)$ is the Moebius function, defined for $d \in \mathbb{N}$ by $\mu(1) = 1$, and for $d = p_1^{s_1} p_2^{s_2} \dots p_l^{s_l}$ with the primes p_i as

$$\mu(d) = \begin{cases} 0, & if\ \exists i \in \{1, \dots, l\} : s_i > 1 , \\ (-1)^k & \forall i \in \{1, \dots, l\} : s_i = 1 \end{cases}$$

The below table contains the numbers N_m for different n ($n, m = 1, \dots, 7$)

n \ m	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	2	1	2	3	6	9	18
3	3	3	8	18	48	116	312
4	4	6	20	60	204	670	2340
5	5	10	40	125	624	2580	11160
6	6	15	70	315	1554	7735	19544
7	7	21	112	588	3360	39990	117648

The asymptotic behavior

$$N_m \sim \frac{m^n}{n} \quad (m \longrightarrow \infty,\ n - fixed)$$

reveals a very fast increase of N_m with m. Though the presence of non-trivial defining relations (Section 2) dumps the growth of basis elements, their computation even for relatively small m, n can not be usually done in practice without the use of a computer for algebraic manipulation.

2.4 Commutators of Basis Elements and Jacobi Identities

Let $R(X)$ be a basis of a free Lie algebra $L(X)$ and $u, v \in R(x), u < v,\ w = [u, v]$.

Definition. w is said to be a *proper pair* if $w \in R(X)$. Otherwise, if $v = [v_1, v_2]$ and $u < v_1$, then w is said to be an *improper pair*.

To express an improper pair in terms of the basis elements, i.e. to determine the structure constants of $L(X)$, one needs to use the Jacobi identities.

It turns out that it is sufficient to consider only those identities which include at least one of free generators.

Theorem 1. Let $\tilde{L}$ be a free K-algebra with the condition

$$\forall u \in \tilde{L} :\ [u, u] = 0 ,$$

and let $\exists a, b \in \tilde{L}$ such that

$$\forall u, v \in \tilde{L} :\ J(a, u, v) = J(b, u, v) = 0 . \qquad (1)$$

Then $J(p(a, b), u, v) = 0$ where $p(a, b)$ is the arbitrary (non-associative) polynomial in a, b.

Proof. Because $b, [u, v] \in \tilde{L}$, from (1) it follows $J(a, b, [u, v]) = 0$. Hence, the straightforward computation with use of bilinearity and skew-symmetry gives

$$\begin{aligned} [[a, b], [u, v]] &= [a, [b, [u, v]]] - [b, [a, [u, v]]] \\ &= [a, ([[b, u], v] + [u, [b, v]])] - [b, ([[a, u], v] + [u, [a, v]])] \\ &= [[a, [b, u]], v] + [[b, u], [a, v]] + [[a, u], [b, v]] + [u, [a, [b, v]]] \\ &\quad - [[b, [a, u]], v] - [[a, u], [b, v]] - [[b, u], [a, v]] - [u, [b, [a, v]]] \\ &= [([a, [b, u]] - [b, [a, u]]), v] + [u, ([a, [b, v]] - [b, [a, v]])] \\ &= [[[a, b], u], v] + [u, [[a, b], v]] \end{aligned}$$

It means than $J([a, b], u, v) = 0$ and, by induction, the statement of the theorem is valid for any polynomial $p(a, b)$ □.

Corollary. If in a free algebra $\tilde{L}(X)$

$$\forall x_i \in X\ \wedge\ \forall u, v \in \tilde{L} :\ J(x_i, u, v) = 0 ,$$

then $\forall u, v, w \in \tilde{L} :\ J(u, v, w) = 0$, i.e. $\tilde{L}$ is a free Lie algebra.

3 Defining Relations

3.1 Formulation of the Problem

Let $L(X)$ be a free Lie algebra over K, $X = \{x_1, \ldots, x_n\}$, and let $P = \{p_1, \ldots, p_m\}$ be a finite set of (Lie) polynomials in X, i.e. $p_i = p_i(X) \in L(X)$, $i = \{1, \ldots, m\}$

Definition. If L is a Lie algebra generated by set X which obey the polynomial equations (*defining relations*)

$$p_i(X) = 0 \quad (i = \{1, \ldots m\}),$$

then L is called a *finitely generated and finitely defined* or *finitely presented.*

Below we study the following fundamental problem:

Problem. Given finite sets generators X and relators P find a Lie algebra L such that $X \subseteq L$ under the conditions $p_i(X) = 0$, $p_i \in P$.

In other words, we search for solutions of polynomial equations in the class of Lie algebras.

Such a problem arises, for example, as the most principal part of the integrability analysis of nonlinear partial differential equations by the Wahlquist-Estabrook method [6]. Different computer algebra aspects of the problem w.r.t. this concrete application have been intensively studied in [7, 8, 9]. We consider the problem in its general form, though illustrate the approach to its solution at the examples from that particular application field.

3.2 Example 1. Defining relations for the Korteveg-de Vries prolongation algebra

In the framework of the Wahlquist-Estabrook method a given *nonlinear partial differential equation*, for instance, the evolution one of the form

$$u_t = \phi(u, u_x, u_{xx}, \ldots), \quad u = u(t, x)$$

is considered as *the compatibility condition*

$$\frac{\partial \hat{F}}{\partial t} - \frac{\partial \hat{G}}{\partial x} + [\hat{G}, \hat{F}] = 0, \tag{2}$$

for a system of *linear differential equations* of the form

$$\begin{cases} \vec{y}_x = \hat{F}(u, u_x, u_{xx}, \ldots)\vec{y}, \\ \vec{y}_t = \hat{G}(u, u_x, u_{xx}, \ldots)\vec{y} \end{cases}$$

where $\vec{y}$ are called by *pseudo-potentials.*

The explicit representation for $\hat{F}$, $\hat{G}$ is sought in the form which leads to the defining relations in X_i, Y_j.

As the first example let us consider the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} - 3uu_x$$

and assume

$$\begin{cases} \hat{F} = \hat{F}(u), \\ \hat{G} = \hat{G}(u, u_x, u_{xx}). \end{cases} \tag{3}$$

Substitution of (3) into (2) gives

$$(u_{xxx} - 3uu_x)\frac{\partial \hat{F}}{\partial u} - u_x\frac{\partial \hat{G}}{\partial u} - u_{xx}\frac{\partial \hat{G}}{\partial u_x} - u_{xxx}\frac{\partial \hat{G}}{\partial u_{xx}} + [\hat{G}, \hat{F}] = 0. \tag{4}$$

Setting the coefficient at u_{xxx} in (4) equal to zero, we obtain

$$\frac{\partial \hat{F}}{\partial u} - \frac{\partial \hat{G}}{\partial u_{xx}} = 0,$$

and, therefore,

$$\hat{G} = u_{xx}\frac{\partial \hat{F}}{\partial u} + \hat{G}_1(u, u_x).$$

Then (4) is rewritten in the form

$$-3uu_x\frac{\partial \hat{F}}{\partial u} - u_x u_{xx}\frac{\partial^2 \hat{F}}{\partial u^2} - u_x\frac{\partial \hat{G}_1}{\partial u} - u_{xx}\frac{\partial \hat{G}_1}{\partial u_x} + u_{xx}[\frac{\partial \hat{F}}{\partial u}, \hat{F}] + [\hat{G}_1, \hat{F}] = 0. \tag{5}$$

Selection of the coefficient at u_{xx} in (5) yields

$$-u_x\frac{\partial^2 \hat{F}}{\partial u^2} - \frac{\partial \hat{G}_1}{\partial u_x} + [\frac{\partial \hat{F}}{\partial u}, \hat{F}] = 0.$$

Hence,

$$-3uu_x\frac{\partial \hat{F}}{\partial u} - u_x\frac{\partial \hat{G}_1}{\partial u} + [\hat{G}_1, \hat{F}] = 0, \tag{6}$$

and

$$\hat{G}_1 = -\frac{1}{2}u_x^2\frac{\partial^2 \hat{F}}{\partial u^2} + u_x[\frac{\partial \hat{F}}{\partial u}, \hat{F}] + \hat{G}_2(u).$$

Substitution of this expression into (6) gives

$$-3uu_x\frac{\partial \hat{F}}{\partial u} + \frac{1}{2}u_x^3\frac{\partial^3 \hat{F}}{\partial u^3} - \frac{3}{2}u_x^2[\frac{\partial^2 \hat{F}}{\partial u^2}, \hat{F}] - u_x\frac{\partial \hat{G}_2}{\partial u} + u_x[[\frac{\partial \hat{F}}{\partial u}, \hat{F}], \hat{F}] + [\hat{G}_2, \hat{F}] = 0. \tag{7}$$

Collecting the coefficients at u_x^3, we come to the equality

$$\frac{\partial^3 \hat{F}}{\partial u^3} = 0,$$

and, hence,

$$\hat{F} = \hat{X}_1 + u\hat{X}_2 + u^2\hat{X}_3. \tag{8}$$

After that, equating in (7) the coefficient at u_x^2 to zero, we obtain

$$[\frac{\partial^2 \hat{F}}{\partial u^2}, \hat{F}] = 0,$$

or $[X_3, X_1] + u[X_3, X_2] = 0$. The latter equality is equivalent to

$$[X_1, X_3] = [X_2, X_3] = 0. \tag{9}$$

Then, the next coefficient, i.e. one at u_x, in (7) leads to the equation

$$-3u(\hat{X}_2 + 2u\hat{X}_3) - \frac{\partial \hat{G}_2}{\partial u} + [[\hat{X}_2, \hat{X}_1], \hat{X}_1] + u[[\hat{X}_2, \hat{X}_1], \hat{X}_2] = 0,$$

which yields the following expression for $\hat{G}_2$

$$\hat{G}_2 = -\frac{3}{2}u^2\hat{X}_2 - 2u^3\hat{X}_3 + u[[\hat{X}_2, \hat{X}_1], \hat{X}_1] + \frac{1}{2}u^2[[\hat{X}_2, \hat{X}_1], \hat{X}_2] + \hat{X}_4. \tag{10}$$

Further, setting $u_x = 0$ in (7), we find

$$[\hat{G}_2, \hat{F}] = 0.$$

Taking (8-9) into account, we find

$$\frac{3}{2}u^2[\hat{X}_2, \hat{X}_1] - u[[[\hat{X}_2, \hat{X}_1], \hat{X}_1], \hat{X}_1] - u^2[[[\hat{X}_2, \hat{X}_1], \hat{X}_1], \hat{X}_2] -$$
$$\frac{1}{2}u^2[[[\hat{X}_2, \hat{X}_1], \hat{X}_2], \hat{X}_1] - \frac{1}{2}u^3[[[\hat{X}_2, \hat{X}_1], \hat{X}_2], \hat{X}_2] - [\hat{X}_4, \hat{X}_1] -$$
$$u[\hat{X}_4, \hat{X}_2] - u^2[\hat{X}_4, \hat{X}_3] = 0.$$

Collecting the coefficients at $u^k, 0 \le k \le 4$ and using the Jacobi identities and skew-symmetry, we obtain the defining relations

$$\begin{aligned} &[[[\hat{X}_1, \hat{X}_2], \hat{X}_2], \hat{X}_2] = 0, \\ &-\frac{3}{2}[\hat{X}_1, \hat{X}_2] + \frac{3}{2}[[[\hat{X}_1, \hat{X}_2], \hat{X}_1], \hat{X}_2] + [\hat{X}_3, \hat{X}_4] = 0, \\ &[[[\hat{X}_1, \hat{X}_2], \hat{X}_1], \hat{X}_1] + [\hat{X}_2, \hat{X}_4] = 0, \\ &[\hat{X}_1, \hat{X}_4] = [\hat{X}_1, \hat{X}_3] = [\hat{X}_2, \hat{X}_3] = 0 \end{aligned} \tag{11}$$

with $\hat{F}$ and $\hat{G}$ represented as

$$\begin{aligned} \hat{F} = {} & \hat{X}_1 + u\hat{X}_2 + u^2\hat{X}_3 \\ \hat{G} = {} & \hat{X}_4 + u[[\hat{X}_2, \hat{X}_1], \hat{X}_1] + \frac{1}{2}u^2[[\hat{X}_2, \hat{X}_1], \hat{X}_2] - \frac{3}{2}u^2\hat{X}_2 \\ & -2u^3\hat{X}_3 - u_x^2\hat{X}_3 + u_x[\hat{X}_2, \hat{X}_1] + u_{xx}(\hat{X}_2 + 2u\hat{X}_3). \end{aligned}$$

Before constructing the Lie algebra solutions, it makes sense to simplify the defining relations (11) as follows.

Theorem 2. Let L be a Lie algebra. If $z, u, v \in L$ and $[z, u] = [z, v] = 0$, then $[z, P(u, v)] = 0$ where P is any Lie polynomial in u, v.

Proof. Under the conditions of the theorem $J(z, u, v) = 0$ implies, obviously, $[z, [u, v]] = 0$. Hence, by induction $[z, P(u, v)] = 0$ □

Corollary. If an element $z \in L$ of a Lie algebra L commutes with all the generators, then z belongs to the center of L ($z \in Z(L)$).

Using an computer, one can show that the polynomial

$$[[[\hat{X}_1, \hat{X}_2], \hat{X}_1], \hat{X}_2] + [\hat{X}_1, \hat{X}_2]$$

commutes with $\{\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4\}$. By this reason we can put

$$[[[\hat{X}_1, \hat{X}_2], \hat{X}_1], \hat{X}_2] + [\hat{X}_1, \hat{X}_2] = 0.$$

Then

$$[\hat{X}_3, \hat{X}_1] = [\hat{X}_3, \hat{X}_2] = [\hat{X}_3, \hat{X}_4] = 0.$$

It justifies the setting $\hat{X}_3 = 0$ as generally accepted in the KdV analysis. Therefore, we come to defining relations of the form

$$\begin{aligned} &[[[\hat{X}_1, \hat{X}_2], \hat{X}_2], \hat{X}_2] = 0, \\ &[[[\hat{X}_1, \hat{X}_2], \hat{X}_1], \hat{X}_2] - [\hat{X}_1, \hat{X}_2] = 0, \\ &[[[\hat{X}_1, \hat{X}_2], \hat{X}_1], \hat{X}_1] + [\hat{X}_2, \hat{X}_4] = 0, \end{aligned} \tag{12}$$
$$[\hat{X}_1, \hat{X}_4] = 0. \tag{13}$$

3.3 Genetic Code

To use an algorithm of the next subsection, the initial data, i.e. the set of generators and relators, must be

- graded by the weight has been chosen in advance, as described in Sect.2.1-2.2, that induces the corresponding graded lexicographical ordering;
- supplied with all the basis elements and the commutator relations as they are resulted by verification of the Jacobi identities for all the triples of summary weight not exceeding the maximal one among the generators and relators.

Definition. In such a way the graded, arranged and extended set of generators and relators is said to be a *genetic code (GC)* of a Lie algebra to be constructed.

Example 1. The genetic code for the prolongation algebra of KdV defined by the relations (12) can be represented in the table form

Weight	No.of basis element	Genetic Code
1	1	X_1
	2	X_2
2	3	$[X_1, X_2]$
3	4	$[X_1, [X_1, X_2]]$
	5	$[X_2, [X_1, X_2]]$
	6	X_4
4	7	$[X_1, [X_1, [X_1, X_2]]]$
		$[X_2, [X_1, [X_1, X_2]]] = [X_1, X_2]$
		$[X_1, [X_2, [X_1, X_2]]] = [X_1, X_2]$
		$[X_2, [X_2, [X_1, X_2]]] = 0$
		$[X_1, X_4] = 0$
		$[X_2, X_4] = -[X_1, [X_1, [X_1, X_2]]]$

In the last column, in addition to the initial set of the graded generators and relators there are also their algebraic consequences modulo Jacobi identities of weight 4.

Given GC, it is possible to proceed the further supplement with the algebraic consequences of higher weights. Below an algorithmic prescription for doing that is given.

4 Algorithm Description

4.1 Basic Structures

Let R be a basis set of monomials for a Lie algebra L under construction. The grading gives

$$R = \cup_l R^l = \cup_l (X^l \cup S^l)\ ,$$

where X and S are sets of generators and the basis elements being the proper pairs, respectively.

Using the order has been chosen we introduce an auxiliary linearly ordered set $\tilde{R}$, such that

$$f:\ R \longrightarrow \tilde{R}$$

is a bijective map. Hence,

$$\tilde{R} = \cup_l \tilde{R}^l = \cup_l (\tilde{X}^l \cup \tilde{S}^l)\ ,$$

and

$$f^{-1}(\tilde{r}) = \begin{cases} x \in X, & if \quad \tilde{r} \in \tilde{X}, \\ (r_i, r_j) \in S, & if \quad \tilde{r} \in \tilde{S}, \end{cases}$$

Set $\tilde{R}$ can be considered as a numbering of R.

Definition. A proper pair is said to be *a bound one* if it is expressed as a linear combination of basis elements by virtue of the Jacobi identities and the defining relations and *a free one*, otherwise.

One should note that a Jacobi identity verification for a higher weight triple may produce an extra relation for the lower weight Lie monomials which have been earlier considered as the basis elements. We call such a relation *a reciprocal phrase*, because it must be taken into account in the further computational steps which are forced by that phrase to start again with lower weight level to reconstruct all the next levels. In the below algorithm such reciprocal phrases are accumulated in the special set P.

Let $B = \cup_l B^l$ is a set of the bound pairs. It means that

$$b \in B \quad iff \quad b = \sum_i \alpha_i r_i, \quad \alpha_i \in K,\ r_i \in R$$

Denote by h the map $h:\ B \longrightarrow Span(R, K)$ and introduce an auxiliary set H accumulating all the intermediate proper Lie pairs. In the further analysis those ones which are bound are moved from H to B. Then an algorithm for computation R^n in terms of $R^k\,(k < n)$ can be written as follows.

4.2 Main Algorithm

Input: $\cup_{k<n} R^k$, $\cup_{k<n} \tilde{R}^k$, $\cup_{k<n} B^k$, $\cup_{k<n} h(B^k)$;
Output: R^n, S^n, B^n, $h(B^n)$, P^n;

$H^n = \emptyset$, $S^n = \emptyset$, $B^n = \emptyset$, $h(B^n) = \emptyset$, $P^n = \emptyset$;
for each $x_i \in X$ **such that** $w(x_i) < n$ **do**
  $l := n - w(x_i)$;
  for each $x_q \in X^l$ **do**   % words of length two
    if $\tilde{x}_i < \tilde{x}_q$ **then** $H^n := \{(x_i, x_q)\} \cup H^n$
    else if $\tilde{x}_i > \tilde{x}_q$ **then** $H^n := \{(x_q, x_i)\} \cup H^n$;
  end;
  for each $r_q = (r_{1q}, r_{2q}) \in S^l$ **do**   % triples with two the same generators
    if $\tilde{x}_i = \tilde{r}_{1q}$ **or** $\tilde{x}_i = \tilde{r}_{2q}$ **then**
      if $\tilde{x}_i < \tilde{r}_q$ **then** $H^n := \{(x_i, r_q)\} \cup H^n$
      else $H^n := \{(r_q, x_i)\} \cup H^n$
    else
    if $\tilde{x}_i < \tilde{r}_{1q}$ **or** $(\tilde{r}_{1q} < \tilde{x}_i < \tilde{r}_{2q}$ **and** $r_{1q} \in S)$
      or $(\tilde{x}_i > \tilde{r}_{2q}$ **and** $r_{1q}, r_{2q} \in S)$
      then Jacobi(x_1, r_{1q}, r_{2q});   % J computes JH, JB, JhB, JP
      $H^n := H^n \cup JH$; $B^n := B^n \cup JB$; $h(B^n) := h(B^n) \cup JhB$;
      replace JB **in** $h(B^n)$ **by** JhB;
      $P^n := P^n \cup JP)$;
  end;
  for each $r_q \in B^l$ **do**

if $\tilde{x}_i < \tilde{r}_{1q}$ or ($\tilde{r}_{1q} < \tilde{x}_i < \tilde{r}_{2q}$ and $r_{1q} \in S$)
or ($\tilde{x}_i > \tilde{r}_{2q}$ and $r_{1q}, r_{2q} \in S$)
then Jacobi(x_1, r_{1q}, r_{2q});
$H^n := H^n \cup JH$; $B^n := B^n \cup JB$; $h(B^n) := h(B^n) \cup JhB$;
replace JB in $h(B^n)$ by JhB;
$P^n := P^n \cup JP$);
end;
end;
$S^n := H^n \setminus B^n$; $\tilde{S}^n = f(S^n)$;
for each $r_q \in S^n$ **replace** r_q **in** $h(B^n)$ **by** $\tilde{r}_q$.

On should note that Theorem 1 is essentially used in the body of the algorithm. Indeed, in the main loop only generators are selected together with the Lie pairs have been obtained in order to construct new triples for the Jacobi identity verification. Three internal loops of the main one create the pairs of generators, the triples with two coinciding generators, the triples of the generator and free proper pairs, and the triples of the generator and bound proper pair, respectively. Unlike the two first internal loops where no necessity to verify the Jacobi identities, the last two ones contain subalgorithm **Jacobi**, presented below, which does such a verification. In so doing, generally, new elements for the sets H, B, $h(B)$ and P are produced by subalgorithm **Jacobi** and collected in sets JH, JB, JhB and JP.

4.3 Subalgorithm Jacobi

Input : $u, v, w \in R$,
Output : JH, JB, JhB, JP
% These are sets of all the proper pairs, the bound pairs
% and the reciprocal phrases, respectively, arising at
% the Jacobi identity verification for u, v, w
$JH := \emptyset$;
Arrange(u, v, w); % the order is to be $\tilde{u} < \tilde{v} < \tilde{w}$
$p_1 :=$ **Simplify**$((u, v), w)$;
$p_2 :=$ **Simplify**$((v, w), u)$;
$p_3 :=$ **Simplify**$((u, w), v)$;
$p := p_1 + p_2 - p_3$;
$JH := JH \cup$**ProperPairs**$(p_1)\cup$**ProperPairs**$(p_2)\cup$ **ProperPairs**(p_3);
% collection of the proper pairs
$lip :=$ **LeadingImproperPair**(p);
% selection of the leading improper pair of p
if $lip \neq 0$ **then** $JB := \{lip\}$; $JhB :=$ **Solve**(p, lip);
% solving equation $p = 0$ w.r.t. lip
else ($lpp :=$**LeadingProperPair**(p);
% selection of the leading proper pair of p
if $lpp \neq 0$ **then** $JB := \{lpp\}$; $JhB :=$ **Solve**(p, lpp);
% solving equation $p = 0$ w.r.t. lpp
else $JP := \{p\}$.

This subalgorithm includes the following modules

Arrange (u, v, w) does arrangement of the triple (u, v, w) in ascending order of their numbers $\tilde{u} < \tilde{v} < \tilde{w}$.

Simplify $((u, v), w)$ is a recursive procedure which

- Applies the bijective map f to the internal pair (u, v) if $(u, v) \in S$ and the map h otherwise, i.e. when $(u, v) \in B$.
- In the latter case one obtains a linear combination of the pairs which are transformed, if necessary, to those with the less first element than the second one.
- The first step being applied to each pair has obtained at the second step, and the process continues recursively until the complete simplification is achieved.

ProperPairs (p) collects all the proper pairs of Lie polynomial p.

LeadingImproperPair (p) selects the leading improper pair of (p).

LeadingProperPair (p) selects the leading proper pair of (p).

Solve (p, k) resolves the equation $p = 0$ to express term k of polynomial p through other its terms.

The described algorithm is an improved version of that of paper [9]. The main improvement is based on the use of Theorem 1. It allows to decrease sharply the number of evaluated Jacobi identities at large number N of basis elements. When $(N \to \infty)$ at fixed number of generators n it means the following reduction of the number of identities under verification

$$\binom{N}{3} \Longrightarrow n \cdot \binom{N}{2}.$$

If one applies the above algorithm to compute the structure of next weight level for the KdV prolongation algebra then one obtains as follows.

4.4 Example 1 (continuation)

Basis elements and their commutators of weight 5

No.	Basis Elements and Algebraic Consequences
8	$[X_1,[X_1,[X_1,[X_1,X_2]]]]$
9	$[X_2,[X_1,[X_1,[X_1,X_2]]]]$
	$[[X_1,X_2],[X_1,[X_1,X_2]]] = -[X_2,[X_1,[X_1,[X_1,X_2]]]] + [X_1,[X_1,X_2]]$
	$[[X_1,X_2],[X_2,[X_1,X_2]]] = -[X_2,[X_1,X_2]]$
	$[[X_1,X_2],X_4] = -[X_1,[X_1,[X_1,[X_1,X_2]]]]$

The further computations up to the elements of weight 20 allow one to guess the following recurrence formulae determining an infinitely dimensional Lie algebra [14] and then to verify that it is indeed a solution of the problem (12) checking up a finite number of the Jacobi identities.

Basis of Lie algebra: $X_1,\ ad\,^kX_1(X_2) = Y_k,\ [Y_0,Y_{2k+1}] = Z_k,\ X_3,\ \{k\}_0^\infty$.

Lie algebra:

$$\begin{aligned}
&[X_1,Y_k] = Y_{k+1}\ ,\ [X_1,X_4] = 0\ ,\ [X_1,Z_k] = Y_{2k+1}\ ,\ [Y_0,Z_q] = 0\ ,\\
&[Y_0,Y_{2k+1}] = Z_k\ ,\ [Y_0,Y_{2n}] = Y_{2n-1}\ ,\ [Y_n,Y_p] = 0\quad (n+p=2m)\ ,\\
&[Y_n,Y_p] = (-1)^nZ_m + (-1)^pY_{2m}\quad (n+p=2m+1)\ ,\ [Y_{2n+1},Z_q] = -Z_{q+n}\ ,\\
&[Y_k,X_4] = -Y_{k+3}\ ,\ [Y_{2p},Z_q] = -Y_{2q+2p-1}\ ,\ [Z_q,Z_k] = 0\ ,\ [X_4,Z_k] = Y_{2k+3}\ ,\\
&\{k\}_0^\infty\ ,\ \{m\}_1^\infty\ ,\ \{n\}_1^\infty\ ,\ \{p\}_1^\infty\ ,\ \{q\}_0^\infty\ .
\end{aligned}$$

There are 6 different types of basis elements

$$X_1, X_4, Y_0, Y_{2n}, Y_{2k+1}, Z_q.$$

By this reason, it is sufficient to verify 38 Jacobi identities for all these types of (indexed) basis elements.

Note, that here we obtain a number of the *reciprocal phrases*, which arise in each row of even weight, starting with the 10th row. Moreover, each time one must return back just in 4 rows. For instance, the first reciprocal phrase, arising in the row of weight 10, has the form

$$[X_2,[X_1,[X_1,[X_1,[X_1,X_2]]]]] = [X_1,[X_1,[X_1,X_2]]]\ .$$

Having the leading word (monomial) of this phrase of weight 6, we are obliged to come back to the row of weight 6 and start the computations once more, taking the above expression into account .

If we are lucky in introducing additional defining relations which cut the further increase of basis elements, then we can find a finite-dimensional Lie algebra as a solution of our

problem. The simplest is to assume that some basis element is a linear combination of the previous ones. For instance, if to introduce the relation

$$Y_7 = \sum_{i=0}^{6} \alpha_i Y_i\ ,$$

then the Jacobi identities verification shows that

$$\alpha_0 = \alpha_2 = \alpha_4 = \alpha_6 = 0,\quad \alpha_1,\alpha_3,\alpha_5 \in \mathbf{C}.$$

It leads to the following 12-dimensional Lie algebra:

$$\begin{aligned}
&[X_1,Y_0] = Y_1,\ [X_1,Y_1] = Y_2,\ [X_1,Y_2] = Y_3,\ [X_1,Y_3] = Y_4,\ [X_1,Y_4] = Y_5,\\
&[X_1,Y_5] = Y_6,\ [X_1,Y_6] = aY_5 + bY_3 + cY_1,\ [X_1,Z_0] = Y_1,\ [X_1,Z_1] = Y_3,\\
&[X_1,Z_2] = Y_5,\ [X_1,X_4] = 0,
\end{aligned}$$

$$\begin{aligned}
&[Y_0,Y_1] = Z_0,\ [Y_0,Y_2] = Y_1,\ [Y_0,Y_3] = Z_1,\ [Y_0,Y_4] = Y_3,\ [Y_0,Y_5] = Z_2,\\
&[Y_0,Y_6] = Y_6,\ [Y_0,Z_0] = 0,\ [Y_0,Z_1] = 0,\ [Y_0,Z_2] = 0,\ [Y_0,X_4] = -Y_3,
\end{aligned}$$

$$\begin{aligned}
&[Y_1,Y_2] = -Z_1 + Y_2,\ [Y_1,Y_2] = -Z_1 + Y_2,\ [Y_1,Y_3] = 0,\ [Y_1,Y_4] = -Z_2 + Y_4,\\
&[Y_1,Y_5] = 0,\ [Y_1,Y_6] = -aZ_2 + Y_6 - bZ_1 - cZ_0,\ [Y_1,Z_0] = -Z_0,\ [Y_1,Z_1] = -Z_1,\\
&[Y_1,Z_2] = -Z_2,\ [Y_1,X_4] = -Y_4,
\end{aligned}$$

$$\begin{aligned}
&[Y_2,Y_3] = Z_2 - Y_4,\ [Y_2,Y_4] = 0,\ [Y_2,Y_5] = aZ_2 - Y_6 - bZ_1 + cZ_0,\ [Y_2,Y_6] = 0,\\
&[Y_2,Z_0] = -Y_1,\ [Y_2,Z_1] = -Y_3,\ [Y_2,Z_2] = -Y_5,\ [Y_2,X_4] = -Y_5,
\end{aligned}$$

$$\begin{aligned}
&[Y_3,Y_4] = -aZ_2 + Y_6 - bZ_1 - cZ_0,\ [Y_3,Y_5] = 0,\ [Y_3,Y_6] = -(a^2+b)Z_2 + aY_6\\
&\qquad -(ab+c)Z_1 - acZ_0 + cY_2,\ [Y_3,Z_0] = -Z_1,\ [Y_3,Z_1] = -Z_2,\\
&[Y_3,Z_2] = -aZ_2 - bZ_1 - cZ_0,\ [Y_3,X_4] = -Y_6,
\end{aligned}$$

$$\begin{aligned}
&[Y_4,Y_5] = (a^2+b)Z_2 - aY_6 + (ab+c)Z_1 - bY_4 + acZ_0 - cY_2,\ [Y_4,Y_6] = 0,\\
&[Y_4,Z_0] = -Y_3,\ [Y_4,Z_1] = -Y_5,\ [Y_4,Z_2] = -aY_5 - bY_3 - cY_1,\\
&[Y_4,X_4] = -aY_5 - bY_3 - cY_1,
\end{aligned}$$

$$\begin{aligned}
&[Y_5,Y_6] = -(a^3+2ab+c)Z_2 + (a^2+b)Y_6 - (a^2b+ac+b^2)Z_1 + (ab+c)Y_4\\
&\qquad -c(a^2+c)Z_0 + acY_2,\ [Y_5,Z_0] = -Z_2,\ [Y_5,Z_1] = -aZ_2 - bZ_1 - cZ_0,\\
&[Y_5,Z_2] = -(a^2+b)Z_2 - (ab+c)Z_1 - acZ_0,\ [Y_5,X_4] = -aY_6 - bY_4 - cY_2,
\end{aligned}$$

$$\begin{aligned}
&[Y_6,Z_0] = -Y_5,\ [Y_6,Z_1] = -aY_5 - bY_3 - cY_1,\ [Y_6,Z_2] = -(a^2+b)Y_5\\
&\qquad -(ab+c)Y_3 - acY_1,\ [Y_6,X_4] = -(a^2+b)Y_5 - (ab+c)Y_3 - acY_1,
\end{aligned}$$

$$[Z_0,Z_1] = 0,\ [Z_0,Z_2] = 0,\ [Z_0,X_4] = Y_3,$$

$$[Z_1,Z_2] = 0,\ [Z_1,X_4] = Y_5,$$

$$[Z_2,X_4] = -aY_5 - bY_3 - cY_1,$$

which is a particular solution of (12).

5 Example 2. Prolongation Algebra for Partial Differential Equations Describing One-Dimensional Langmuir Turbulence

5.1 Defining Relations

Investigation of Langmuir turbulence in plasma excited by strong electromagnetic fields is a very difficult physical problem. It is a topical one in many different applications connected with the necessity of the plasma heating by high energetic radiation and, first of all, for the laser thermonuclear synthesis.

The process is described by the following system of nonlinear partial differential equations, being the simplest one for the problem, namely, without the source and dissipation [15].

$$i\frac{\partial E}{\partial t}+\frac{1}{2}\frac{\partial^2 E}{\partial x^2}=nE, \quad \frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial x^2}=2\frac{\partial^2 |E|^2}{\partial x^2}.$$

Transformation to the polar coordinates $E=\rho e^{i\phi}$ leads

$$\begin{aligned}
\rho_t &= -\frac{1}{2}\rho\varphi_{xx}-\rho_x\varphi_x,\\
\phi_t &= \frac{1}{2\rho}\rho_{xx}-\frac{1}{2}\varphi_x^2-n,\\
n_{tt}-n_{xx} &= 4\rho\rho_{xx}+4\rho_x^2.
\end{aligned}$$

Here functions $E=E(x,t)$ and $n(x,t)$ describe the electric field and the plasma density, respectively.

We use the Whalquist-Estabrook method under the hypothesis

$$\left\{\begin{array}{lcl}
\hat F &=& \hat x_1 n_t+\tilde F(n,\phi,\rho),\\
\hat G &=& \hat x_1 n_x+\tilde G(n,\phi,\rho,\phi_x,\rho_x).
\end{array}\right.$$

Then the compatibility condition (2) yields

$$\begin{aligned}
\hat F &= \hat x_1 n_t+(\hat x_2\sin\varphi+\hat x_3\cos\varphi)\rho+\hat x_4\rho^2+[\hat x_1,\hat x_6]n+x_5,\\
\hat G &= \hat x_1 n_x+\frac{1}{2}(\hat x_2\cos\varphi-\hat x_3\sin\varphi)\rho_x+4\hat x_1\rho\rho_x-\frac{1}{2}(\hat x_2\sin\varphi+\hat x_3\cos\varphi)\rho\varphi_x\\
&\quad -\hat x_4\rho^2\varphi_x+\frac{1}{2}([\hat x_2,\hat x_5]\cos\varphi-[\hat x_3,\hat x_5]\sin\varphi)\rho+(\frac{1}{4}[\hat x_2,\hat x_3]+2[\hat x_1,\hat x_5])\rho^2\\
&\quad +[\hat x_1,\hat x_5]n+\hat x_6.
\end{aligned}$$

It leads to the following defining relations having the form

$$\begin{aligned}
&[\hat x_1,\hat x_2]=[\hat x_1,\hat x_3]=[\hat x_1,\hat x_4]=[\hat x_2,\hat x_4]=[\hat x_3,\hat x_4]=[\hat x_5,\hat x_4]=[\hat x_5,\hat x_6]=0,\\
&[\hat x_1,[\hat x_1,\hat x_6]]=[\hat x_1,[\hat x_1,\hat x_5]]=[\hat x_1,[\hat x_2,\hat x_5]]=[\hat x_1,[\hat x_3,\hat x_5]]=0,\\
&[\hat x_4,[\hat x_1,\hat x_6]]=[\hat x_2,[\hat x_1,\hat x_6]]=[\hat x_3,[\hat x_1,\hat x_6]]=0,\\
&[\hat x_2,[\hat x_2,\hat x_3]]=[\hat x_3,[\hat x_3,\hat x_2]]=0,\\
&[\hat x_5,[\hat x_5,\hat x_1]]-[\hat x_6,[\hat x_6,\hat x_1]]=0,\\
&[\hat x_2,[\hat x_3,\hat x_5]]+[\hat x_3,[\hat x_2,\hat x_5]]=0,\\
&[\hat x_2,[\hat x_2,\hat x_5]]-[\hat x_3,[\hat x_3,\hat x_5]]=0,\\
&2[\hat x_4,\hat x_6]+[\hat x_5,[\hat x_2,\hat x_3]]+4[\hat x_5,[\hat x_1,\hat x_5]]=0,\\
&2[\hat x_3,\hat x_6]+[\hat x_5,[\hat x_2,\hat x_5]]=0,\\
&2[\hat x_2,\hat x_6]+[\hat x_5,[\hat x_5,\hat x_3]]=0,\\
&[[\hat x_1,\hat x_6],[\hat x_1,\hat x_5]]=0,\\
&[[\hat x_1,\hat x_6],[\hat x_2,\hat x_5]]+2\hat x_2=0,\\
&[[\hat x_1,\hat x_6],[\hat x_3,\hat x_5]]+2\hat x_3=0
\end{aligned}$$

for six generators $\{\hat x_i\}(1\le i\le 6)$.

5.2 Prolongation Algebra

To investigate the above defining relations we applied our implementation of the algorithm of Sect.3.4 in Reduce and discovered that

$$\hat x_2=\hat x_3=0$$

and the corresponding prolongation Lie algebra is a finitely dimensional one of dimension nine, with the following structure of the basis elements

$$\begin{aligned}
&e_1=\hat x_1,\quad e_2=\hat x_4,\quad e_3=\hat x_5,\quad e_4=\hat x_6,\\
&e_5=[\hat x_1,\hat x_5],\quad e_6=[\hat x_1,\hat x_6],\quad e_7=[\hat x_4,\hat x_6],\\
&e_8=[\hat x_5,[\hat x_1,\hat x_6]],\quad e_9=[\hat x_6,[\hat x_4,\hat x_6]]
\end{aligned}$$

and their table of non-zero commutators

$$\begin{aligned}
&[e_1,e_3]=e_5,\quad [e_1,e_4]=e_6,\quad [e_2,e_4]=e_7,\\
&[e_3,e_5]=-1/2\,e_7,\quad [e_3,e_6]=e_8,\quad [e_3,e_8]=-1/2\,e_9,\\
&[e_4,e_5]=e_8,\quad [e_4,e_6]=-1/2\,e_7,\quad [e_4,e_7]=e_9.
\end{aligned}$$

The computation with our implementation in Reduce took about two hours on an 25 Mhz MS-DOS based AT/386 computer.

6 Conclusion

One should note that another method of algebraic manipulation over the finitely presented Lie algebras, described in [8] and also implemented in Reduce, provides much better timing for the example of Sect.5. One of the reasons of a relative slowness of the algorithm described above is, typically, a very rapid growth of a number of the reciprocal phrases with increase of the monomial length. By this reason the computational process is forced to restart repeatedly.

Now a new version of the algorithm is under development [16] which allows to avoid much of that trouble and to construct the basis elements as well as their commutators using the Lie differentiations of the defining relations together with their mutual reductions. It looks like quite similar to an algorithmic scheme of the involutive approach, proposed in [17] for the commutative algebra. An implementation of the new version is to be done in C, and the run of a preliminary C code have already shown its much higher efficiency.

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