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LINEAR TRANSPORTS ALONG PATHS IN VECTOR BUNDLES.

IV. Consistency with Bundle Metrics

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1. INTRODUCTION

The present work investigates certain connections between linear transports along paths and bundle metrics, defined on one and the same vector bundle, which arise from the question of their consistency. More precisely, we shall consider the following main problem.

Let (E,π,B) be a real vector bundle with a base B, total bundle space E and projection $\pi:E\longrightarrow B$ [1]. Let g be a bundle metric on it [7], i.e. it is a map $g:x\longmapsto g_x$, $x\in B$, where the map

$$g_x:\pi^{-1}(x)\times\pi^{-1}(x)\longrightarrow\mathbb{R} \tag{1.1}$$

is bilinear, nondegenerate and symmetric for every $x\in B$. By definition the scalar product of $u,v\in\pi^{-1}(x)$ is $u\cdot v:=g_x(u,v)$, $x\in B$. Let L be a linear transport (L-transport) along paths in (E,π,B), i.e. if $\gamma:J\longrightarrow B$, J being an arbitrary real interval, is a path in B, then $L:\gamma\longmapsto L^\gamma$, where L^γ is the L-transport along γ and $L^\gamma:(s,t)\longmapsto$ $\longmapsto L^\gamma_{s\longrightarrow t}$, $s,t\in J$ is the L-transport along γ from s to t having the described in [2] properties.

Definition 1.1. The transport along paths L and the bundle metric g are consistent (resp. along a path $\gamma:J\longrightarrow B$) if L preserves the scalar products of the vectors along every (resp. along the given) path $\gamma:J\longrightarrow B$, i.e. if the scalar product of $u,v\in\pi^{-1}(\gamma(s))$, $s\in J$ is equal to the scalar product of the vectors obtained from u and v by L-transportation along γ at an arbitrary point $\gamma(t)$, $t\in J$:

$$g_{\gamma(s)}(u,v)=g_{\gamma(t)}(L^\gamma_{s\longrightarrow t}u,L^\gamma_{s\longrightarrow t}v),\quad s,t\in J. \tag{1.2}$$

Due to the arbitrariness of u and v and the nondegeneracy of g the equality (1.2) is equivalent to

$$g_{\gamma(s)}=g_{\gamma(t)}\circ(L^\gamma_{s\longrightarrow t}\times L^\gamma_{s\longrightarrow t}),\quad s,t\in J. \tag{1.3}$$

Important examples for transports along paths (in this case along curves) consistent with some metric are the parallel and Fermi-Walker transports in a Riemannian manifold which are consistent with the defining them Riemannian metric. This is proved, for instance, in [3,4] (see also below section 3).

The purpose of this work is to find necessary and/or sufficient conditions for consistency, or one may say compatibility, between L-transports along paths and bundle metrics. Analogous problems have been investigated (in a slightly different notation) in [5], where they were studied in the special case of the tangent bundle to a differentiable manifold.

The organization of the material is the following. In Sect. 2 attention is focussed on finding necessary and/or sufficient conditions for local (i.e. along a fixed path) or global (i.e. along any path) consistency between L-transports along paths and real bundle metrics. Also the corresponding problems of existence are considered. In Sect. 3 are investigated problemso of consistency concerning the specific case of generated by derivations of tensor algebras L-transports of vectors. In Sect. 4 the results of Sect. 2 are transferred to complex vector bundles endowed with Hermitian bundle metrics. Some remarks on the presented in this paper material are made in Sect. 5.

At the end of this introduction we shall note that in a field of bases $\{e_i,\ i=1,\dots,\dim(\pi^{-1}(x)),\ x\in B\}$ along a path $\gamma:J\longrightarrow B$, J being an arbitrary $\mathbb{R}$-interval, any L-transport along γ from s to t, $s,t\in J$ is uniquely characterized by its matrix $H(t,s;\gamma)=\|H^i{}_j(t,s;\gamma)\|$. Hereafter the Latin indices run from 1 to $n:=\dim(\pi^{-1}(x))$, $x\in B$ and further the usual summation rule from 1 to n is assumed over the repeated on different levels indices. The general form of $H(t,s;\gamma)$ is

$$H(t,s;\gamma)=F^{-1}(t;\gamma)F(s;\gamma) \tag{1.4}$$

for a nondegenerate matrix function F, defined up to a left multiplication with a nondegenerate matrix depending only on γ, i.e. up to the transformation

$$F(s;\gamma)\longmapsto D(\gamma)F(s;\gamma),\ \det(D(\gamma))\neq 0,\infty. \tag{1.5}$$

For further details of notation and results concerning L-transports along paths in vector bundles the reader is referred to [2].

2. GENERAL CONDITIONS FOR CONSISTENCY

Let in the real vector bundle (E,π,B) be given an L-transport along paths L and a real bundle metric g. Let $\gamma:J\longrightarrow B$ and in every fibre $\pi^{-1}(\gamma(s))$, $s\in J$ be fixed a basis $\{e_i(s):\ i=1,\dots,n=\dim(\pi^{-1}(x)),\ x\in B\}$. Let in $\{e_i\}$ the transport $L^\gamma_{s\to t}$ along γ from s to t be given by its matrix $H(t,s;\gamma)=\|H^i{}_j(t,s;\gamma)\|$ [2].

The components of the metric g in $\{e_i(s)\}$ at $\gamma(s)$ are

$$(g_{\gamma(s)})_{ij}:=g_{\gamma(s)}(e_i(s),e_j(s)). \tag{2.1}$$

Let $G(\gamma(s)):=\|(g_{\gamma(s)})_{ij}\|$. The nondegeneracy of g means $\det(G(\gamma(s)))\neq 0,\infty$, $s\in J$, and its symmetry – $G^T(\gamma(s))=G(\gamma(s))$, $s\in J$, where the superscript T means a transposition of matrices.

Proposition 2.1. A necessary and sufficient condition for a global (resp. local) consistency between a bundle metric g and an L-transport along paths L is

$$G(\gamma(s))=H^T(t,s;\gamma)G(\gamma(t))H(t,s;\gamma),\ s,t\in J \tag{2.2}$$

for every (resp. for the given) path $\gamma:J\longrightarrow B$.

Proof. If $u=u^ie_i(s)$ and $v=v^ie_i(s)$ (a summation from 1 to n is understood over repeated on different levels indices) are vectors from $\pi^{-1}(\gamma(s))$, then, due to the bilinearity of the metric, we have

$$g_{\gamma(s)}(u,v)=u^iv^jg_{\gamma(s)}(e_i(s),e_j(s))=u^iv^j(g_{\gamma(s)})_{ij}. \tag{2.3}$$

Substituting this equality into (1.2) and taking into account the linearity of L, the definition of its matrix H [2], and the arbitrariness of u and v, we see (1.2) to be equivalent to

$$(g_{\gamma(s)})_{ij}=H^k{}_{.i}(t,s;\gamma)(g_{\gamma(t)})_{kl}H^l{}_j(t,s;\gamma), \tag{2.4}$$

which is simply a component form of (2.2).∎

Using the general form of the matrices of L-transports along paths, which is given by (1.4), we can "simplify" (2.2) by putting it into "one point" form which is the contents of

Proposition 2.2. If the nondegenerate matrix function $F(s;\gamma)$ defines the L-transport along paths L by (1.4), then a necessary and sufficient condition for a global (resp. local) consistency of L with the bundle metric g is the existence of a nondegenerate, symmetric (resp. and constant) matrix function C of γ such that

$$(F^{-1}(s;\gamma))^TG(\gamma(s))F^{-1}(s;\gamma)=C(\gamma),\ s\in J,\ C^T=C. \tag{2.5}$$

Proof. Substituting (1.4) into (2.2) and multiplying the so obtained equality on the left by $(F^{-1}(s;\gamma))^T$ and on the right by $F^{-1}(s;\gamma)$, we find that (2.2) is equivalent to

$$(F^{-1}(s;\gamma))^T G(\gamma(s)) F^{-1}(s;\gamma)=(F^{-1}(t;\gamma))^T G(\gamma(t)) F^{-1}(t;\gamma). \quad (2.6)$$

If the metric and transport are consistent, we can put here $t=t_0$ for a fixed $t_0\in J$, so we get (2.5) with $C(\gamma)=(F^{-1}(t_0;\gamma))^T\times \times G(\gamma(t_0))F^{-1}(t_0;\gamma)$, besides $C^T=C$ due to $G^T=G$. On the contrary, if (2.5) is fulfilled, then (2.6) is identically valid and, consequently, (2.2) is also true. From here we conclude that (2.2) and (2.6) are equivalent, i.e. propositions 2.1 and 2.2 are equivalent, the former of which was already proved.■

The arbitrariness in the choice of F is described by the transformation (1.5). As a consequence of this the matrix $C(\gamma)$ in (2.5) is also not uniquely defined. Evidently, the above transformation leads to $C(\gamma)\longmapsto(D^{-1}(\gamma))^T C(\gamma)D^{-1}(\gamma)$. It is easy to check the validity of the inverse to this statement, i.e. we have

$$F(s;\gamma)\longmapsto D(\gamma)F(s;\gamma) \Leftrightarrow C(\gamma)\longmapsto(D^{-1}(\gamma))^T C(\gamma)D^{-1}(\gamma). \quad (2.7)$$

At this place naturally arises the question when and under what conditions there exist bundle metrics (resp. transports along paths) which are consistent with a given transport along paths (resp. bundle metric). This question can be put in two variants: global, when consistency along every path is investigated, and local, when consistency along a given path is studied. The following propositions give a solution of the above question from different viewpoints.

Proposition 2.3. If an L-transport along paths is defined by the matrix function F through (1.4), then any consistent with it along $\gamma:J\longrightarrow B$ bundle metric along γ has a matrix of the form

$$G(\gamma(s);\gamma)=(F(s;\gamma))^T C(\gamma)F(s;\gamma),\ s\in J,\ C^T=C, \quad (2.8)$$

where $C(\gamma)$ is nondegenerate, symmetric and depending on γ matrix.

Proof. This proposition is a corollary from proposition 2.2: in fact (2.8) is the general solution of (2.5) with respect to G when F is given.■

Proposition 2.3 shows that locally, i.e. along a given path, every L-transport along paths defines (see (2.8)) a unique class of consistent along that path with it metrics, uniquely defined only on the same path. The global variant of this problem is more difficult and will be treated below in proposition 2.6.

Proposition 2.4. A necessary and sufficient condition for the existence of globally (resp. locally) consistent with a given bundle metric L-transport along paths is the independence of the signature of the bundle metric, i.e. of the matrix G(x), from the point of the base B (resp. the path of transport) at which it is evaluated, i.e. from $x\in B$ (resp. $x\in\gamma(J)$).

Remark. If p(x) and q(x) are the numbers, respectively, of the positive and negative eigenvalues of G(x), the signature of G(x) is $s(x):=p(x)-q(x)$ (see [6]). The proposition states that the bundle metric admits a (globally or locally) consistent with it L-transport along paths iff $s(x)=\text{const}$, which due to $p(x)+q(x)=n:=\dim(\pi^{-1}(x))$, $x\in B$ is equivalent to $p(x)=\text{const}$ and/or $q(x)=\text{const}$, i.e. to the independence of the number of the positive (negative) eigenvalues of the metric from the point at which they are evaluated.

Proof. Let the bundle metric of (E,π,B) be (globally or locally) consistent with an L-transport along paths described in some basis along $\gamma:J\longrightarrow B$ by the matrix $F_0(s;\gamma)$ through (1.4) (with $F=F_0$). Then, by proposition 2.2, there exists a nondegenerate and symmetric matrix $C_0(\gamma)$ such that $(F_0^{-1}(s;\gamma))^T G(\gamma(s))F_0^{-1}(s;\gamma)=C_0(\gamma)$, $s\in J$. Because of $C_0^T(\gamma)=C_0(\gamma)$ there exists an orthogonal matrix $D_0(\gamma)$ for which $D_0^T(\gamma)C_0(\gamma)D_0(\gamma)$ is <u>constant</u> diagonal matrix [6]. Then (see (1.5) and (2.7)) the matrix $F(s;\gamma):=D_0(\gamma)F_0(s;\gamma)$ describes the same L-transport and due to the last equality it satisfies (2.5) with $C(\gamma)=D_0^T(\gamma)C_0(\gamma)D_0(\gamma)=\text{diag}(c_1,\dots,c_n)$, $c_1,\dots,c_n\in\mathbb{R}$. This means that in (2.5) the matrix $F(s;\gamma)$ may be chosen in such a way as $C(\gamma)$ to be a constant diagonal matrix. If such a choice is already made, then, due to (2.5), $G(\gamma(s))$ with the help of a transformation of a form $D^T G(\gamma(s))D$, where $D=F^{-1}(s;\gamma)$, can be transformed into the diagonal matrix $\text{diag}(c_1,\dots,c_n)$ which does not depend either on $\gamma(s)$ or on γ. On the other hand, as $G^T=G$, there exists an orthogonal matrix $D_1(\gamma(s))$ such that

$$D_1^T(\gamma(s))\times G(\gamma(s))D_1(\gamma(s))=\text{diag}(g_1(\gamma(s)),\dots,g_n(\gamma(s))),$$

where $g_i(\gamma(s))\neq 0,\infty$, $i=1,\dots,n$ are the eigenvalues of $G(\gamma(s))$ [6]. From the last two results, as a consequence of the Jacobi-Sylvester inertia law [6], it follows that the number p of the positive and the number q=n-p of the negative eigenvalues of $G(\gamma(s))$, i.e. of the bundle metric, are equal, respectively, to the number of positive and negative diagonal elements of $D^T G(\gamma(s))D=\text{diag}(c_1,\dots,c_n)=\text{const}$. Consequently the numbers p and q(=n-p) do not depend on the point $\gamma(s)$, $s\in J$ in the local case (consistency along a path) or on the point of B in the global case at which they are calculated, i.e. the signature of the bundle metric is $s=p-q=\text{const}$.

On the contrary, let in (E,π,B) be given a bundle metric g whose signature $s=p-q$, and consequently, the number p of its positive and the number $q=n-p$ of its negative eigenvalues, do not depend on the point x at which they are calculated ($x\in B$ in the global case or $x\in\gamma(J)$ in the local case for some path $\gamma:J\longrightarrow B$). Because of $G^T=G$ there exists an orthogonal matrix $D_1(x)$ such that $D_1^T(x)G(x)\times D_1(x)=\text{diag}(g_1(x),\dots,g_n(x))$, where $g_i(x)\neq 0,\infty$, $i=1,\dots,n$ are the eigenvalues of $G(x)$ [6]. If we put

$$D_2(x):=\text{diag}(|g_1(x)|^{-1/2},\dots,|g_n(x)|^{-1/2})$$

and $D(x):=D_1(x)D_2(x)$, then due to the above acceptance, we get $D^T(x)G(x)D(x)=\text{diag}(\varepsilon_1,\dots,\varepsilon_n)$, where p of the numbers $\varepsilon_1,\dots,\varepsilon_n$ are equal to +1, the remaining $q=n-p$ ones being equal to -1. Then from proposition 2.2 for $C(\gamma)=\text{diag}(\varepsilon_1,\dots,\varepsilon_n)=\text{const}$ (see also (2.5)) and (1.4) for $F(s;\gamma)=(D(\gamma(s)))^{-1}$, $s\in J$ for every path $\gamma:J\longrightarrow B$ (global case) or for some path $\gamma:J\longrightarrow B$ (local case), we conclude that the given bundle metric is, respectively, globally or locally consistent with the L-transport along paths defined by the matrix (1.4) with the above definition of F.■

The next proposition describes the general form and the way of construction of L-transports along paths consistent with a given bundle metric admitting such transports along paths.

Proposition 2.5. Let in (E,π,B) be given a bundle metric whose signature does not depend on the point $\gamma(s)$ at which it is evaluated for every (resp. some) path $\gamma:J\longrightarrow B$. Let there be chosen bases $\{e_i(s)\}$, $s\in J$ along γ in such a way that the first p eigenvalues of the matrix $G(\gamma(s))$, defining the metric in them (see (2.1)), be positive. Then one L-transport along paths is consistent with this bundle metric along every (resp. a given) path γ if and only if some of the defining its matrix (1.4) matrix functions F has the form

$$F(s;\gamma)=Y(\gamma)Z(s;\gamma)(D(\gamma(s)))^{-1},\quad s\in J \tag{2.9}$$

for every (resp. the given) path γ. In this equality: $Y(\gamma)$ is $n\times n$ nondegenerate matrix function of γ; $Z(s;\gamma)$ is a pseudo-orthogonal matrix of type (p,q), $q=n-p$, i.e. $Z(s;\gamma)\in O(p,q)$, or

$$(Z(s;\gamma))^T G_{p,q} Z(s;\gamma)=G_{p,q}:=\text{diag}(\underbrace{1,\dots,1}_{p\text{ times}},\underbrace{-1,\dots,-1}_{q\text{ times}}) \tag{2.10}$$

and $D(\gamma(s))$ is a fixed (orthogonal) matrix such that

$$(D(\gamma(s)))^T G(\gamma(s))D(\gamma(s))=G_{p,q}. \tag{2.11}$$

Remark. The case when in some basis not all of the first p eigenvalues of the bundle metric are positive is obtained from the above one by transformation (renumbering) of the basis $\{e_i(s)\}$.

Proof. To prove the necessity we have to solve the equation (2.5), when G and C are given, with respect to $F(s;\gamma)$. From $G^T=G$, the choice of $\{e_i(s)\}$ and the independence of p and $q=n-p$ from $\gamma(s)$ (because of $p-q=s=\text{const}$) follows the existence of a satisfying (2.11) matrix $D(\gamma(s))$: e.g. we can put $D(\gamma(s))=D_1(\gamma(s))D_2(\gamma(s))$, where D_1 and D_2 were defined in the proof of proposition 2.4.

Let $F(s;\gamma)=:F_1(s;\gamma)(D(\gamma(s)))^{-1}$. From (2.5) and (2.11), we get $(F_1^{-1}(s;\gamma))^T G_{p,q} F_1^{-1}(s;\gamma)=C(\gamma)$. Putting here $F_1(s;\gamma)=:Y(\gamma)Z(s;\gamma)$, where $Y(\gamma)$ is arbitrary nondegenerate matrix for which $(Y(\gamma))^T C(\gamma)\times Y(\gamma)=G_{p,q}$, we see that $Z(s;\gamma)$ satisfies (2.10). (The existence of $Y(\gamma)$ follows from (2.5): from it and the law of Jacobi-Sylvester it follows that $C(\gamma)$ has p positive eigenvalues, due to which the needed matrix $Y(\gamma)$, which is orthogonal, exists [6].) All that proves that $F(s;\gamma)$ has the form (2.9) if the considered L-transport along paths is consistent along every (resp. some) path γ with the given bundle metric.

On the contrary, the sufficiency of the proposition is almost evident: if (2.9) is valid, then it is easy to check the validity of (2.5) for $C(\gamma)=(Y^{-1}(\gamma))^T G_{p,q} Y^{-1}(\gamma)$ and according to proposition 2.2 the L-transport along paths and the bundle metric are consistent along every (resp. some) path γ.■

Corollary 2.1. For a given L-transport along paths L there locally exists a consistent with it along γ bundle metric if and only if along γ exists a basis in which the defining it by (1.4) matrix $F(s;\gamma)$ has the form (2.9), in which $Y(\gamma)$ and $D(s;\gamma)$ are arbitrary $n\times n$ nondegenerate matrices and $Z^T(s;\gamma)G_{p,q}Z(s;\gamma)=G_{p,q}$ for some $p,q\geq 0$, $p+q=n$ which may depend on γ.

Proof. If along γ there exists a consistent with L bundle metric g, then the expansion (2.9) follows from proposition 2.5. Conversely, if (2.9) is valid, then substituting (2.9) into (2.8), we get, in accordance with proposition 2.3, a class of consistent with L along γ metrics defined along γ. In particular, choosing Y and C in such a way that $Y^T CY = G_{p,q}$, we obtain

$$G(\gamma(s);\gamma)=((D(\gamma(s)))^{-1})G_{p,q}(D(\gamma(s)))^{-1},$$

where p and q may depend on γ.■

Now we shall go back to the question of the global existence of a bundle metric globally consistent with a given L-transport along paths (see the comment after the proof of proposition 2.3).

Proposition 2.6. A necessary and sufficient condition for the existence of globally consistent with a given L-transport along paths bundle metrics is the existence of a local basis along any path $\gamma:J\longrightarrow B$ in which the matrix $F(s;\gamma)$, defining this transport along $\gamma:J\longrightarrow B$ by (1.4), has the form (2.9) in which: $Y(\gamma)$ and $D(\gamma(s))$ are arbitrary nondegenerate $n\times n$ matrices and $Z(s;\gamma)$ is a pseudo-orthogonal matrix of type (p,q), with $p+q=n$. Besides, if for a given L-transport along paths are fulfilled these conditions, then in the above mentioned bases all globally consistent with the L-transport along paths bundle metrics are defined by the matrix

$$G(\gamma(s))=((D(\gamma(s)))^{-1})^T G_{p,q}(D(\gamma(s)))^{-1}, \tag{2.12}$$

which depends only on the point $\gamma(s)$, but not on the path γ.

Proof. Let there be given an L-transport along paths for which there exists a globally consistent with it bundle metric. Now we shall prove that under this condition is fulfilled an equality like (2.9). In arbitrary bases along every path γ this L-transport along γ defines a class of consistent with it bundle metrics along γ which are defined by (2.8). Let in (2.8) the matrix $C(\gamma)$ have $k(\gamma)$ positive and $l(\gamma)=n-k(\gamma)$ negative eigenvalues. Then (see [6]) there exists a matrix $Y(\gamma)$ such that $Y^T(\gamma)C(\gamma)Y(\gamma)=G_{k(\gamma),l(\gamma)}$, as a consequence of which (2.8) can be represented as

$$G(\gamma(s);\gamma)=F^T(s;\gamma)(Y^{-1}(\gamma))^T G_{k(\gamma),l(\gamma)} Y^{-1}(\gamma)F(s;\gamma).$$

If the consistent with an L-transport along paths metric, which by assumption exists, is described in some basis with the matrix $G_0(x)$ having for every $x\in B$ p positive and $q=n-p$ negative eigenvalues, then by proposition 2.3 along γ this bundle metric belongs to the above class of bundle metrics and, hence, there exists a matrix $Y_0(\gamma)$ (or the corresponding matrix $C_0(\gamma)$; see (2.8)) such that

$$G_0(\gamma(s);\gamma)=F^T(s;\gamma)(Y_0^{-1}(\gamma))^T G_{k_0(\gamma),l_0(\gamma)} Y_0^{-1}(\gamma)F(s;\gamma).$$

From here, due to the Jacobi-Sylvester law [6], it follows that $k_0(\gamma)=p=\text{const}$ and $l_0(\gamma)=n-p=\text{const}$. If we put $F(s;\gamma)=:Y_0(\gamma)\times Z_0(s;\gamma)D_0^{-1}(\gamma(s))$, where $D_0(x)$ is a matrix for which $D_0^T(x)G_0(x)\times D_0(x):=G_{p,q}$, $x\in B$ [6], then from the same equality, we see that

$$G_0(\gamma(s))=((D_0(\gamma(s)))^{-1})^T Z_0^T(s;\gamma)G_{p,q}Z_0(s;\gamma)(D_0(\gamma(s)))^{-1}.$$

From here, as a consequence of the definition of D_0, we conclude that $Z_0(s;\gamma)$ is a pseudo-orthogonal matrix of type (p,q) (see (2.10)). This result proves the existence of a representation like (2.9) for the considered L-transport along paths.

The sufficiency of the proposition is almost evident. In fact, if (2.9) is valid for one L-transport along paths, then substituting (2.9) into (2.8), we see the class of consistent with it along γ bundle metrics to be defined by

$$G(\gamma(s);\gamma)=((D(\gamma(s)))^{-1})^T Z^T(s;\gamma)Y^T(\gamma)C(\gamma)Y(\gamma)Z(s;\gamma)(D(\gamma(s)))^{-1},$$

where $C(\gamma)$ is nondegenerate, symmetric ($C^T=C$) matrix of type $n\times n$, which plays a role of a parameter whose change describes the considered class of bundle metrics along γ. If p and q are arbitrary nonnegative integers and $p+q=n$, then, due to the last equality and (2.10), the choice

$$C(\gamma)=(Y^{-1}(\gamma))^T G_{p,q} Y^{-1}(\gamma)=C^T(\gamma)$$

defines a set of n+1 (resp. for $p=0,\ \dots\ ,p=n;\ q=n-p$) bundle metrics along γ, given by the equality

$$G(\gamma(s);\gamma;p,q)=(D^{-1}(\gamma(s)))^T G_{p,q} D^{-1}(\gamma(s)) \tag{2.12'}$$

and, evidently, depending only on the point $\gamma(s)$, but not on the map $\gamma:J\longrightarrow B$. Consequently these bundle metrics are globally consistent with the given L-transport along paths.■

3. THE CASE OF GENERATED BY DERIVATIONS OF TENSOR ALGEBRAS LINEAR TRANSPORTS OF VECTORS

In this section we shall concentrate our attention on the tangent bundle $(T(M),\pi,M)$ of a given manifold M. In particular we are going to consider in it some special questions concerning the consistency of bundle metrics and S-transports (linear transports along paths generated by derivations of tensor algebras [8]) of vectors along (smooth) paths. In this case the bundle metric g is a nondegenerate symmetric section (tensor field) of type (0,2). The bundle metrics in this fibre bundle are used to be called simply metrics [3,7], because of which the adjective "bundle", as applied to metric(s), will be omitted till the end of the present section.

The details concerning S-transports along paths and their properties can be found in [8].

In accordance with definition 1.1 the metric g and the S-transport S^γ along $\gamma:J\longrightarrow M$ are (locally) consistent along γ if

$$g_{\gamma(s)}(A_0,B_0)=g_{\gamma(t)}(S^\gamma_{s\longrightarrow t}A_0,S^\gamma_{s\longrightarrow t}B_0),\quad s,t\in J \tag{3.1}$$

for every $A_0,B_0\in T_{\gamma(s)}(M)$. The metric and the S-transport are (globally) consistent if this equality is fulfilled for every path γ.

Proposition 3.1. The C^1 metric g and the S-transport S are consistent (resp. along a given path γ) iff the equality

$$g_{\gamma(t)}=S^\gamma_{s\longrightarrow t}g_{\gamma(s)},\quad s,t\in J \tag{3.2}$$

is valid for every (resp. the given) path γ.

Proof. Using the contraction operator C^1_1 on the first superscript and the first subscript, we find

$$g(A,B)=(C^1_1)^2(A\otimes g\otimes B),\quad A,B\in\mathrm{Sec}(T(M),\pi,M). \tag{3.3}$$

If we apply $S^\gamma_{t\longrightarrow s}$ to (3.1) and take into account (3.3), the properties of the S-transports [8], and the arbitrariness of A_0 and B_0, we get that (3.1) is equivalent to $g_{\gamma(s)}=S^\gamma_{t\longrightarrow s}g_{\gamma(t)}$ which, due to the arbitrariness of $s,t\in J$, is equivalent to (3.2).∎

Let us note that by definition 5.1 from [8] the equality (3.2) means that the metric g is (S-)transported along γ section, hence proposition 3.1 is equivalent to

Proposition 3.1'. The C^1 metric g and an S-transport S are consistent (resp. along a given path γ) iff g is S-transported along every (resp. the given) path γ section.

Proposition 3.2. The S-transport S^γ along $\gamma:J\longrightarrow M$ is consistent along γ with the C^1 metric g iff

$$(\mathcal{D}^\gamma g)(\gamma(s))=\mathcal{D}^\gamma_s g=0,\quad s\in J, \tag{3.4}$$

where $\mathcal{D}^\gamma$ is the defined by or defining S^γ derivation of the tensor algebra over $\gamma(J)$ according to [8], proposition 8.

Proof. This proposition is a consequence of proposition 3.1' and proposition 5.3. of [9].∎

The equality (3.4) is useful and effective tool for practical check of the question of consistency of S-transports and C^1 metrics. A typical example for this is the Riemannian parallel transport (in (pseudo-)Riemannian manifolds) which is defined from the Levi-Cevita connection and for which (3.4) is identically satisfied [3, 4]: in this case $\nabla^{\{\}}_X g=0$, where $\nabla^{\{\}}_X$ is the defined by the Cristoffel symbols from g covariant differentiation along the vector field X and as now $\mathcal{D}^\gamma=\nabla_{\dot\gamma}$, $\dot\gamma$ being the tangent to γ vector field (cf. [2], Sect. 5), the eq. (3.4) holds identically. In (pseudo-)Riemannian manifolds the connection for which $\nabla_X g=0$ is called metric preserving as it preserves the scalar product of the (parallel) transported with its help vectors along any path [10].

For some purposes it is convenient to write the conditions (3.2) and (3.4) in local coordinates. For this goal along a path $\gamma:J\longrightarrow M$ are introduced bases $\{E_i|_{\gamma(s)}\}$ in the tangent spaces $T_{\gamma(s)}(M)$ to M at $\gamma(s)$, $s\in J$ in which the metric along γ is given from the matrix $G(\gamma(s))$ with components (2.1), and the derivation $\mathcal{D}^\gamma$ and the S-transport S^γ along γ are described with the help of the matrix $\Gamma_\gamma(s):=\|\Gamma^i_{\ .j}(s;\gamma)\|$ of the coefficients of $\mathcal{D}^\gamma$ and S in $\{E_i\}$ (see [2,8]).

Proposition 3.3. The C^1 metric g and the S-transport S are consistent (resp. along a given path γ) iff one of the following two equalities is valid

$$G(\gamma(t))=Y(t,s;\Gamma_\gamma^{\ T})G(\gamma(s))Y^T(t,s;\Gamma_\gamma^{\ T}),\quad s,t\in J, \tag{3.5}$$

$$\frac{dG(\gamma(s))}{ds}-G(\gamma(s))\Gamma_\gamma(s)-\Gamma_\gamma(s)G(\gamma(s))=0,\quad s\in J, \tag{3.6}$$

where Y is the solution of the initial-value problem (3.9) of [8], for every (resp. the given) path γ. The equalities (3.5) and (3.6) are equivalent between each other as well as, respectively, to

(3.2) and (3.4). Besides, (3.5) is the general solution of (3.6) with respect to G.

Proof. With the help of (3.4), (2.16) and [8], eq. (3.8) it is easy to check that in the chosen bases (3.5) and (3.6) are component form in matrix notation of (3.2) and (3.4) respectively and, consequently, (3.5) and (3.6) are equivalent, respectively, to (3.2) and (3.4). From here and propositions 3.1 and 3.2 follows the first part of the proposition. From the same propositions follows also the equivalence of (3.2) and (3.4), and hence the equivalence of (3.5) and (3.6). If we look on (3.6) as an equation with respect to G (resp. along γ), then from the definition of Y it is clear that its general solution is given by (3.5) in which s has to be fixed and $G(\gamma(s))$ must be replaced with an arbitrary constant matrix C (resp. a matrix function $C(\gamma)$ of γ).∎

Proposition 3.4. If in a given basis one S-transport is defined by the matrix Γ_γ of its coefficients and the metric g - by the matrix G, then the S-transport and the metric are consistent (resp. along a given path γ) iff there exists a nondegenerate symmetric matrix $C(\gamma)$ such that

$$Y(s_0,s;\Gamma_\gamma{}^T)G(\gamma(s))Y(s,s_0;-\Gamma_\gamma)=C(\gamma)=C^T(\gamma) \qquad (3.7)$$

for Y defined in [8], eq. (3.9), $s\in J$, fixed $s_0\in J$ and every (resp. the given) path γ.

Proof. According to the proof of proposition 3.4 of [8] the S-transport is uniquely defined by the matrices

$$F(s;\gamma)=Y(s_0,s;-\Gamma_\gamma)=[Y(s,s_0;-\Gamma_\gamma)]^{-1},\ s\in J \qquad (3.8)$$

through (1.4). Substituting (3.8) into (2.5) (with the help of the properties of Y), we get (3.7), which shows that proposition 3.4 is a special case of proposition 2.2.∎

Proposition 3.5. If an S-transport is fixed in a given basis through the matrix Γ_γ, then any consistent along $\gamma:J\longrightarrow M$ with it metric is obtained in the same basis by the formula

$$G(\gamma(s);\gamma)=Y(s,s_0;\Gamma_\gamma{}^T)C(\gamma)Y(s_0,s;-\Gamma_\gamma), \qquad (3.9)$$

where Y is defined in [8], eq. (3.9), $s_0\in J$ is fixed and $C(\gamma)$ is nondegenerate symmetric matrix. A necessary condition for the obtained by this formula metrics to be globally consistent with the initial S-transport is $C(\gamma)$ to be independent of γ, i.e. $C(\gamma)=\text{const}$.

Proof. The first part of the proposition follows, analogously to the proof of proposition 3.4., from proposition 2.3 for $F(s;\gamma)=Y(s_0,s;-\Gamma_\gamma)$ and the fact that (3.9) is the general solution of (3.6) with respect to G (see proposition 3.3). The second part of the proposition follows from the circumstance that if in (3.9) we put $s=s_0$, we get

$$G(\gamma(s_0);\gamma)=C(\gamma). \qquad (3.10)$$

Hence, if the given by (3.9) metrics and the defined by Γ_γ S-transport are (globally) consistent, then $C(\gamma)$ will not depend on γ as $G(\gamma(s_0);\gamma)$ is simply the value of the matrix G(x), which represents g in the given basis, at the point $\gamma(s_0)$, but, evidently, $G(\gamma(s_0))$ does not depend on γ.∎

Proposition 3.6. Let $\gamma:J\longrightarrow M$ be a C^1 path and g be a C^1 metric (resp. along γ) which in a basis $\{E_i|_{\gamma(s)}\}$ along γ is represented by the matrix G(x), $x\in\gamma(J)$, whose signature does not depend on the point x. Let D(x) be an orthogonal matrix such that [6]

$$D^T(x)G(x)D(x)=\tilde{G}(x):=\text{diag}(g_1(x),\dots,g_n(x)),\ D^T=D^{-1}, \qquad (3.11)$$

where $g_i(x)\neq 0,\infty$, $i=1,\dots,n$ are the eigenvalues of G(x) and

$$K(\gamma(s)):=D^T(\gamma(s))\cdot\frac{dG(\gamma(s))}{ds}\cdot D(\gamma(s))=K^T(\gamma(s)),\ s\in J. \qquad (3.12)$$

Then in a considered basis the matrix $\Gamma_\gamma(s)$ of the coefficients of all consistent (resp. along γ) with the metric g S-transports has the form

$$\Gamma_\gamma(s)=D(x)[P(x)+Q(x)\tilde{G}(x)+R(x)]D^T(x)=$$

$$=D(x)[P(x)+R(x)]D^T(x)+D(x)Q(x)D^T(x)G(x),\ x=\gamma(s),\ s\in\gamma(J), \qquad (3.13)$$

where the matrices P, Q, R have the following symmetries:

$$P^T=P,\ Q^T=-Q,\ R^T=-R \qquad (3.14a)$$

and their components are:

$$\left.\begin{aligned} &Q_{ij}(x):=0,\\ &R_{ij}(x):=K_{ij}(x)/2g_i(x), \end{aligned}\right\}\ \text{for } g_i(x)+g_j(x)=0, \tag{3.14b}$$

$$\left.\begin{aligned} &P_{ij}(x):=K_{ij}(x)/(g_i(x)+g_j(x)),\\ &R_{ij}(x):=0, \end{aligned}\right\}\ \text{for } g_i(x)+g_j(x)\neq 0, \tag{3.14c}$$

the remaining of which can be chosen arbitrarily (only if (3.14a) are satisfied).

Proof. In fact we have to prove that (3.13) is the general solution of (3.6) along γ with respect to Γ_γ when G is given.

Multiplying (3.6) on the left by $D^T(\gamma(s))$ and on the right by $D(\gamma(s))$, and using (3.11), we get

$$\tilde{G}\tilde{\Gamma}+\tilde{\Gamma}\tilde{G}=K, \tag{3.15}$$

where $\tilde{\Gamma}:=D^T\Gamma_\gamma D$ and here, as well as below in this proof, for brevity we omit everywhere the arguments s and $\gamma(s)$, $s\in J$. When written in component form (see (3.11)) this equation will be equivalent to

$$g_i\tilde{\Gamma}_{ij}+g_j\tilde{\Gamma}_{ji}=K_{ij}. \tag{3.15′}$$

(Do not sum here over i and j!)

Let us consider the pairs (i,j) for which $g_i+g_j=0$. Then (3.15′) reduces to $g_i(\tilde{\Gamma}_{ij}-\tilde{\Gamma}_{ji})=K_{ij}$, hence we can find only the antisymmetric part of the element $\tilde{\Gamma}_{ij}$. So, using the identity $\tilde{\Gamma}_{ij}=\frac{1}{2}(\tilde{\Gamma}_{ij}+\tilde{\Gamma}_{ji})+\frac{1}{2}(\tilde{\Gamma}_{ij}-\tilde{\Gamma}_{ji})$, we get $\tilde{\Gamma}_{ij}=P_{ij}+R_{ij}+Q_{ij}g_j$, where $R_{ij}:=$ $:=K_{ij}/2g_i=:-R_{ji}$, $Q_{ij}:=-Q_{ji}:=0$ and the quantities $P_{ij}:=P_{ij}$ are arbitrary.

Now we shall consider the pairs (i,j) for which $g_i+g_j\neq 0$. In this case we define the quantities $P_{ij}=P_{ji}$, i.e. the remaining components of the matrix $P=P^T$, as the symmetric solution of the equation (3.15′), i.e. $g_iP_{ij}+g_jP_{ji}=K_{ij}$, and hence $P_{ij}=K_{ij}/(g_i+g_j)=$ $=P_{ji}$. Then, if we put $\tilde{\Gamma}_{ij}=:P_{ij}+Q_{ij}g_j+R_{ij}$, $R_{ij}:=-R_{ji}:=0$, we see that (3.15′) reduces to $Q_{ij}+Q_{ji}=0$, i.e. the only restriction on the quantities Q_{ij} is their antisymmetry.

Thus we proved that $\tilde{\Gamma}=P+Q\tilde{G}+R$, where P, Q and R are defined by (3.14), is the general solution of (3.15) with respect to $\tilde{\Gamma}:=D^T\Gamma_\gamma D$. From here it follows that $\Gamma_\gamma=(D^T)^{-1}\tilde{\Gamma}D^{-1}=D\tilde{\Gamma}D^T$ is given by (3.13) and it is the general solution of (3.6) with respect to Γ_γ.∎

4. CONSISTENCY BETWEEN LINEAR TRANSPORTS ALONG PATHS AND HERMITIAN BUNDLE METRICS

In this section with (E,π,B) is denoted an arbitrary complex vector bundle [1,11].

By a Hermitian bundle metric g in (E,π,B) we understand (see [1] and [11], ch. I, §8) a map $g:x\longmapsto g_x$, $x\in B$, where the maps

$$g_x:\pi^{-1}(x)\times\pi^{-1}(x)\longrightarrow\mathbb{C} \tag{4.1}$$

have the properties: $1\frac{1}{2}$ linearity, nondegeneracy and Hermiticity, i.e.

$$g_x(u+v,w)=g_x(u,w)+g_x(v,w),\quad u,v,w\in\pi^{-1}(x), \tag{4.2a}$$

$$g_x(\lambda u,\mu v)=\lambda\bar{\mu}g_x(u,v),\quad \lambda,\mu\in\mathbb{C},\quad u,v\in\pi^{-1}(x), \tag{4.2b}$$

$$g_x(u,v)\neq 0 \text{ for } u,v\neq 0\in\pi^{-1}(x), \tag{4.2c}$$

$$g_x(u,v)=\overline{g_x(v,u)}, \tag{4.2d}$$

where the bar over a complex number or a matrix means complex conjugation. In this definition we neglect the usual [11] condition $g_x(u,u)>0$ for $u\neq 0$ as insignificant for the following investigation.

The studied below L-transports along paths in the complex vector bundle (E,π,B) are supposed to be $\mathbb{C}$-linear (cf. [2], Sect. 2 or [14], Subsect. 2.3).

Evidently, in the real case, i.e. when $\mathbb{C}$ is replaced with $\mathbb{R}$, a Hermitian bundle metric reduces to the defined in section 1 real bundle metric. Therefore any result for Hermitian bundle metrics is also valid for real bundle metrics. The contrary is mutatis mutandis true, i.e. with certain changes in the formulations of the definitions and propositions of sections 1 and 2 they remain valid also in the Hermitian case. The present section is intended for the description of these changes.

The basic definition 1.1 now reads.

Definition 4.1. The linear transport along paths L and the Hermitian bundle metric g are called consistent (resp. along the path

γ) if L preserve the Hermitian scalar products of the vectors along every (resp. along the given) path $\gamma:J\longrightarrow B$, i.e. if

$$g_{\gamma(s)}=g_{\gamma(t)}\circ(L^{\gamma}_{s\longrightarrow t}\times L^{\gamma}_{s\longrightarrow t}),\quad s,t\in J \tag{4.3}$$

for every (resp. for the given) path γ.

Let $\gamma:J\longrightarrow B$ and in every fibre $\pi^{-1}(\gamma(s))$, $s\in J$ be fixed a (complex) basis $\{e_i(s):\ i=1,\dots,n=\dim_{\mathbb{C}}(\pi^{-1}(x)),\ x\in B\}$. In them the L-transport $L^{\gamma}_{s\longrightarrow t}$ along γ from s to t is uniquely defined by its, generally complex, matrix $H(t,s;\gamma)=\|H^{i}_{\ j}(t,s;\gamma)\|$ (see [2]); the metric g at the point $\gamma(s)$ is defined by the matrix $G(\gamma(s)):=\|(g_{\gamma(s)})_{ij}\|$, $i,j=1,\dots,n=\dim_{\mathbb{C}}(\pi^{-1}(\gamma(s))$ with the defined by (2.1) elements. In terms of $G(\gamma(s))$ the nondegeneracy and Hermiticity of g mean, respectively, $\det(G(\gamma(s)))\neq 0,\infty$ and $G^{*}(\gamma(s))=G(\gamma(s))$, where $*$ means Hermitian conjugation of matrices ($G^{*}:=\bar{G}^{T}=(\bar{G})^{T}$; see [6]), i.e. G is a nondegenerate Hermitian matrix function. This follows from the fact that if $u=u^{i}e_{i}(s)$ and $v=v^{i}e_{i}(s)$, then due to (4.2), we have

$$u^{i}\overline{v^{j}}g_{\gamma(s)}(e_i(s),e_j(s))=g_{\gamma(s)}(u,v)=\overline{g_{\gamma(s)}(v,u)}=$$

$$=\overline{v^{j}\overline{u^{i}}g_{\gamma(s)}(e_j(s),e_i(s))}, \tag{4.4}$$

i.e. $\det\|(g_{\gamma(s)})_{ij}\|\neq 0$ and $(g_{\gamma(s)})_{ij}=\overline{(g_{\gamma(s)})_{ji}}$.

Below we present the analogs of propositions 2.1-2.6, respectively, with numbers 4.1-4.6 for Hermitian bundle metrics.

Proposition 4.1. A necessary and sufficient condition for global (resp. local) consistency of a Hermitian bundle metric g and an L-transport along paths L is the equality

$$G^{T}(\gamma(s))=H^{*}(t,s;\gamma)G^{T}(\gamma(t))H(t,s;\gamma),\quad s,t\in J \tag{4.5}$$

for every (resp. for a given) path $\gamma:J\longrightarrow B$.

Proof. Applying (4.3) to (u,v), $u,v\in B$, using (4.4) and the arbitrariness of (u,v), we get

$$(g_{\gamma(s)})_{ij}=(g_{\gamma(t)})_{kl}H^{k}_{\ .i}(t,s;\gamma)\overline{H^{l}_{\ \cdot j}(t,s;\gamma)}, \tag{4.5'}$$

from where taking a complex conjugate and taking into account $G^{T}=\bar{G}$, due to $G^{*}=G$, we obtain the component form of (4.5).■

Proposition 4.2. If the nondegenerate (complex) matrix function $F(s;\gamma)$ defines an L-transport along paths L by (1.4) then a necessary and sufficient condition for the global (resp. local) consistency of L with a Hermitian bundle metric g is the existence of a nondegenerate Hermitian matrix function C of γ, such that

$$(F^{-1}(s;\gamma))^{*}G^{T}(\gamma(s))F^{-1}(s;\gamma)=C(\gamma),\quad s\in J,\quad C^{*}=C \tag{4.6}$$

for every (resp. a given) path γ.

Proof. The proof of this proposition is an exact copy of the proof of proposition 2.2 and it reduces to the substitution of (1.4) into (4.5) and the separation in the obtained equality the terms at $\gamma(s)$ to the left of the equality sign and those at $\gamma(t)$ - to its right side.■

As is known [2] the function F, appearing in (4.6), is defined up to the transformation (1.5), so the function C in (4.6) is not uniquely defined and analogously to (2.7) one finds the implication

$$F(s;\gamma)\longmapsto D(\gamma)F(s;\gamma)\ \Leftrightarrow\ C(\gamma)\longmapsto(D^{-1}(\gamma))^{*}C(\gamma)D^{-1}(\gamma). \tag{4.7}$$

Proposition 4.3. If an L-transport along paths is defined by the matrix function F through (1.4), then any consistent with it along $\gamma:J\longrightarrow B$ bundle Hermitian metric along γ is given by

$$G^{T}(\gamma(s);\gamma)=(F(s;\gamma))^{*}C(\gamma)F(s;\gamma),\quad s\in J,\quad C^{*}=C, \tag{4.8}$$

where C is nondegenerate Hermitian matrix function of γ.

Proof. Solving (4.6) with respect to $G(\gamma(s))$, when F is fixed and C is arbitrary, we get (4.8).■

Proposition 4.4. A necessary and sufficient condition for the existence of globally (resp. locally) consistent with a given Hermitian bundle metric L-transport along paths is the independence of the signature (and consequently, of the number of positive (and/or negative) eigenvalues) of that metric, i.e. of the matrix G(x), from the point of the manifold M (resp. of the path) at which it is evaluated, i.e. of $x\in B$ (resp. $x\in\gamma(J)$).

Proof. The proof of this proposition in form coincides with the one of proposition 2.4 and as the latter is simple but long enough here we shall present only the changes which must be done in it for obtaining the needed proof in the Hermitian case.

1. The matrix $C_0(\gamma)$ is nondegenerate, Hermitian and such that $(F_0^{-1}(s;\gamma))^{*}G^{T}(\gamma(s))F_0^{-1}(s;\gamma)=C_0(\gamma)=C_0^{*}(\gamma)$.

2. The matrix $D_0(\gamma)$ is unitary ($D_0^{-1}=D^{*}$) and such that $C(\gamma)=$

$=D_0^*(\gamma)C_0(\gamma)D_0(\gamma)=\mathrm{diag}(c_1,\dots,c_n)$, $c_1,\dots,c_n\in\mathbb{C}$, $n=\dim_{\mathbb{C}}(\pi^{-1}(x))$, $x\in B$.

3. Further in the proof the matrix functions D and D_1 are unitary (not orthogonal), D^T and D_1^T must be replaced, respectively, with D^* and D_1^* (due to $G^*=G$, but not $G^T=G$) and one must have in mind that the eigenvalues of the Hermitian matrices (in this case C_0 and G) are real (e.g. $c_1,\dots,c_n\in\mathbb{R}\subset\mathbb{C}$) [6].∎

Proposition 4.5. Let in (E,π,B) be given a Hermitian bundle metric g the signature of which is independent of the point $\gamma(s)$ at which it is evaluated for every (resp. some) path $\gamma:J\longrightarrow B$. Let there be chosen bases $\{e_i(s)\}$, $s\in J$ along γ such that the first p eigenvalues of the Hermitian matrix $G(\gamma(s))$, defining the metric in them (see (2.1)), be positive. Then one L-transport along paths is consistent with this Hermitian metric if and only if some of the defining it by (1.4) matrix functions F has the form

$$F(s;\gamma)=Y(\gamma)Z(s;\gamma)(D(\gamma(s)))^{-1},\quad s\in J \tag{4.9}$$

for every (resp. the given) path γ. In this equality: $Y(\gamma)$ is $n\times n$, $n:=\dim(\pi^{-1}(x))$, $x\in B$ nondegenerate depending only on γ matrix; $Z(s;\gamma)$ is a pseudo-unitary matrix of type (p,q), $q=n-p$, i.e.

$$(Z(s;\gamma))^*G_{p,q}Z(s;\gamma)=G_{p,q}:=\mathrm{diag}(\underbrace{1,\dots,1}_{p\ \text{times}},\underbrace{-1,\dots,-1}_{q\ \text{times}}), \tag{4.10}$$

and $D(\gamma(s))$ is a fixed matrix such that

$$(D(\gamma(s)))^*G^T(\gamma(s))D(\gamma(s))=G_{p,q}. \tag{4.11}$$

Proof. To prove the necessity we have to solve the equation (4.6) with respect to F when G and C are given. From $G^*=G$ it follows $(G^T)^*=G^T$ which combined with the choice of $\{e_i(s)\}$ and the independence of p (or/and q) from $\gamma(s)$ leads to the existence of a satisfying (4.11) unitary matrix $D(\gamma(s))$ [6].

Let $F(s;\gamma)=:F_1(s;\gamma)(D(\gamma(s)))^{-1}$. Then from (4.6) and (4.11), we get $(F_1^{-1}(s;\gamma))^*G_{p,q}F_1^{-1}(s;\gamma)=C(\gamma)$. Putting here $F_1(s;\gamma)=:Y(\gamma)Z(s;\gamma)$, where $Y(\gamma)$ is an arbitrary nondegenerate (unitary) matrix for which $Y^*(\gamma)C(\gamma)Y(\gamma)=G_{p,q}$, we see that $Z(s;\gamma)$ satisfies (4.10). (The existence of $Y(\gamma)$ is a consequence of (4.6): from it and the inertial law of Jacobi-Sylvester it follows that $C(\gamma)$ has p positive eigenvalues, hence the sought matrix $Y(\gamma)$ exists [6].) All this proves that $F(s;\gamma)$ has the form (4.9) under the condition that the considered L-transport along paths is consistent along every (resp. some) path γ with the given Hermitian bundle metric.

On the contrary, the sufficiency of the proposition is almost evident: if (4.9) is valid, then an elementary checking shows that (4.6) is true for $C(\gamma)=(Y^{-1}(\gamma))^*G_{p,q}Y^{-1}(\gamma)$ and according to proposition 4.2 the L-transport along paths and the Hermitian bundle metric are consistent along every (resp. some) path γ.∎

Proposition 4.6. A necessary and sufficient condition for the existence of a globally consistent with a given L-transport along paths Hermitian bundle metrics is the existence of a local basis in which the matrix $F(s;\gamma)$, defining this transport along $\gamma:J\longrightarrow B$ by (1.4), has the form (4.9) in which: $Y(\gamma)$ and $D(\gamma(s))$ are arbitrary nondegenerate $n\times n$, $n=\dim_{\mathbb{C}}(\pi^{-1}(x))$, $x\in B$ matrices and $Z(s;\gamma)$ is a pseudo-unitary matrix of type (p,q), $p+q=n$. Besides, if for a given L-transport along paths these conditions are fulfilled, then in the above basis all globally consistent with it Hermitian bundle metrics are described by the matrix

$$G^T(\gamma(s))=((D(\gamma(s)))^{-1})^*G_{p,q}(D(\gamma(s)))^{-1}, \tag{4.12}$$

which depends only on the point $\gamma(s)$, but not on the path γ.

Proof. The proof of this proposition is an exact copy of the one of proposition 2.6 and it can be obtained from it with the following changes: metric $\longmapsto$ Hermitian metric; transposition sign $(T)\longmapsto$ Hermitian conjugation sign $(*)$; $(2.8\text{-}10)\longmapsto(4.8\text{-}10)$; $G\longmapsto G^T$ and $G_0\longmapsto G_0^T$.∎

5. REMARKS AND COMMENTS

(1) From the proof of proposition 2.1 it is clear that (2.2) is a matrix form of (1.3) in the basis $\{e_i(s)\}$, $s\in J$.

(2) In (2.8) the uncertainty in the choice of F (see (1.5)) is taken by C (see (2.7)), because of which $G(\gamma(s);\gamma)$ does not depend on the concrete choice of F.

(3) Said in another way, proposition 2.5 means that if there exist consistent with a given bundle metric L-transports along paths (along a given path), then they have in some basis a matrix (1.4) in which the matrix $F(s;\gamma)$ has the form (2.9).

(4) In proposition 3.1 the condition that the metric must be of class of smoothness C^1 follows from the necessity for the equality (3.2) to have a sense and vice versa (cf. [8,9]).

(5) Proposition 3.4 follows also from proposition 3.3: it is sufficient to put $t=s_0$ into (3.5) and to denote $G(\gamma(t))$ with $C(\gamma)$.

(6) In a general form a necessary and sufficient condition for the existence of globally consistent with an S-transport metrics is given by proposition 2.6 in which eq. (3.8) has to be taken into account, due to which the needed variant of proposition 2.6 can be formulated in terms of Γ_γ, but we are not going to do this here.

(7) In proposition 3.6 the condition for the independence of the signature of the metric from the point at which it is calculated is necessary for the existence of consistent with the metric (S-)transports along paths (see proposition 2.4).

(8) If $g_i(x)+g_j(x)\neq 0$ for every $i,j=1,\dots,n$, of which type are, in particular, the Euclidean metrics, then (3.13) may be written equivalently as

$$\Gamma_\gamma(s)=\Gamma_1(x)+\Gamma_2(x)G(x),\ x=\gamma(s),\ s\in J, \tag{5.1}$$

where $\Gamma_2=-\Gamma_2^T$ is arbitrary antisymmetric matrix and $\Gamma_1=\Gamma_1^T$ is arbitrary fixed symmetric solution of (3.6) with respect to Γ_γ which under certain conditions (see below) admits the representation (see [6], chapter 12, sec. 13)

$$\Gamma_1(\gamma(s))=-\int_0^\infty(\exp(G(\gamma(s))t)\cdot\frac{dG(\gamma(s))}{ds}\cdot(\exp(G(\gamma(s))t))dt,\ s\in J. \tag{5.2}$$

Evidently, a necessary and sufficient condition for the existence of the representation (5.1) is the existence of a symmetric solution $\Gamma_1=\Gamma_1^T$ of the equation (3.6) with respect to Γ_γ. The use of the described in the proof of proposition 3.6 method gives the possibility to prove that such a solution exists iff for every pair (i,j) for which $g_i(x)+g_j(x)=0$ the equality $K_{ij}(x)=0$ is satisfied simultaneously. If (3.6) admits a symmetric solution Γ_1 and there exists at least one pair (i,j) for which $g_i(x)+g_j(x)=K_{ij}(x)=0$, then the integral in the right-hand side of (5.2) does not exist and Γ_1 admits the representation

$$\Gamma_1(x)=D(x)\Gamma_0(x)D^T(x),\ x\in\gamma(J), \tag{5.3}$$

where

$$\Gamma_0^T:=\Gamma_0 \tag{5.4a}$$

$$(\Gamma_0(x))_{ij}=K_{ij}(x)/(g_i(x)+g_j(x))\ \text{for}\ g_i(x)+g_j(x)\neq 0, \tag{5.4b}$$

and the remaining components of Γ_0, for every (i,j) such that $g_i(x)+g_j(x)=K_{ij}(x)=0$, if any, are arbitrary.

(9) The fact that all results for the consistency of real (symmetric, Riemannian) metrics and linear transports along paths are true mutatis mutandis also in the case of Hermitian metrics is not random. In fact, if we denote by $h:x\longmapsto h_x$, $x\in B$ an arbitrary Hermitian metric in the complex vector bundle (E,π,B), then

$$g=\mathrm{Re}(h)=\frac{1}{2}(h+h^T),\ \omega=\mathrm{Im}(h)=\frac{1}{2i}(h-h^T),\ i=+\sqrt{-1}, \tag{5.5}$$

where $h_x^T(u,v):=h_x(v,u)$, $\pi(u)=\pi(v)=x$, define, respectively, symmetric (Riemannian) and symplectic metrics in (E,π,B). The definition of h is equivalent to the definition of g or ω, which is a corollary from

$$\omega_x(u,v)=g_x(u,Jv), \tag{5.6}$$

where the complex structure J on (E,π,B) is defined by $Ju=iu$. It is important to note that

$$h=h\circ(J\times J),\ g=g\circ(J\times J),\ \omega=\omega\circ(J\times J). \tag{5.7}$$

Due to this the definition of an arbitrary symmetric (complex) metric g with the property $g=g\circ(J\times J)$ allows a Hermitian metric $h=g+ig\circ(id_E\times J)$ to be introduced. (In [12] g itself is called a Hermitian metric.) The existence of g with the needed property follows from the known fact that if g_0 is a symmetric metric, then the metric $g_0+g_0\circ(J\times J)$ has the pointed property.

Namely the fact that the definition of a Hermitian metric h is equivalent to the definition of a symmetric metric g with the property $g=g\circ(J\times J)$ is the reason that any result concerning symmetric metrics can be formulated mutatis mutandis also for Hermitian metrics.

The above connections between Hermitian, Riemannian and symplectic metrics are not new and, for instance, can be found in the article "Hermitian Metric" in [13].

(10) The presented here material admits a generalization concerning arbitrary transports along paths in fibre bundles and bundle morphisms between them which will be a subject of other work.

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