объединенный
институт
ядерных
исследований
дубна

E5-94-281

K.V.Rerikh*

# NON-ALGEBRAIC INTEGRABILITY OF THE CHEW-LOW REVERSIBLE DYNAMICAL SYSTEM OF THE CREMONA TYPE AND THE RELATION WITH THE 7th HILBERT PROBLEM (NON-RESONANT CASE)

Submitted to the Bogoliubov International Symposium, August 18—21, Dubna, 1994 and accepted for «Physica D»

*e-mail: rerikh@theor.jinrc.dubna.su

1994

" Dedicated to the blessed memory
of unforgettable dear
Nikolai Nikolaevich Bogoliubov,
with deep gratitude
for constant support and kindness."

# 1 Introduction

## 1.1 General survey

Impressive development of the theory of dynamical systems in the XX-th century (see, e.g. monographs and surveys [1]–[17]), well known as a powerful and wide in its applications mathematical instrument, has provided prerequisites for integration and discoveries of relations of this part of mathematics with other ones such as differential forms theory [18], algebraic geometry [19], [20], number theory [21], [22], automorphic forms theory [23]–[25], theory of functional equations in single and several variables [10]–[12], theory of geometric invariants [26], mathematical methods of classical mechanics [3], [27] and others.

This paper is concerned with an interesting example of relations between reversible dynamical systems of the Cremona type [28] and the transcendental number theory, more precisely, the 7th Hilbert problem [22]. These relations are very important for solving the difficult nonlinear problem of the Chew–Low equations of $p$-wave $\pi N$-scattering in the static model [32] within the dispersion approach in the strong interaction quantum theory [33].

We hope to discuss in subsequent papers a very interesting problem of the dynamics of rational quadratic Cremona mappings in the context of the general problem of integrability, which is very topical [34] in view of a new concept — the complexity of a dynamical system, introduced by V.I. Arnold [35], [36].

Reversible dynamical systems (RDS) are qualitatively very similar to Hamiltonian systems [37]–[52]. In particular, the existence of Kolmogorov–Arnold–Moser tori in reversible non-Hamiltonian flows [37]–[42], [45]–[48] and non-symplectic mappings [40]–[43], [49]–[51] has been proved. However, the reversible KAM theory possesses some features which have no analogues for Hamiltonian systems [51]. Reversible dynamical systems are of great importance in physics, which is a consequence of the time-reversible invariance of many physical laws (see surveys [47], [52]).

## 1.2 History of the problem

RDS are exemplified by the Chew–Low equations [53] for $p$-waves of $\pi N$-scattering, which are the simplest example of a non-trivial model [32], where fundamental requirements of the dispersion approach, such as analyticity, unitary property and crossing-symmetry [33] are taken into account simultaneously.

As is known [54] (see also [55]), conversion to a nonlinear boundary value problem for matrix elements of the $S$-matrix and introduction of uniformizing variable $w = \pi^{-1} \arcsin \omega$, where $\omega$ is the energy of the $\pi$-meson in the laboratory system, allow one to rewrite these equations as a system of nonlinear functional (difference) equations. The latter define a reversible mapping, that is the composition of two involutions: the standard Cremona transformation [28] and the crossing-symmetry mapping. Dynamical systems (DS) of this type, distinguished only by the crossing-symmetry matrix $A$ and defined by the quadratic Cremona mappings, have been investigated in papers [54] –[72]. Interest in integrating these DS is related, besides the physical one, with a problem of integrating general Cremona mappings which, according to the M. Noether theorem, can be constructed from different quadratic Cremona mappings [28]. Moreover, such DS do not belong to the Quispel 18-parametric family of integrable mappings [52], [73], and were not considered in the survey on integrable maps [74]. The problem of integrating these two-dimensional DS with the second-order matrix $A$ was solved long ago in [56], [57], but only recently the 3-dimensional DS with the third-order matrix $A(1,1)$ has been integrated in [58]. The latter is interesting due to the fact that it has three invariant algebraic curves, which provides the Cremona mapping found in [58] with an equivalence of this DS to an area-preserving mapping with a fixed Poincaré-resonant hyperbolic point. The use of the famous Birkhoff substitution [75] and the Moser theorem [76] for such mappings provides the non-algebraic integrability of the system [58]. A general approach to integrating such $n$-dimensional DS was developed in [59]. In [54], [60], [61], [65], [72], particular solutions for the third-order matrices $A(1,1)$ and $A^{Chew-Low}$ were obtained.

This paper is devoted to the problem of integrating a dynamical system with the matrix $A^{Chew-Low}$. The problem has a thirty-year history. In [55], it has been shown that the necessary physical solution can be obtained only from the general solution of the problem. In [64], it was proved that the system cannot be transformed into a semiseparable, easily integrable, form by any Cremona transformation of a finite order. In [68], [69], a differential equation corresponding to the system has been obtained from the consideration of the Abelian group of iterations of the initial difference DS. Enough convincing arguments were formulated for holomorphy of the functions defining the differential equation, the proof of which turns out to be very difficult. On this basis, in [70] the general solution of this DS was obtained in a neighborhood of the parabolic fixed point. It was supposed, in accordance with [68], [69], to be holomorphic at the point. In [67], on the basis of [71] where a functional equation for the functions defining the differential equation from [69] was obtained, it has been proved that these functions are not holomorphic and, consequently, the general solution from [70] is irregular at the parabolic fixed point and, therefore, is formal.

The next important step in this direction was made in [62], [63], where the first integral of the system was derived in a neighborhood of the parabolic fixed point. However, it follows from [66] that the parabolic fixed point is an essential singular point for the first integral. In this paper, the problem is solved in a neighborhood of the hyperbolic fixed point of the system on the basis of Siegel's theorem ([77], see §28) on holomorphic linearization of a mapping at a non-resonant fixed point. The applicability of the Siegel theorem depends on the known restrictions on "small denominators" considered below: the inequality

$$|\lambda_j - \lambda^m| \geq C|m|^{-\nu}, \quad (|m| = m_1 + \ldots + m_n, \quad \lambda^m = \lambda_1^{m_1} \ldots \lambda_n^{m_n})$$



is satisfied for all $j \in (1,2,\ldots,n), m_i \in \mathbf{Z}_+, |m| \geq 2$, $(C > 0, \nu > 0)$, where $(\lambda_1,\ldots,\lambda_n) \in \mathbf{C}^n$ is the set of eigenvalues of the mapping at the fixed non-resonant point. Since the eigenvalues of the DS under consideration are algebraic numbers (which is probably a typical case), the problem of "small denominators" for the applicability of Siegel's theorem is solved by means of the Feldman theorem on the evaluation of linear forms of logarithms of algebraic numbers. The latter theorem is one of the classical results [78], [22] §4.10 which relies upon the solution of the 7th Hilbert problem by A.I. Gel'fond and T. Schneider and the new powerful methods of A. Baker in the theory of transcendental numbers. Later A. Baker [29] (see sect. 3) has proved much more general theorem and has improved Feldman's estimate [30] . The best estimate was obtained recently by A. Baker and G. Wüstholz in [31].

# 2 The general approach to integration of 3-dimensional Cremona reversible DS

## 2.1 General formulation of the problem

**Definition 1** Let $X$ be an arbitrary set. A one-to-one mapping $T : X \to X$ is said to be reversible if there exists another mapping $G : X \to X$ for which $T^{-1} = G \circ T \circ G$ and $G$ is an involution: $G^2 = id$ [42], [44], [52].

These conditions imply that $T \circ G$ is also an involution and $T = (T \circ G) \circ G$ is the composition of two involutions. Conversely, the composition of any two involutions is reversible with respect to each of them.

Consider the following two involutions of the complex plane $\mathbf{C}$: the linear one

$$A_1 : \mathbf{C} \to \mathbf{C}, \qquad A_1 : w \mapsto -w,$$

and the affine one

$$I_1 : \mathbf{C} \to \mathbf{C}, \qquad I_1 : w \mapsto 1 - w.$$

These involutions do not commute. Their compositions $A_1 \circ I_1$ and $I_1 \circ A_1$ define a couple of two reciprocal reversible dynamical systems on $\mathbf{C}$.

On the other hand, consider the following two involutions of the space $\mathbf{C}^3$: the linear one, defined by the Chew–Low crossing symmetry involutive matrix

$$z \mapsto Az, \qquad A = \frac{1}{9}\begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix} \tag{1}$$

with $\det A = -1$, and the nonlinear one

$$I : \mathbf{C}^3 \to \mathbf{C}^3, \qquad I(z_1, z_2, z_3) = \left(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}\right)$$

(the latter being the standard Cremona transformation). These involutions do not commute either. Their compositions $A \circ I$ and $I \circ A$ define a couple of two reciprocal reversible dynamical systems on $\mathbf{C}^3$.

**Remark 1** The dynamical systems we have met here are discrete rather than continuous (i.e., they are cascades rather than flows). Recall that a dynamical system is an action of a group on a set [1], [79]. In our case, the group which acts on C or $C^3$ is the additive group of integers Z. However, one can embed the dynamical systems in question in flows (see Remark 3 below).

We look for a meromorphic mapping $\mathbf{C} \to \mathbf{C}^3$ which realizes the equivalence of these two couples of reversible dynamical systems. In other words, we seek a meromorphic mapping $S : \mathbf{C} \to \mathbf{C}^3$ satisfying the following system of functional equations (FE):

$$S(-w) = AS(w), \tag{2}$$

$$S(w+1) = I \circ AS(w), \tag{3}$$

where $w \in \mathbf{C}$. This system describes $p$-wave scattering of $\pi$-mesons on nucleons within the framework of the static model.

Taking the liberty of speech, we will sometimes call the system of FE (2)-(3) a dynamical system (DS).

One can say that the involutions $A$ and $I$ of $\mathbf{C}^3$ are representations of the involutions $A_1$ and $I_1$ of $\mathbf{C}$, respectively. Meromorphic mappings $S : \mathbf{C} \to \mathbf{C}^3$ satisfying FE (2)-(3) realize these representations.

Each meromorphic mapping $S : \mathbf{C} \to \mathbf{C}^3$ defines some curve in $\mathbf{C}^3$. The system of FE (2)-(3) completely determines this curve up to the group of automorphisms. One can easily verify that every transformation of the form

$$S(w) \to S\left(w + \beta(w)\right) \exp\left(\alpha(w - 1/2)\right), \tag{4}$$

where

$$\begin{array}{llll} \beta(w+1) = & \beta(w), & \beta(-w) = & -\beta(w), \\ \alpha(w+1) = & -\alpha(w), & \alpha(-w) = & -\alpha(w). \end{array}$$

is an automorphism of the solution space for DS (2)-(3) ($\beta$ defining an inner automorphism of the curve). Obviously, there should exist also an automorphism which depends on another arbitrary function in $w$ with period equal to 1.

Since the strip $|\mathrm{Re}\, w| \leq 1/2$ is a fundamental domain for infinite discrete Cremona group $(I \circ A)^n$ (a subgroup of the group generated by $I$ and $A$), if $S_i(w)$ $(i = 1, 2, 3)$ are rational functions in the strip then they are rational everywhere except at infinity. On the contrary, if $S_i(w)$ have an essential singularity in the strip then they have infinitely many essentially singular points.

As $A^2 = id$, the matrix $A$ is of simple structure (i.e., diagonalizable):

$$A = B\Lambda^a B^{-1}, \qquad \Lambda^a = \mathrm{diag}(\lambda_1^a, \lambda_2^a, \lambda_3^a), \qquad \lambda_1^a = -1, \ \lambda_{2,3}^a = 1, \tag{5}$$

$$B_{ij} = \mu_i^{(j)}, \qquad A\mu^{(j)} = \lambda_j^a \mu^{(j)}, \quad i, j = 1, 2, 3, \tag{6}$$

where $B$ is the fundamental matrix for $A$ in the basis of eigenvectors $\mu^{(j)}$ of the matrix $A$ [80], which are the following:

$$\mu^{(1)} = (-4, -1, 2), \quad \mu^{(2)} = (4, -2, 1), \quad \mu^{(3)} = (1, 1, 1).$$



According to the general approach [59] introduce the functions $z : \mathbf{C} \to \mathbf{C}^3$, $j : \mathbf{C}^3 \to \mathbf{C}^3$, $\pi : \mathbf{C}^3 \to \mathbf{C}$,

$$\begin{aligned} z(w) &= B^{-1} S(w), && (7) \\ j(z) &= ABz, && (8) \\ \pi(z) &= \prod_{i=1}^{3} j_i(z). \end{aligned}$$

Define the mapping $\Phi : \mathbf{C}^3 \to \mathbf{C}^3$:

$$\Phi : z \mapsto \tilde{z} = \Phi(z) = \phi(z)/\pi(z), \tag{9}$$

where $\phi : \mathbf{C}^3 \to \mathbf{C}^3$ is the mapping

$$\phi(z) = \pi(z) B^{-1} \left(Ij(z)\right).$$

The DS (2)-(3) may be rewritten as:

$$\begin{aligned} z(-w) &= \Lambda^a z(w), && (10) \\ z(w+1) &= \phi(z(w))/\pi(z(w)). && (11) \end{aligned}$$

**Remark 2** Note that mapping (9) is a birational Cremona transformation in $\mathbf{C}^3$. Transformations of the form (9) with different denominators $\pi(z)$ (but the same numerator $\phi(z)$) induce the same projective Cremona transformation in $\mathbf{CP}^2$ [28]:

$$\tilde{z}_1 : \tilde{z}_2 : \tilde{z}_3 = \phi_1 : \phi_2 : \phi_3.$$

**Remark 3** Note that the FE (11) (and also a FE of general type) can be associated [71] with the differential equations generated by the Abelian group of analytical iterations

$$\begin{aligned} \frac{dz(w)}{dw} &= P(z(w)), && (12) \\ P(z) &= \left.\frac{d\Phi^{(s)}(z)}{ds}\right|_{s=0}, \quad s \in \mathbf{R} && (13) \end{aligned}$$

where $\Phi^{(n)} : z(w) \mapsto z(w+n)$, $\Phi^{(1)} \equiv \Phi$, $\Phi^{(0)} \equiv id$ and $P$ are solutions of the following functional equation:

$$P(\Phi(z)) = \nabla\Phi(z) \cdot P(z). \tag{14}$$

It is clear that $\nabla\Phi(z)$ is a $3 \times 3$ matrix while $P(z)$ is a 3-vector. For the one-dimensional variant of equations (12), (14), see the passage in [10] about P. Erdős' and E. Jabotinsky's paper, p. 209–212, and for the two-dimensional variant, [71].

According to (5) – (8) and taking into account that $A^2 = id$, we have

$$\Lambda^a = B^{-1}AB, \qquad S(w) = j\left(\Lambda^a z(w)\right) \stackrel{\mathrm{def}}{=} j'\left(z(w)\right). \tag{15}$$

## 2.2 Partial automorphic forms

Let $m = (m_1, m_2, m_3) \in \mathbf{Z}_+^3$ be a multiindex, $m_i \in \mathbf{Z}_+$, $|m| = m_1 + m_2 + m_3$.

**Definition 2** A polynomial $P : \mathbf{C}^3 \to \mathbf{C}$ is said to be invariant if $P(\Phi(z)) = 0$ whenever $P(z) = 0$.

**Theorem 1** [59], (see also [62] For any invariant irreducible homogeneous polynomial $P(z)$ of degree $k$ there exists a multiindex $m$ with $|m| = k$ such that $P(z)$ satisfies the following system of FE:

$$P(\phi(z)) = \varepsilon P(z) j^m(z), \qquad j^m = j_1^{m_1} j_2^{m_2} j_3^{m_3}, \qquad \varepsilon = \pm 1, \tag{16}$$

$$P(\Lambda^a z) = \nu P(z), \qquad \nu = \pm 1. \tag{17}$$

We will denote any solution of these equations by $P_m(z)$ provided that it is an irreducible homogeneous polynomial of degree $|m|$.

As the matrix $A$ (1) has the only eigenvalue equal to $-1$, we can assume that $\nu = +1$ because for $\nu = -1$ any solution $P$ of FE (16)–(17) will be reducible:

$$P(z) = z_1^{2l+1} \hat{P}(z), \quad \deg \hat{P} = |m| - 2l - 1.$$

Following [59], we shall call polynomials $P_m(z)$ partial automorphic forms (PAF) for the DS (2)–(3) of weight $m = (m_1, m_2, m_3)$.

**Definition 3** PAF $P_m(z)$ is said to be an automorphic form (AF) of weight $q$ if

$$P_m(\Phi(z)) \equiv P_m(z) J^q(z)$$

where $J$ denotes the Jacobian of the mapping $\Phi$ (9).

One can easily verify that the Jacobian $J$ of the mapping $\Phi$ (9) is equal to $\pi^{-2}(z)$. Indeed,

$$J \doteq -\det\left(B^{-1} \bar{J} B \Lambda^a\right) = \det(AJ) = \pi^{-2}(z),$$

where $\bar{J} = \mathrm{diag}\,(1/j_1^2(z), 1/j_2^2(z), 1/j_3^2(z))$.

A mapping $\Phi$ has a polynomial AF of weight $q$ if and only if it possesses a PAF of weight $m = \{q, q, q\}$. It has a rational AF of weight $q$ if there exists a rational function of the PAF that satisfies the equations (16)–(17) with $m_1 = m_2 = m_3 = q$.

According to [62] the DS (2)–(3) possesses the only PAF ($z_1/z_3 = x$, $z_2/z_3 = y$):

$$P_{(1,1,0)}(z) = z_3 z_2 - z_1^2 = z_3^2 (y - x^2),$$

— the solution of (16) with $\varepsilon = -1$. Therefore, according to the classification in [59], the DS (2)–(3) is not algebraically integrable, i.e. it has no first algebraic integral, or, in other words, AF of weight zero. Similarly, the system of FE under consideration is not algebraically equivalent to an area-preserving mapping (APM) because it has no AF of weight one.

**Remark 4** The DS in [58] has an AF of weight one (recall that it has three PAF) and is equivalent to an APM but it is non-algebraically integrable, i.e., it possesses a first non-algebraic integral.



## 2.3 Fundamental points and principal lines of a quadratic Cremona mapping

Let $\tilde{z} = z(w+1)$, then according to (15), (11), (8) we have

$$j'(\tilde{z}) = I j(z).$$

Consider some special points and lines on the projective plane $\mathbf{CP}^2$ of homogeneous coordinates $z$ known from the theory of quadratic Cremona transformations. Introduce six principal lines (P-lines) $J_k$, $J_k'$ ($k = 1, 2, 3$) of direct mapping $\Phi$ specified by (9) and inverse one $\Phi^{-1}$:

$$J_k = \{z : \ j_k(z) = 0\}, \qquad J_k' = \{z : \ j_k'(z) = 0\} \qquad (k = 1, 2, 3),$$

and six points of pairwise intersections of these lines $\{O_1, O_2, O_3\}$ and $\{O_1', O_2', O_3'\}$ (see Fig. 1):

$$O_i = J_k \bigcap J_l, \qquad O_i' = J_k' \bigcap J_l'.$$

They are called fundamental points (F-points) [28] of direct mapping (11) and the inverse one. Here $\{ilk\}$ is a permutation of $\{123\}$.

Under the direct mapping (11) and its inverse

$$z(w) = \frac{\Lambda^a \phi(\Lambda^a \tilde{z})}{\pi(\Lambda^a \tilde{z})},$$

the images of the F-points $O_i$, $O_i'$ and those of the P-lines $J_i$, $J_i'$ are

$$\Phi : O_i \mapsto J_i', \qquad \Phi : J_i \mapsto O_i', \qquad \Phi^{-1} : O_i' \mapsto J_i, \qquad \Phi^{-1} : J_i' \mapsto O_i.$$

Thus the F-points and P-lines $\{O_i, J_i\}$ and $\{O_i', J_i'\}$ are the only elements of the mapping (11) where the one-to-one correspondence is violated.

Notice that the mappings $\Phi$ and $\Phi^{-1}$ realize blowing up and contraction (see the Codaira theorem [19], Ch. 1, §4) and also the concept of $\sigma$-process of blowing up of singularities in the theory of ordinary differential equations [77], §2.

The F- and P-elements have the form

$$J_i = \left\{z : \ j_i(z) = \mu_i^{(l)} \lambda_l^{(a)} z_l = 0\right\}, \qquad J_i' = \left\{z : \ j_i'(z) = \mu_i^{(l)} z_l = 0\right\},$$

$$O_i = \left( \frac{\mu_k^{(2)} - \mu_l^{(2)}}{\mu_k^{(1)} \mu_l^{(2)} - \mu_l^{(1)} \mu_k^{(2)}}, \frac{\mu_k^{(1)} - \mu_l^{(1)}}{\mu_k^{(1)} \mu_l^{(2)} - \mu_l^{(1)} \mu_k^{(2)}}, 1 \right), \qquad O_i' = \Lambda^a O_i. \tag{18}$$

**Remark 5** Similarly to [62], [58], the multiindex $m = (m_1, m_2, m_3)$ has the meaning of multiplicities $m_i$ of algebraic invariant curve $P_m(z) = 0$ at the F-points $O_i$ (see (16), (17)), and consequently at the points $O_i'$, according to the symmetry of (17).

Let us proceed from homogeneous projective coordinates $z(w) \in \mathbf{C}^3$ to the coordinates

$$((x_1(w), x_2(w), z_3(w)), \qquad x_i = \frac{z_i}{z_3} \quad (i = 1, 2), \qquad x(w) = (x_1(w), x_2(w)) \in \mathbf{C}^2. \tag{19}$$

Then instead of (10), (11) we obtain

$$x_i(-w) = \Lambda^a_{ii} x_i(w) \quad (i=1,2), \qquad z_3(-w) = z_3(w), \tag{20}$$

$$x_i(w+1) = \frac{\Phi_i(xz_3, z_3)}{\Phi_3(xz_3, z_3)} \stackrel{\text{def}}{=} f_i(x) \quad (i=1,2), \qquad z_3(w+1) = \Phi_3(xz_3, z_3), \tag{21}$$

where $\Phi_i$ are defined by (9) and

$$f_i = \frac{\Phi_i(x,1)}{\Phi_3(x,1)} = \frac{\sum_{i=1}^3 B^{-1}_{1i} j_k(x,1) j_l(x,1)}{\sum_{i=1}^3 B^{-1}_{3i} j_k(x,1) j_l(x,1)}, \qquad (i \neq k \neq l, \quad l > k).$$

In the sequel, we set $x_1 \equiv x$, $x_2 \equiv y$.

## 2.4 Structure of the general solution

The following theorem on the structure of the general solution of the FE (20)-(21) is valid.

**Theorem 2** The general solution of FE (20)-(21) has the form:

$$x(w) = \frac{2(F(w+\frac{1}{2}) + F(w-\frac{1}{2}))}{1 + 4F(w+\frac{1}{2})F(w-\frac{1}{2})}, \qquad y(w) = \frac{2(1 + F(w+\frac{1}{2}) - F(w-\frac{1}{2}))}{1 + 4F(w+\frac{1}{2})F(w-\frac{1}{2})}, \tag{22}$$

where the meromorphic function $F(w)$ is a solution of the functional equations

$$F(w+1) = \frac{4 + 2F(w)F(w-1) + F(w-1) - 14F^2(w) - 4F(w-1)F^2(w)}{1 - 2F(w) - 2F(w-1) - 4F^2(w)}, \tag{23}$$

$$F(-w) = -F(w). \tag{24}$$

*Proof.* Introduce the quadratic Cremona mapping casting the functions $x(w), y(w)$ to $u_1(w), u_2(w)$:

$$u_1 = \frac{(y+x)(y-1)}{y^2 - x^2}, \qquad u_2 = \frac{(y-x)(y-1)}{y^2 - x^2}. \tag{25}$$

The inverse mapping is single-valued and has the form:

$$x = \frac{u_1 - u_2}{u_1 + u_2 - 2u_1u_2}, \qquad y = \frac{u_1 + u_2}{u_1 + u_2 - 2u_1u_2}. \tag{26}$$

Using (25) and (26) one can obtain a simpler expression for the system (21):

$$u_1(w+1) = \frac{(9u_1 + u_2 - 2u_1u_2)(1 - u_1)}{u_1 + u_2 - 2u_1^2}, \qquad u_2(w+1) = 1 - u_1, \tag{27}$$

where in the right hand side we have set $u_1 = u_1(w)$, $u_2 = u_2(w)$. According to (20) and (25) we have:

$$u_1(w) = u_2(-w). \tag{28}$$

Substituting (28) into the second eq. of (27) and assuming

$$u_2(w) = 1/2 - F(w - 1/2), \tag{29}$$

we get (24). Eqs. (26), (28) and (29) imply (22). Substituting (28), (29) into the first eq. of (27) we obtain equation (23) for $F(w)$.

This equation has a particular solution $F(w) = w$ corresponding to the solution in [60].



# 3 Integration of the equation for function $F(w)$

## 3.1 Spectra of hyperbolic fixed points

Let $F(w-1) = u(w)$, $F(w) = v(w)$ and consider the mapping $T$ corresponding to equation (23) for $F(w)$ $\left(\tilde{U} = (\tilde{u}, \tilde{v}) = (u(w+1), v(w+1)),\ U = (u,v)\right)$

$$T: \tilde{U} = T(U), \qquad \tilde{u} = v, \qquad \tilde{v} = \frac{4 + u + 2uv - 14v^2 - 4uv^2}{1 - 2u - 2v - 4v^2}. \tag{30}$$

The mapping $T$ is reversible since it can be represented in the form $T = R \circ W$ where $R$ and $W$ are involutions:

$$R: U' = R(U), \quad u' = -u, \quad v' = \frac{4 - 14u^2 - (1 - 2u - 4u^2)v}{1 + 2u - 4u^2 + 2v};$$

$$W: U' = W(U), \quad u' = -v, \quad v' = -u. \tag{31}$$

On Fig.2 there are shown some special points and curves of the mapping $T$ and the involutions $R$ and $W$ such as: the unique algebraic invariant line $v = u + 1$ of mapping $T$, the fixed points $u = 0$, $v = 1$ and $u = 0$, $v = -2$ of involution $R$, the line $v = -u$ of fixed points of involution $W$ (the symmetry line of $W$), the antisymmetry lines of involutions $R$ and $W$ defined by the conditions $R(U) = -U$ and $W(U) = -U$, the singularity lines of involution $R$ and mapping $T$ (i.e. lines where involution $R$ or mapping $T$ are not defined). Besides the parabolic fixed point $d_1: u = v = \infty$, the mapping (30) has two symmetrical hyperbolic fixed points

$$d_2: u = v = \frac{1}{\sqrt{2}}, \qquad d_3: u = v = -\frac{1}{\sqrt{2}}, \tag{32}$$

$$R(d_2) = d_3, \qquad R(d_3) = d_2, \qquad W(d_2) = d_3, \qquad W(d_3) = d_2. \tag{33}$$

The eigenvalues $\lambda_1, \lambda_2$ of the linear part of the mapping (30) at the fixed point $d_2$ (32) and $\lambda_3, \lambda_4$ at $d_3$ are the roots of the fundamental polynomial of algebraic number $\lambda$:

$$P_4(\lambda) = 7\lambda^4 - 96\lambda^3 + 306\lambda^2 - 96\lambda + 7,$$

$$\lambda_{1,2} = \frac{6\sqrt{2} \pm \sqrt{65}}{2\sqrt{2} + 1} \quad (+ \text{ for } \lambda_1); \quad \lambda_3 = \lambda_1^{-1}, \quad \lambda_4 = \lambda_2^{-1}, \tag{34}$$

$$\lambda_1 \approx 4.322, \qquad \lambda_2 \approx 0.1105, \qquad \lambda_3 \approx 0.2314, \qquad \lambda_4 \approx 9.050.$$

Notice that the numbers $\lambda_i$ are called conjugate algebraic numbers (see [81]). The eigenvalues $\lambda_1, \lambda_2$ of the mapping (30) at $d_2$ belong to Siegel's domain ($0 < \lambda_2 < 1 < \lambda_1$). The same holds for the eigenvalues $\lambda_3, \lambda_4$.

**Theorem 3** The set of eigenvalues $\lambda_1, \lambda_2$ at the fixed point $d_2$ and the set $\lambda_3, \lambda_4$ at the fixed point $d_3$ (34) are not resonant, i.e.

$$\lambda_j - \lambda_1^{m_1}\lambda_2^{m_2} \neq 0 \quad \text{for } j = 1,2 \text{ and } |m| = m_1 + m_2 \geq 2, m_i \in \mathbf{Z}_+,$$

$$\lambda_j - \lambda_3^{m_1}\lambda_4^{m_2} \neq 0 \quad \text{for } j = 3,4 \text{ and } |m| = m_1 + m_2 \geq 2, m_i \in \mathbf{Z}_+.$$

*Proof.* We have to prove that algebraic numbers $\chi_1 = 1 - \lambda_1^{m_1-1}\lambda_2^{m_2}$ and $\chi_2 = 1 - \lambda_1^{m_1}\lambda_2^{m_2-1}$ are not equal to zero for any integers $m_1$ and $m_2$, $|m| \geq 2$. Consider the number $\chi_1$. As $\lambda_1, \lambda_2 \in \mathbf{Q}(\sqrt{2}, \sqrt{65})$, a quadratic extension of the rational number field [81] (see §17), then $\chi_1$ belongs to $\mathbf{Q}(\sqrt{2}, \sqrt{65})$ and has relative degree $\leq 4$. The equality $\chi_1 = 0$ implies $\chi_1 \in \mathbf{Q}(\sqrt{2})$. Since $\lambda_{1,2} = R_1 \pm R_2\sqrt{65}$, where $R_1, R_2 \in \mathbf{Q}(\sqrt{2})$, then, obviously, $\chi_1 \in \mathbf{Q}(\sqrt{2})$ if and only if $m_2 = m_1 - 1$. Let $m_2 = m_1 - 1$. Then $\chi_1 = 1 - ((9 - 4\sqrt{2})/7)^{m_2}$. It is clear that for $|m| = 2m_2 + 1 \geq 2$ $(m_2 \geq 1)$ one has $\chi_1 \notin \mathbf{Q}$ and, consequently, $\chi_1 \neq 0$. Analogously, one can prove that $\chi_2 \neq 0$. It is obvious that also the set $\lambda_3$, $\lambda_4$ is non-resonant at the fixed point $d_3$. ◃

## 3.2 Algebraic addition concerning the Siegel theorem and the 7th Hilbert problem

The problem of linearization of the mapping (30) will be solved on the basis of the following Siegel theorem.

**Definition 4** A set $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ is of the multiplicative type $(C, \nu)$, if the following inequality:

$$|\lambda_j - \lambda^m| \geq C|m|^{-\nu}, \quad (|m| = m_1 + \ldots + m_n, \quad \lambda^m = \lambda_1^{m_1} \ldots \lambda_n^{m_n}) \tag{35}$$

is satisfied for all $j \in (1, 2, \ldots, n), m_i \in \mathbf{Z}_+, |m| \geq 2 \quad (C > 0, \nu > 0)$.

**Theorem 4 (C.L. Siegel** for $n = 1$, **E. Zehnder**'s generalization for $n > 1$ [82] )
If the set $\{\lambda\}$ of eigenvalues of the linear part of a mapping, holomorphic near a fixed point, is of the multiplicative type $(C, \nu)$ for some $C > 0, \nu > 0$, then the mapping is biholomorphically equivalent to its linear part in some neighborhood of the fixed point (see [77], §28).

**Remark 6** The set $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ is also of the multiplicative type $(C_1, \nu)$ with some $C_1$, if instead of inequality (35) a similar one

$$|\lambda_j - \lambda^m| \geq C(|m| + 1)^{-\nu} \tag{36}$$

is satisfied under the same conditions.

*Proof.* Indeed, from (36) it follows that for $|m| \geq 2$

$$|\lambda_j - \lambda^m| > C(2|m|)^{-\nu} = C_1|m|^{-\nu},$$

where $C_1 = 2^{-\nu}C$, and we obtain (35) with $C_1$ instead of $C$. ◃

Let us give necessary definition from the algebraic number theory (see [22], Addition D, p. 267-268)

**Definition 5** A number $\alpha \in C$ is an algebraic number if there exists such a polynomial $P(z) \in Q[z], P(z) \not\equiv 0$, that $P(\alpha) = 0$, where $Q[z]$ is the set of all polynomials over Q. The set of all algebraic numbers $A$ is an algebraically closed number field. For every $\alpha \in A$ there exists the single polynomial (the fundamental polynomial of number $\alpha$) $P(z) = a_n z^n + \ldots + a_0$, $P(z) \in Z[z]$, with the following properties: 1) $P(z)$ is irreducible over Q; 2) $P(\alpha) = 0$; 3) $a_n \geq 1$; 4) the greatest common divisor $(a_0, a_1, \ldots, a_n) = 1$; 5) if $Q(z) \in Q[z]$ and $Q(\alpha) = 0$ then $P(z)|Q(z)$. $L(\alpha)$ is the length of algebraic number $\alpha$, $L(\alpha) = |a_n| + \ldots + |a_0|$. ◃



The following theorem on sets $\{\lambda\}$ of the multiplicative type $(C, \nu)$ is valid (see also [59]).

**Theorem 5** A set $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ is of the multiplicative type $(C, \nu)$, if this set is multiplicatively non-resonant and numbers $\lambda_j \in \mathbf{A}$, where $\mathbf{A}$ is an algebraic number field.

This theorem is an immediate corollary of the following theorem in [22] (see chapter 10, §4.10) about the evaluation of linear forms of the logarithms of algebraic numbers.

**Theorem 6 (N.I. Feldman)** Suppose $n \in \mathbf{N}$; the numbers $\alpha_1, \ldots, \alpha_m \in \mathbf{A}$; and $\ln\alpha_1, \ldots, \ln\alpha_m$ are arbitrary, but fixed and linearly independent over $\mathbf{Q}$ values of their logarithms. Then there exists such an effective (i.e., there is a recipe of its calculation) constant

$$\nu = \nu(\alpha_1, \ldots, \alpha_m; \ln\alpha_1, \ldots, \ln\alpha_m; n) > 0$$

that for any $\beta_k$, $k = 0, 1, \ldots, m$:

$$\beta_0, \beta_1, \ldots, \beta_m \in \mathbf{A}; \quad |\beta_0| + \ldots + |\beta_m| > 0,$$

$$\deg \mathbf{Q}(\alpha_1, \ldots, \alpha_m; \beta_0, \ldots, \beta_m) \leq n$$

the following inequality:

$$|\beta_0 + \beta_1 \ln\alpha_1 + \ldots + \beta_m \ln\alpha_m| > L^{-\nu}$$

with

$$L = \max_{0 \leq k \leq m} L(\beta_k)$$

is satisfied. Here $\mathbf{Q}(\alpha_1, \ldots, \alpha_m; \beta_0, \ldots, \beta_m)$ is the field of rational functions in the numbers $(\alpha_1, \ldots, \alpha_m; \beta_0, \ldots, \beta_m)$. ◃

*Proof* of Theorem (5). Let $\min_{1 \leq j \leq n} |\lambda_j| = c_1 > 0$, $\imath$ is imaginary unit. Since $|e^z - 1| \geq \frac{1}{2}|z - 2\imath\pi k|$ for $|z - 2\imath\pi k| \leq \frac{1}{2}$, $k \in Z$, one has

$$|\lambda_j - \lambda^m| \geq c_1 \left|1 - \exp\left(\sum_{i=1}^{n}(m_i - \delta_{ij})\ln\lambda_i\right)\right| \geq \frac{c_1}{2}|l(m) - 2\imath\pi k|,$$

provided that

$$|l(m) - 2\imath\pi k| \leq \frac{1}{2}, \tag{37}$$

where

$$l(m) = \sum_{i=1}^{n}(m_i - \delta_{ij})\ln\lambda_i, \quad k = \left[\frac{\operatorname{Im} l(m)}{2\pi}\right],$$

$$|k| \leq p|m|, \quad p = \left|\left[\max_{1 \leq i \leq n} |\ln\lambda_i|\right]\right| + 1 \geq 1$$

(it obviously suffices to consider only those $m \in \mathbf{Z}_+^n$ satisfying this condition). Here $\delta_{ij}$ is the Kronecker delta. Since the set $\{\lambda\}$ is multiplicatively non-resonant, $\ln\lambda_j$ are linearly independent over $\mathbf{Q}$ and we can choose such branch of ln that $2\imath\pi \equiv \ln(1), 1 \in \Lambda$, and the conditions of Feldman's theorem from [22] are satisfied. Letting

$$\beta_0 = 0, \quad \beta_i = m_i - \delta_{ij} \quad (1 \le i \le n), \quad \beta_{n+1} = -k, \quad \alpha_{n+1} = 1,$$
$$L(\beta_{n+1}) = |k| + 1, \quad L(\beta_i) = |m_i - \delta_{ij}| + 1 \le \max(2, m_i + 1)$$

and observing that

$$L = \max_{1 \le i \le n+1} L(\beta_i) \le 1 + \max\left(\left(\max_{1 \le i \le n} m_i\right), |k|\right) \le p(|m| + 1),$$

we obtain inequality (36). Now, taking into account Remark 6, we conclude that the set $\{\lambda\}$ is of the multiplicative type $(C, \nu)$. ◁

**Remark 7** One may wonder whether it is possible to determine the multiplicative type of the set $\{\lambda\}$ on the base of estimate of rate of growth of the partial fractions set for some, relative with $\lambda_i$, transcendental number, that would be the other formulation of the Feldman theorem. Let $\lambda_i (i = 1, \cdots, n)$ are real and positive. Let again $c_1 = \min_{1 \le j \le n} \lambda_j$. If a multiindex $m \in \mathbf{Z}^n_+$ and an index $j \in (1, 2, \ldots, n)$ satisfy (37) then for $\chi_j = \lambda_j - \lambda^m$ we have

$$\begin{aligned} |\chi_j| &= |\lambda_j - \lambda^m| \ge \frac{c_1}{2}\left|\sum_{i=1}^{n}(m_i - \delta_{ij})\ln\lambda_i\right| \\ &= \frac{c_1}{2}|\ln\lambda_n|\left|\sum_{i=1}^{n-1}(m_i - \delta_{ij})\frac{\ln\lambda_i}{\ln\lambda_n} + (m_n - \delta_{nj})\right| \\ &= \frac{c_1}{2}|\ln\lambda_n|\left|\sum_{i=1}^{n-1}(m_i - \delta_{ij})\frac{\ln\frac{\lambda_i}{\lambda_n}}{\ln\lambda_n} + |m| - 1\right|. \end{aligned} \tag{38}$$

Then we obtain for $n = 2$ and $0 < \lambda_2 < 1 < \lambda_1$

$$\begin{aligned} |\chi_j| &\ge \frac{c_1}{2}\ln\frac{\lambda_1}{\lambda_2}(|m| - 1)\left|\frac{m_1 - \delta_{1j}}{|m| - 1} - \frac{\ln\frac{1}{\lambda_2}}{\ln\frac{\lambda_1}{\lambda_2}}\right| \\ &= \frac{c_1}{2}\ln\frac{\lambda_1}{\lambda_2}(|m| - 1)\left|\mu - \frac{m_1 - \delta_{1j}}{|m| - 1}\right|, \end{aligned} \tag{39}$$

where $\mu = \frac{\ln(1/\lambda_2)}{\ln(\lambda_1/\lambda_2)} = \left(1 - \frac{\ln\lambda_1}{\ln\lambda_2}\right)^{-1} > 0$, as far as $\lambda_2 < 1 < \lambda_1$. Notice that the statement "$\mu$ is a transcendental number" is a particular case of the 7th Hilbert problem. Then using Chebyshev's theorem on the best approximation of the second kind [84] (see theorems 13 and 17), we have:

$$\min|\chi_j| \ge \frac{c_1}{2}\ln\frac{\lambda_1}{\lambda_2}q_k\left|\mu - \frac{p_k}{q_k}\right| \ge \frac{c_1}{2}\ln\frac{\lambda_1}{\lambda_2}\frac{1}{(q_{k+1} + q_k)} \tag{40}$$

or (see theorem 12 in [84])

$$\min|\chi_j| > \frac{c_1}{2}\ln\frac{\lambda_1}{\lambda_2}\frac{1}{(a_{k+1} + 2)q_k}, \tag{41}$$



where $p_k = m_1 - \delta_{1j}$, $q_k = |m| - 1$, $p_k/q_k$ is a convergent to the continued fraction for $\mu$, $a_k$ is the partial fraction of transcendental number

$$\mu = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}.$$

Then the conditions of the Siegel theorem are satisfied if the following estimate is valid:

$$a_{k+1} + 2 \lesssim A q_k^{\nu - 1}, \tag{42}$$

where $A \in \mathbf{N}$, $A$ does not depend on $k$, $\nu \ge 1$. Note that the inequality (42) is the sufficient condition. For $\nu = 1$ the spectrum is bounded (this is true only for quadratic algebraic numbers [81] if we have in mind only algebraic numbers ). It is known (see e.g. [79], Vol. 3, pp. 94–95), that the problem of description of the spectrum of $\{a_k\}$ has not been solved even for cubic irrationalities $\mu$. Comparing our inequality with Feldman's one, where $\nu$ is an effective constant (i.e. a formula for its value has been obtained), we conclude that the inequality (42) is satisfied apparently. In more detail this question will be discussed elsewhere.

Now we can make a general proposition about solutions of equations (23)–(24) for the function $F(w)$, defining the general solution (22) of the initial FE (21).

**Theorem 7** The function $F_\delta(w)$ — is a function of a one-parameterical family of solutions of the system (23)–(24) — is a holomorphic, in certain finite neighborhoods of the origin in $\mathbf{C}^2$, function of variables $z_1$, $z_2$ or $z_3$, $z_4$:

$$z_i(w) = \delta(w)\exp\left(\left(w + \beta(w) - \frac{1}{2}\right)\ln\lambda_i\right), \quad 1 \le i \le 4, \tag{43}$$

where $\lambda_i$ are determined by (34) while arbitrary functions $\delta(w)$ and $\beta(w)$ (cf. (4)) have the following properties:

$$\begin{aligned} \delta(-w) &= \delta(w), & \delta(w + 1) &= \delta(w), \\ \beta(-w) &= -\beta(w), & \beta(w + 1) &= \beta(w). \end{aligned} \tag{44}$$

The family $F_\delta(w)$ is defined by the Taylor expansion in $z_1, z_2$

$$F(w) = \sum_k f_k z^k, \quad k \in \mathbf{Z}^2_+, \quad f_k = f_{k_1, k_2}, \quad z^k = z_1^{k_1} z_2^{k_2} \tag{45}$$

with the coefficients $f_k$ determined by the recurrent relation

$$\begin{aligned} f_k = &\left\{\sum_{|m|=1}^{|k|-1} f_{k-m}\left[\left(-(4f_0 + 1)\lambda^{2k-m} + (4f_0 - 1)\lambda^m - \lambda^{2k-2m} + 7\lambda^k\right)f_m\right.\right. \\ &\left.\left. -2(\lambda^{2k-m} - \lambda^m)\sum_{|l|=1}^{|m|-1} f_{m-l}f_l\right]\right\}\left[\frac{4f_0 + 1}{2}(\lambda^k - \lambda_1)(\lambda^k - \lambda_2)\right]^{-1}, \end{aligned} \tag{46}$$

where $\lambda^k = \lambda_1^{k_1}\lambda_2^{k_2}$, $f_0 = 1/\sqrt{2}$, $f_{0,1}$ and $f_{1,0}$ are arbitrary, and the sum is taken over all the permissible values. The coefficients $\tilde{f}_k$ of the Taylor expansion of $F$ in $z_3, z_4$ satisfy the same recurrent relation with $\lambda_1$, $\lambda_2$ replaced by $\lambda_3$, $\lambda_4$ and they are

$$\tilde{f}_0 = -1/\sqrt{2}, \qquad \tilde{f}_{0,1} = -\lambda_3 f_{0,1}, \qquad \tilde{f}_{1,0} = -\lambda_4 f_{1,0}, \qquad \tilde{f}_k = -\lambda^{-k} f_k, \quad \lambda^{-k} \equiv \lambda_1^{-k_1}\lambda_2^{-k_2}.$$

Each local solution of the family $F_\delta(w)$ (the germ), defined by (43)-(46), can be extended up to the global one using $F_\delta(w), F_\delta(w-1)$ and iterations of the equation (23) and lies on the invariant manifold $\Gamma_\delta$:

$$\Gamma_\delta: \qquad z_1^{\gamma_1}(w+1) z_2^{\gamma_2}(w+1) = z_1^{\gamma_1}(w) z_2^{\gamma_2}(w) = \delta^2(w), \tag{47}$$

where

$$\gamma_1 = \frac{-2\ln\lambda_2}{\ln\lambda_1 - \ln\lambda_2}, \qquad \gamma_2 = \frac{2\ln\lambda_1}{\ln\lambda_1 - \ln\lambda_2}.$$

*Proof.* According to Theorems 3 and 5, the set $(\lambda_1, \lambda_2)$ from (34) of the eigenvalues of the mapping $T$ (30) at the hyperbolic fixed point $d_2$ given by (32) is of the multiplicative type $(C, \nu)$. Therefore Theorem 4 guarantees the existence of a biholomorphic mapping $U = G(z)$, $U = (u, v)$, $z = (z_1, z_2)$, $G(0) = d_2$, which transforms a neighborhood of the origin in $\mathbf{C}^2$ to a neighborhood of the point $d_2$ and reduces the mapping $T$ to the normal form at $d_2$

$$G^{-1}TG: \ (z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2).$$

On the other hand, the biholomorphic mapping $WG$, where $W$ is involution (31), transforms a neighborhood of the origin in $\mathbf{C}^2$ to a neighborhood of the point $d_3$ (32) and reduces the mapping $T$ to the normal form at $d_3$:

$$G^{-1}WTWG: (z_3, z_4) \mapsto (\lambda_3 z_3, \lambda_4 z_4).$$

Indeed, since involution $W$ reverses mapping $T$, the mappings $G^{-1}TG$ and $G^{-1}WTWG$ are inverse to each other.

Set $(z_3, z_4) = \zeta$ and

$$G(z) = (g_1(z), g_2(z)), \qquad WG(\zeta) = (-g_2(\zeta), -g_1(\zeta)).$$

If some functions $z_i(w)$, $1 \le i \le 4$, satisfy the FE

$$z_i(w+1) = \lambda_i z_i(w) \tag{48}$$

then the functions $F(w) = g_2(z(w))$ or $F(w) = -g_1(\zeta(w))$ satisfy FE (23). Moreover, one will have $g_1(z(w)) = F(w-1)$ or $-g_2(\zeta(w)) = F(w-1)$, respectively.

Now we should determine the relations between $z(w)$ and $\zeta(w)$ which make $F(w)$ satisfy FE (24), i.e., make $F(w)$ odd. Let $F(w) = g_2(z(w))$ be close to $1/\sqrt{2}$. Then we have $F(-w) = -g_2(\zeta(1-w))$. The equality $F(-w) = -F(w)$ is therefore ensured by

$$\zeta(1-w) = z(w). \tag{49}$$

If

$$z_i(w) = \delta_i(w)\exp((w+\alpha_i)\ln\lambda_i), \qquad 1 \le i \le 4, \tag{50}$$



then (48) is equivalent to $\delta_i(w+1) = \delta_i(w)$ and (49) is equivalent to $\zeta(w) = \Lambda z(-w)$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$, i.e.,

$$\delta_{i+2}(-w) = \lambda_i \delta_i(w)\exp((\alpha_i + \alpha_{i+2})\ln\lambda_i), \qquad i = 1, 2.$$

For $\alpha_i = -1/2$ $(1 \le i \le 4)$ we have

$$\delta_3(-w) = \delta_1(w), \qquad \delta_4(-w) = \delta_2(w), \tag{51}$$

From (48), (50) it follows the existence of the invariant manifold $\Gamma_\delta$ (cf. (47))

$$\Gamma_\delta: \quad \tilde{z}_1^{\gamma_1}\tilde{z}_2^{\gamma_2} = z_1^{\gamma_1} z_2^{\gamma_2} = \delta^2(w), \qquad \tilde{z}_3^{\gamma_3}\tilde{z}_4^{\gamma_4} = z_3^{\gamma_3} z_4^{\gamma_4} = \delta^2(w), \tag{52}$$

where $\tilde{z} \equiv z(w+1)$, exponents $\gamma_1, \gamma_2$ are determined by (47) and are invariant relatively on the change $\lambda_1, \lambda_2 \to \lambda_3, \lambda_4$. Substituting (50), (51) into (52) we have

$$\delta_1^{\gamma_1}(w)\delta_2^{\gamma_2}(w) = \delta^2(w), \qquad \delta_1^{\gamma_1}(-w)\delta_2^{\gamma_2}(-w) = \delta^2(w). \tag{53}$$

From (53) we have

$$\left(\frac{\delta_1(w)}{\delta_1(-w)}\right)^{\gamma_1}\left(\frac{\delta_2(w)}{\delta_2(-w)}\right)^{\gamma_2} = 1. \tag{54}$$

Since $\delta_i(w+1) = \delta_i(w)$, then supposing

$$\frac{\delta_i(w)}{\delta_i(-w)} = \exp(\beta_i(w)\ln\lambda_i), \qquad i = 1, 2, \tag{55}$$

we obtain from (54)

$$\exp[(-\beta_1(w) + \beta_2(w))\ln\lambda_1\ln\lambda_2] = 1. \tag{56}$$

From (54)-(56) it follows that

$$\beta_1(w) = \beta_2(w) = 2\beta(w), \tag{57}$$

where $\beta(w)$ is odd function with period equal to 1 (cf. (44)). We have from (55), (57)

$$\delta_i(w) = \delta(w)\exp(\beta(w)\ln\lambda_i), \qquad \delta(-w) = \delta(w)$$

and we obtain (43), (44).

Finally, let

$$F(w) = g_2(z(w)) = \sum_k f_k\,(z(w))^k$$

for $F(w)$ close to $1/\sqrt{2}$ and

$$F(w) = -g_1(\zeta(w)) = \sum_k \tilde{f}_k\,(\zeta(w))^k$$

for $F(w)$ close to $-1/\sqrt{2}$. Substituting these expansions into (23), we derive the desired recurrent relations on $f_k$ and $\tilde{f}_k$. The arbitrariness in the choice of $f_{1,0}$ and $f_{0,1}$ corresponds to the ambiguity in the choice of the normalizing mapping $G$. ◁

**Remark 8** Since the basis of an algebraic number $\alpha$ (a root of the fundamental polynomial $P_n(\alpha)$) is the set $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ [81], all the four numbers $\lambda_i$ (34) can be represented as polynomials of one of them, say $\lambda_1$, with rational coefficients. Thus, the coefficients $f_k$ and $\tilde{f}_k$ are rational functions of $\lambda_1$.

The eigenvalues $\lambda_1, \lambda_2$ are also the roots of the polynomial

$$\lambda^2 - \frac{24 f_0}{4 f_0 + 1}\lambda + \frac{4 f_0 - 1}{4 f_0 + 1}.$$

Carrying over the results of Bryuno [85] (see theorem 7, chapter 2, example 7) on differential equations to mappings and using theorem 9 in chapter 4 from [85], one can state that the domain of holomorphy of expansion (45) is defined by that of mapping $T$ (30). To be more precise, the former is smaller than the latter by a factor of $\epsilon$,

$$\epsilon = \sum_{k=0}^{\infty} \frac{\ln q_{k+1}}{q_k},$$

where $q_k$ is the denominator of the $k$-th convergent for the continued fraction $p_k/q_k$ of the transcendental number $\gamma = \ln \lambda_1 / \ln \lambda_2$ (see [84]).

A numerical investigation of the phase curves of this family by various choices of $\delta(w)$ will be performed elsewhere.

The following theorem is valid.

**Theorem 8** Every curve of the family $F_\delta(w)$ on the manifold $\Gamma_\delta$ intersects other curves of this family and the straight line

$$\Phi(w) = F(w) - F(w-1) - 1 = 0$$

(which corresponds to the particular solution $F(w) = w$) at the F-point $O_3 = (5/2, 3/2)$ on the $(v, u)$-plane (see Fig. 2 and (18), this point is the same as the point $O_3$ on Fig. 1 in the $(x, y)$-plane) and its subsequent points (7/2,5/2), (9/2,7/2), etc. This implies, in particular, that every curve of this family is not algebraic.

*Proof.* Consider the equations for the functions $F(w)$ and $\Phi(w)$ (for brevity we use the notations $F \equiv F(w)$, $\Phi \equiv \Phi(w)$):

$$F(w+1) = -\frac{\Phi \frac{(2F+1)(2F-3)}{(2F-1)(2F+3)}}{1 - \frac{2\Phi}{(2F-1)(2F+3)}} + F + 1,$$

$$\Phi(w+1) = -\Phi \frac{\frac{(2F+1)(2F-3)}{(2F-1)(2F+3)}}{1 - \frac{2\Phi}{(2F-1)(2F+3)}}.$$

Suppose that at some $w^*$ the function $F(w^*) = 3/2$. It is always possible to choose such a $w^*$, since a meromorphic function admits every value except two ones at least once ( see the Picard theorem, [86], chapter IV, §12, p. 220). The such except values for the function $F(w)$ already are $\pm 1/\sqrt{2}$.

Then we have:

$$F(w^* + 1) = \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots, etc.$$



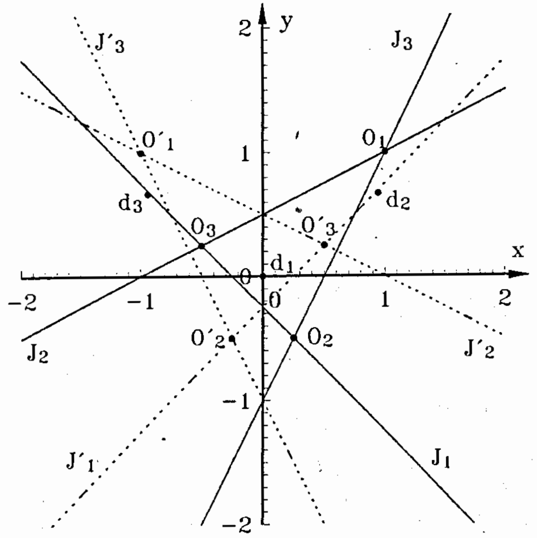


Fig. 1: The principal lines $J_i$ (solid) of direct mapping $\Phi$ and $J'_i$ (dash) of inverse mapping $\Phi^{-1}$, the fundamental points $O_i$ of mapping $\Phi$ and $O'_i$ of mapping $\Phi^{-1}$, the parabolic fixed point $d_1$ of mapping $\Phi$ and the hyperbolic fixed points $d_2$, $d_3$ of mapping $\Phi$.

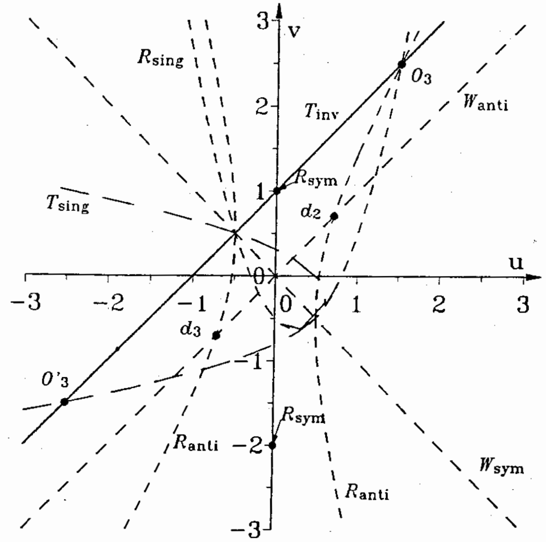


Fig. 2: The special points and curves of the mapping $T$ and involutions $R$ and $W$ (subscripts *sym* are for symmetry points and lines, *anti* for antisymmetry lines, *sing* for singularity lines, $T_{inv}$ is the unique algebraic invariant curve of $T$, $d_2$ and $d_3$ are the fixed points of the mapping $T$, $O_3$ and $O'_3$ are the fundamental points of $T$).

$$\Phi(w^* + 1) = 0, 0, 0, \ldots, etc.$$

◁

An interesting question, whether the function $F(w)$ is a rational function of variables $z_1(w), z_2(w)$, remains open for the present. Of more interest is the general statement of this question: if the initial mapping is rational, then in which cases a biholomorphic change of co-ordinates transforming this mapping into the normal form is also a rational function.


## Acknowledgements

The author is deeply grateful to V.V. Kozlov and V.B. Priezzhev for constant support and useful discussions. He is also deeply thankful to V.I. Arnol'd for support, the interest to the paper and some information useful for future investigations. He wishes to acknowledge the critical assistance of and helpful discussions with M.B. Sevryuk. The author is particularly grateful N.I. Fel'dman for useful discussions at January of 1993 year, when this paper was mainly fulfilled. The author is also thankful to D.V. Anosov, A.D. Bryuno, M.I. Graev , V.A. Iskovskikh, V.G. Kadyshevsky, A.A. Kirillov, A.G. Khovansky, I.M. Krichever, Yu.I. Manin, N.G. Moshchevitin, Yu.V. Nesterenko, D.V. Shirkov, Ya.G. Sinai, A.M. Stěpin, A.P. Veselov, V.S. Vladimirov and V.E. Zakharov for useful discussions and the interest to the paper.

I thank J. Moser for the letter with answers to my questions and with information about [83], [82], received after this paper was accepted for publication. I would like to remark that studying the dynamics of quadratic symplectic mappings, in particular of the Hènon map, J. Moser [83] (see Sect. 3, p. 426-428) came to the similar idea of using the Feldman-Baker theorem from the transcendental number theory for the proof of some Diophantine condition in the Rüssmann theorem.




# References


[1] D.V. Anosov et al., Dynamical Systems, Vol. 1 (Ordinary Differential Equations. Smooth Dynamical Systems, eds. D.V. Anosov and V.I. Arnold) (Encyclopaedia Math. Sciences, Vol. 1, Springer, Berlin, 1988).

[2] L.A. Bunimovich et al., Dynamical Systems, Vol. 2 (Ergodic Theory, ed. Ya.G. Sinai) (Encyclopaedia Math. Sciences, Vol. 2, Springer, Berlin, 1989).

[3] V.I. Arnold, V.V. Kozlov and A.I. Neishtadt, Dynamical systems, Vol. 3 (Mathematical Aspects of Classical and Celestial Mechanics) (Encyclopaedia Math. Sciences, Vol. 3, Springer, Berlin, 1988).

[4] V.I. Arnold et al., Dynamical Systems, Vol. 4 (Symplectic Geometry, Geometrical Quantization, Integrable Systems, eds. V.I. Arnold and S.P. Novikov) (Encyclopaedia Math. Sciences, Vol. 4, Springer, Berlin, 1990).

[5] V.I. Arnold et al., Dynamical Systems, Vol. 5 (Theory of Bifurcations and Catastrophes, ed. V.I. Arnold) (Encyclopaedia Math. Sciences, Vol. 5, Springer, Berlin, 1990).

[6] V.I. Arnold et al., Dynamical Systems, Vol. 6 (Singularities: Local and Global Theory, ed. V.I. Arnold) (Encyclopaedia Math. Sciences, Vol. 6, Springer, Berlin, in press).

[7] A.T. Fomenko et al., Dynamical Systems, Vol. 7 (Nonholonomic Systems, Integrable Systems, eds. V.I. Arnold and S.P. Novikov) (Encyclopaedia Math. Sciences, Vol. 16, Springer, Berlin, in press).

[8] V.I. Arnold et al., Dynamical Systems, Vol. 8 (Singularities: Classification and Applications, ed. V.I. Arnold) (Encyclopaedia Math. Sciences, Vol. 39, Springer, Berlin, in press).

[9] D.V. Anosov et al., Dynamical Systems, Vol. 9 (Dynamical Systems with Hyperbolic Behavior, ed. D.V. Anosov) (Encyclopaedia Math. Sciences, Vol. 66, Springer, Berlin, in press).

[10] M. Kuczma, Functional Equations in a Single Variable (Pol. Scien. Publ., Warsaw, 1968).

[11] J. Aczél, Lectures on Functional Equations and Their Applications (Math. in Science and Engineering, Vol. 19, ed. H. Oser) (Acad. Press, New York, 1966).

[12] J. Aczél, J. Dhombres, Functional Equations in Several Variables (Cambridge University Press, Cambridge, 1989).

[13] M. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform (Society for Industrial and Applied Mathematics, Philadelphia, 1981).

[14] F. Calogero and A. Degasperis, Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations, Vol. 1 and 2 (Studies in Mathematics and its Applications, Vol. 13) (North-Holland Publishing Company, Amsterdam, New York, Oxford, 1982).

[15] A.C. Newell, Solitons in Mathematics and Physics

[16] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaievski, The Theory of Solitons. The Inverse Problem Method (Nauka, Moscow, 1980, in Russian).

[17] L.D. Faddeev and L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons (Springer, Berlin, 1987).

[18] H. Cartan, Calcul Différentiel. Formes Différentielles (Hermann, Paris, 1967).

[19] Ph. Griffiths and J. Harris, Principles of Algebraic Geometry (John Wiley & Sons, New York, 1978).

[20] I.R. Shafarevich, Basic Algebraic Geometry (Springer, Berlin, 1977).

[21] Z.I. Borevich and I.R. Shafarevich, Number Theory (Nauka, Moscow, 1972, in Russian).

[22] N.I. Feldman, Hilbert's Seventh Problem (Moscow University Press, Moscow, 1982, in Russian).

[23] I.M. Gel'fand, M.I. Graev and N.Ya. Vilenkin, Integral Geometry and Related Topics in Representation Theory (Fizmatgiz, Moscow, 1962, in Russian).

[24] I.M. Gel'fand, M.I. Graev and I.I. Pyatetsky-Shapiro, Representation Theory and Automorphic Functions (Nauka, Moscow, 1966, in Russian).

[25] H. Jacquet and R.P. Langlands, Automorphic Forms on GL(2) (Springer, Berlin, 1970).

[26] H. Kraft, Geometrische Methoden in der Invariantentheorie, (Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1985).

[27] V.I. Arnold, Mathematical Methods of Classical Mechanics (Springer, Berlin, 1989).

[28] H. Hudson, Cremona Transformations in Plane and Space (Cambridge University Press, Cambridge, 1927).

[29] A. Baker, Transcendental Number Theory (Cambridge University Press, Cambridge, 1975).

[30] A. Baker, Acta Arithmetica 28 (1975) 247–252.

[31] A. Baker and G. Wüstholz, J. reine anngew. Math. 442 (1993), 19-62.

[32] D.V. Shirkov, V.V. Serebryakov and V.A. Meshcheryakov, Dispersion Theories of Strong Interactions at Low Energies (Nauka, Moscow, 1967, in Russian).



[33] N.N. Bogoliubov, B.V. Medvedev and M.C. Polivanov, Topics of the Theory of Dispersion Relations (Fizmatgiz, Moscow, 1958, in Russian).

[34] A.P. Veselov, Commun. Math. Phys. 145 (1992) 181–193.

[35] V.I. Arnold, Bol. Soc. Bras. Mat. 21 (1990) 1–10.

[36] V.I. Arnold, in: *Proceedings of a Conference in Honour of J. Moser,* eds. P. Rabinowitz and E. Zehnder (Academic Press, New York, 1990) pp. 77–84.

[37] J.K. Moser, SIAM Rev. 8 (1966) 145–172.

[38] J.K. Moser, Math. Ann. 169 (1967) 136–176.

[39] J.K. Moser, Stable and Random Motions in Dynamical Systems, with Special Emphasis on Celestial Mechanics (Princeton University Press, Princeton, NJ, 1973).

[40] V.I. Arnold, in: Nonlinear and Turbulent Processes in Physics, ed. R.Z. Sagdeev (Harwood, Chur, 1984) pp. 1161–1174.

[41] V.I. Arnold and M.B. Sevryuk, in: Nonlinear Phenomena in Plasma Physics and Hydrodynamics, ed. R.Z. Sagdeev (Mir, Moscow, 1986) pp. 31–64.

[42] M.B. Sevryuk, Reversible Systems, Lecture Notes in Mathematics, Vol. 1211 (Springer, Berlin, 1986).

[43] K.M. Khanin and Ya.G. Sinai, in: Nonlinear Phenomena in Plasma Physics and Hydrodynamics, ed. R.Z. Sagdeev (Mir, Moscow, 1986) pp. 93–118.

[44] R.L. Devaney, Trans. Am. Math. Soc. 218 (1976) 89–113.

[45] J. Scheurle, Arch. Rat. Mech. Anal. 97 (1987) 103–139.

[46] Yu.N. Bibikov, Multifrequency Nonlinear Oscillations and their Bifurcations (Leningrad University Press, Leningrad, 1991, in Russian).

[47] M.B. Sevryuk, Chaos 1 (1991) 160–167.

[48] M.B. Sevryuk, Russian Acad. Sci. Dokl. Math. 47 (1993) 129–133.

[49] M.B. Sevryuk, Chaos 3 (1993) 211–214.

[50] A. Lahiri, A. Bhowal, T.K. Roy and M.B. Sevryuk, Physica D 63 (1993) 99–116.

[51] G.R.W. Quispel and M.B. Sevryuk, Chaos 3 (1993) 757–769.

[52] J.A.G. Roberts and G.R.W. Quispel, Physics Reports 216 (1992) 63–177.

[53] G.F. Chew and F.E. Low, Phys. Rev. 101 (1956) 1570–1579.

[54] V.A. Meshcheryakov, Comm. JINR P-2369, Dubna (1965) (in Russian).

[55] V.A. Meshcheryakov and K.V. Rerikh, Ann. Phys. 59 (1970) 408–425.

[56] G. Wanders, Nuovo Cim. 23 (1962) 817–837.

[57] V.A. Meshcheryakov, Zh. Eksp. Teoret. Fiz. 51 (1966) 648–657 (in Russian).

[58] K.V. Rerikh, Physica D 57 (1992) 337–354.

[59] K.V. Rerikh, Prep. JINR P5–93–106, Dubna (1993) (in Russian).

[60] T. Rothelutner, Z. Phys. 177 (1964) 287–299.

[61] V.I. Zhuravlev, V.A. Meshcheryakov and K.V. Rerikh, Yadernaya Fiz. 10 (1969) 168–175 (in Russian).

[62] K.V. Rerikh, Teor. Mat. Fiz. 50 (1982) 251–260 (in Russian).

[63] K.V. Rerikh, in: *Proc. of the XIII Int. Conf. on Differential Geometric Methods in Theoretical Physics*, 170–178, Shumen, Bulgaria, 1984, eds. H.D. Doebner and T.D. Palev (World Scientific, Singapore, 1986).

[64] H. Kaiser, Ann. Physik 27, 7F (1971) 149–160.

[65] K.V. Rerikh, in: *Proc. of the XIX Int. Symp. on Special Topics in Gauge Field Theories*, Ahrenshoop, DDR, 18–22 November 1985, Institut für Hochenergiephysik, PHE 85–15, 236, 1985.

[66] K.V. Rerikh, Comm. JINR P2–82–813, Dubna (1982) (in Russian).

[67] K.V. Rerikh, Comm. JINR P2–83–459, Dubna (1983) (in Russian).

[68] V.A. Meshcheryakov, Comm. JINR P2–5906, Dubna (1971) (in Russian).

[69] V.A. Meshcheryakov, Comm. JINR P2–7047, Dubna (1973) (in Russian).

[70] V.P. Gerdt and V.A. Meshcheryakov, Teor. Mat. Fiz. 24 (1975) 155–163 (in Russian).

[71] K.V. Rerikh, Comm. JINR P2–12714, Dubna (1979) (in Russian).

[72] K.V. Rerikh, Comm. JINR P2–87–514, Dubna (1987) (in Russian).

[73] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, Physica D 34 (1989) 183–192.

[74] A.P. Veselov, Russian Math. Surveys 46, no. 5 (1991) 1–51.

[75] G.D. Birkhoff, Acta Math. 43 (1922) 1–119.

[76] J. Moser, Comm. Pure Appl. Math. 9 (1956) 678–692.

[77] V.I. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations (Springer, New York, 1988).

[78] N.I. Feldman, Mat. Sbornik 77 (119), no. 3 (1968) 423–436 (in Russian).

[79] Mathematical Encyclopaedia, ed. I.M. Vinogradov, Vol. 1–5, (1977-1985), (Sov. Encycl., Moscow, in Russian).



[80] F.R. Gantmacher, Matrix Theory (Nauka, Moscow, 1988, in Russian); Matrizentheorie (Springer, Berlin, 1986).

[81] E. Hecke, Vorlesungen über die Theorie algebraischen Zahlen (Leipzig, 1923).

[82] E. Zehnder, Geometry and Topology, Lecture Notes in Mathematics. Vol. 597 (Springer ,Berlin,1977), 555-566 .

[83] J. Moser, Mathematische Zeitschrift, 216 (1994), 417-430.

[84] A.Ya. Khinchin, Continued Fractions (Nauka, Moscow, 1978, in Russian).

[85] A.D. Bryuno, Analytical form of differential equations, Transactions of the Moscow Mathematical Society, Vol. 25 (1971) 131–288, Vol. 26 (1972) 199–239.

[86] B.V. Shabat, Introduction in complex analysis, (Nauka, Moscow, 1969, in Russian).

**Received by Publishing Department on October 18, 1994.**