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NUMERICAL STUDY OF WAVES OF TOPPLINGS IN ABELIAN SANDPILE MODEL

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The sandpile models introduced by Bak, Tang and Weisenfeld [1] as simple models of self-organized criticality (SOC) have been intensively investigated in recent years. A class of such models called the Abelian sandpile models (ASM) [2] attracted special interest since essential properties were analytically determined and some characteristics were exactly evaluated for these models in two dimensions [3-6]. Nevertheless, the problem of the consistent theoretical description of the dynamics of avalanches for non-trivial lattices remains unsolved even in this simplest case . Recently, Ivashkevich, Ktitarev and Priezzhev [7] have introduced the concept of waves in ASM avalanches. They represented each avalanche as a sequence of waves and evaluated the critical exponent of the size distribution of a general wave. They established the equivalence between waves and inverse avalanches defined by Dhar and Manna [8] and reproduced their result for the critical exponent of the size distribution of the first inverse avalanche. Using special properties of avalanches starting at the boundary they found also the critical exponent for boundary avalanches.

In this Letter we investigate the power-law behavior of wave distributions depending on the "age" of the wave in a given avalanche. Besides, we obtain numerical estimates for the critical exponents found exactly in [7]. The results of our computer simulations are in agreement with the theoretical predictions [7, 8].

We consider the ASM on a square $N \times N$ lattice with open boundary conditions [2]. The dynamics of the model is defined by the toppling $N^2 \times N^2$ matrix Δ with nonzero elements $\Delta_{ii} = 4$ and $\Delta_{ij} = -1$ for adjacent sites i and j. A sandpile configuration of heights is stable iff $0 \le z_i \le \Delta_{ii}$. We add particles at random to the stable configuration until a certain height exceeds 4. Then it topples, giving 4 particles to its neighbors. If some of their heights become greater than 4, they also topple and the process proceeds until a stable configuration is reached. The process of topplings of all nonstable sites is called the avalanche. Some sites may not topple, some may topple once or more times during a given avalanche.

The avalanche may be represented as a sequence of waves in the following way [7]. We add a particle to the site with the height 4, topple it once and then relax all other unstable sites preserving the initial site from the second toppling. The set of

toppled sites is called "the first wave". Then we permit the site to topple a second time, creating a second wave and so on, until the avalanche stops. It is shown [7] that the last wave coincides with the first inverse avalanche [8].

There are several spatial characteristics of an avalanche which have the power-law behavior. Let s be the total number of topplings and s_d be the number of distinct sites where at least one toppling occurs during a given avalanche. The exponents τ_s and τ_d are defined by the following relations for large x:

$$\mathrm{Prob}(s = x) \sim x^{-\tau_s}$$

$$\mathrm{Prob}(s_d = x) \sim x^{-\tau_d}$$

Manna has performed the computer simulations [9] and obtained the same values for these exponents: $\tau_s = \tau_d = 1.22$. In a later paper [10], he has suggested a more accurate estimation, $\tau_s = 1.20$. Based on theoretical considerations and known numerical results, Majumdar and Dhar [6] have predicted the following values for these exponents: $\tau_s = 8/7 \approx 1.14$ and $\tau_d = 7/6 \approx 1.17$.

Dhar and Manna [8] have found that the probability distribution of the first inverse avalanche that coincides with the last wave in our interpretation has the critical exponent $\tau_f = -11/8$:

$$\mathrm{Prob}(s_f = x) \sim x^{-11/8} \text{ for large } x,$$

where s_f is the size of the last wave. They have also argued that the size of the cluster of maximum topplings (the set of sites which toppled more than other sites during the given avalanche) has the same power-law distribution.

Ivashkevich, Ktitarev and Priezzhev [7] considered the dynamics of waves ignoring the pauses between avalanches and called this collection of waves as the "waves of general form". They argued that the average number of general waves of size s varies as $1/s$ for large s. They also showed that the avalanches starting at the boundary consist of the only wave and the probability distribution of their sizes s varies as $s^{-3/2}$ for large s.

Our computer simulations give close estimates of the three exponents found exactly in [8, 7] : $\tau_{Last} = 1.37, \tau_{General} = 0.99, \tau_{Boundary} = 1.51$. In calculating the critical exponents we use the algorithm described by Manna [9]. We produce one million avalanches for square lattices with $N = 100, 150, 200, 300, 500$, constructing the size distributions for each value of N. In the case of general waves we take into account all waves appearing in each avalanche. Open boundary conditions are considered except the investigation of the avalanches starting at the boundary. In the latter case we define two opposite sides of the square as open boundaries and govern the two other sides by periodic boundary conditions. Every 10 avalanches we add at random the particle into one of the sites at the open boundary and collect data on the sizes of these particular avalanches. We construct the size distributions by measuring the number of waves of boundary avalanches of a particular size s. Then we take a double logarithmic plot of this distribution (see, for example, Fig.1) and evaluate the slope of the straight part of the curve. In this way we find numerical values $\tau^{(N)}$ for different sizes N of the lattice. Then we estimate the exponent $\tau = \lim_{N\to\infty} \tau^{(N)}$ (Fig.2).

We also study other power-law distributions arising in the wave structure of the avalanches. We consider the distributions of waves depending not on their number m in an avalanche, but on their index α which we define as the ratio of their number in a given avalanche and the total number of waves in this avalanche:

$$\alpha = \begin{cases} \frac{m-1}{k-1}, & \text{if } k > 1 \\ 0, & \text{if } k = 1 \end{cases}$$

We choose those five families of waves whose indices are the closest to values 0, 1/4, 1/2, 3/4, 1 and find their power-law distributions with the following exponents: $\tau_{First} = \tau_1 = 1.11, \tau_2 = 1.11, \tau_3 = 1.22, \tau_4 = 1.26, \tau_5 = \tau_{Last} = 1.37$.

In conclusion, we summarize the results of our simulations. We have investigated the wave structure of sandpile avalanches and observed the power-law behavior of sizes of the waves at different stages of the avalanche evolution. Our numerical experiments confirm the theoretical estimates of critical exponents for sizes of general waves, sizes of the last waves in an avalanche and sizes of avalanches starting at the open boundary.

We have to point out that all the known estimates of τ_s and τ_d are between the values

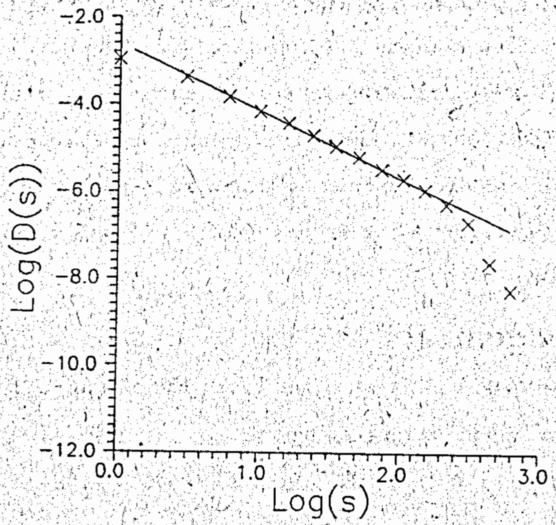

Fig. 1. Double logarithmic plot of the size probability distribution $D(s)$ of boundary avalanches for lattice size $N = 500$ integrated over bins of lengths increasing exponentially.

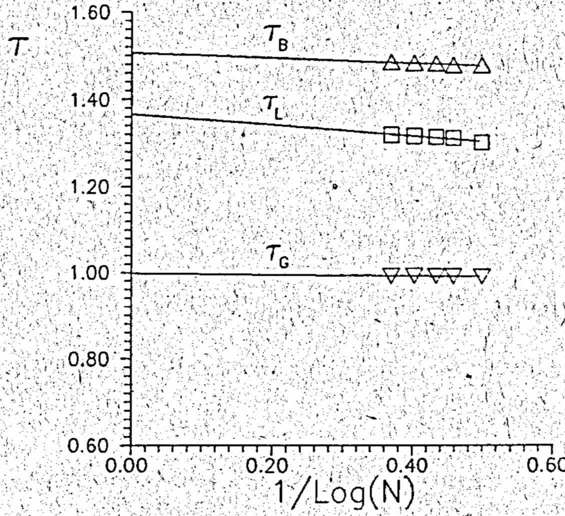

Fig. 2. Plot of estimations of critical exponents $\tau_{General}, \tau_{Last}, \tau_{Boundary}$ for different lattice sizes with $1/\mathrm{Log}N$.

obtained in this Letter τ_{First} and τ_{Last}. One of the interesting questions is to derive the relationship between these exponents. Recent investigations by Dhar and Manna [8] have shown that the values of the exponent of the first inverse avalanche (i.e. the last wave) and the exponent of cluster of maximum topplings probably coincide. We believe that wider and more elaborate computer experiments and theoretical considerations could clarify the connection between critical properties of sandpile avalanches and waves they consist of.

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