94-16

сообщения
объединенного
института
ядерных
исследований
дубна

E5-94-16

B.Z.Iliev*

TRANSPORTS ALONG PATHS IN FIBRE BUNDLES.
II. Ties with the Theory of Connections and Parallel Transports

*Permanent address: Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, blvd. Tzarigradsko Chaussée 72, 1784 Sofia, Bulgaria

1994

1. INTRODUCTION

In the work [1], we have considered certain aspects of the general theory of transports along paths in arbitrary fibre bundles without investigating its ties with the ones of parallel transports and connections, which is the aim of the present paper.

Sect. 2 contains a review of the theory of parallel transports in fibre bundles adapted to suit our purposes. At first, are considered parallel transports generated by connections, after which attention is paid to the axiomatic approach to the concept of parallel transport. The main result of this paper, proved in Sect. 3, is that any parallel transport, axiomatically defined or generated by a connection, is a transport along paths satisfying certain additional conditions. Also some other ties between parallel transports and transports along paths are investigated. In Sect. 4 it is shown how linear transports along paths generated by derivations of tensor algebras [2] can be regarded as (axiomatically defined) parallel transports.

In this paper, we shall use the following notation.

By (E,π,B) we denote an arbitrary fibre bundle with a base B, bundle space E and projection $\pi:E\longrightarrow B$ [25,37,17]. The fibres $\pi^{-1}(x)$, $x\in B$ are supposed to be homeomorphic.

An arbitrary real interval and a path in B are denoted, respectively, by J and $\gamma:J\longrightarrow B$. If B is a manifold, the tangent to γ vector field is written as $\dot\gamma$. The path $\bar\gamma:J\longrightarrow E$ is a lifting of $\gamma:J\longrightarrow B$ (resp. through $u\in\pi^{-1}(\gamma(J))$) if $\pi\circ\bar\gamma=\gamma$ (resp. and $\bar\gamma(J)\ni u$).

By M, $T_x(M)$ and $(T(M),\pi,M)$ we denote, respectively, a differentiable manifold, the tangent to it space at $x\in M$, and the tangent bundle to M, $T(M):=\bigcup_{x\in M} T_x(M)$.

Now for reference purposes, we shall summarize a certain material from [1].

A transport along paths I in (E,π,B) is a map $I:\gamma\longmapsto I^\gamma$, where $I^\gamma:(s,t)\longmapsto I^\gamma_{s\to t}$, $s,t\in J$ in which the maps

$$I^\gamma_{s\to t}:\pi^{-1}(\gamma(s))\longrightarrow\pi^{-1}(\gamma(t)), \tag{1.1}$$

satisfy the equalities (id_X is the identity map of the set X):

$$I^\gamma_{t\to r}\circ I^\gamma_{s\to t}=I^\gamma_{s\to r},\quad r,s,t\in J, \tag{1.2}$$

$$I^{\gamma}_{s\longrightarrow s}=id_{\pi^{-1}(\gamma(s))}, \quad s\in J. \tag{1.3}$$

It is easily seen that

$$\left(I^{\gamma}_{s\longrightarrow t}\right)^{-1}=I^{\gamma}_{t\longrightarrow s}. \tag{1.4}$$

Important special classes of transports along paths are selected by one or both of the conditions

$$I^{\gamma|J'}_{s\longrightarrow t}=I^{\gamma}_{s\longrightarrow t}, \quad s,t\in J', \tag{1.5}$$

$$I^{\gamma\circ\tau}_{s\longrightarrow t}=I^{\gamma}_{\tau(s)\longrightarrow\tau(t)}, \quad s,t\in J'', \tag{1.6}$$

where $\gamma|J'$ is the restriction of γ on the subinterval J' of J and $\tau:J''\longrightarrow J$ is one-to-one map from the interval $J''\subset\mathbb{R}$ onto J.

As for the transports along paths the types of the intervals J, J' and J'' are insignificant, in this work, for purposes which will be cleared up later, all real intervals are supposed to be closed, i.e. of type $[a,b]$ for some $a\le b$, $a,b\in\mathbb{R}$.

If the fibres $\pi^{-1}(x)$, $x\in B$ are differentiable manifolds (e.g. when (E,π,B) is smooth [40]), one can consider the class of smooth transports along paths obeying the condition

$$I^{\gamma}_{s\longrightarrow t}\in \mathrm{Diff}(\pi^{-1}(\gamma(s)),\pi^{-1}(\gamma(t))), \quad s,t\in J, \tag{1.7}$$

where $\mathrm{Diff}(M,N)$ is the set of diffeomorphisms from the manifold M on the manifold N.

Any path $\gamma:[0,1]\longrightarrow B$ is called canonical (canonically defined). Its inverse path is $\gamma_{-}:=\gamma\circ\tau^{c}_{-}$, where $\tau^{c}_{-}:[0,1]\longrightarrow[0,1]$ is given by $\tau^{c}_{-}(s):=1-s$, $s\in[0,1]$ [38,39]. The (canonical) product of the paths $\gamma_1,\gamma_2:[0,1]\longrightarrow B$ is the path $\gamma_1\gamma_2:[0,1]\longrightarrow B$ such that $(\gamma_1\gamma_2)(s):=\gamma_1(2s)$ for $s\in[0,1/2]$ and $(\gamma_1\gamma_2)(s)=\gamma_2(2s-1)$ for $s\in[1/2,1]$ [38,39].

2. REVIEW OF PARALLEL TRANSPORTS IN FIBRE BUNDLES

This section contains a brief review of the concept "parallel transport" in fibre bundles. It will be a basis for comparison of the parallel transport with the transports along paths studied in [1-3].

A common feature of most of the works [4,5,6-8,12,15-18,21-35]

dedicated to that problem is that in them as a basic object is taken the connection (in corresponding fibre bundles) and with its help the parallel transport is defined. In connection with this, one can distinguish the works [9-11,13,14,19,20,36] in which as an initial (axiomatically given) object one takes the parallel transport which, in its turn, defines (and sometimes is identified with) the connection. It has to be noted that in these works in contrast to our considerations in [1-3], main attention is paid to the dependence of the parallel transport on the curve (path) along which it is made.

2.1. PARALLEL TRANSPORT IN DIFFERENTIABLE FIBRE BUNDLES ENDOWED WITH CONNECTION

Let (E,π,B) be locally trivial differentiable and smooth (of class C^1) fibre bundle [17,25,37]. The fibre $\pi^{-1}(\pi(u))$ through $u\in E$ is a manifold the tangent space of which at u is denoted by $T^{v}_{u}(E):=T_u(\pi^{-1}(\pi(u)))$. Evidently $T^{v}_{u}(E)\subset T_u(E)$. By definition $T^{v}_{u}(E)$ consists of vertical vectors [5,25,38,40].

Definition 2.1 (cf. [5,35,40,41]). A connection (of general form) in (E,π,B) is a smooth (of class C^1) $\dim(B)$-dimensional distribution $T^h(E):E\longrightarrow T(E)$ such that the image $T^h(E):u\longmapsto T^h_u(E)$ of $u\in E$ lies in $T_u(E)$ and is a direct complement of $T^v_u(E)$ in $T_u(E)$, i.e.

$$T^{v}_{u}(E)\oplus T^{h}_{u}(E):=T_u(E), \tag{2.1}$$

where $\oplus$ is the direct sum sign. By definition $T^h_u(E)$ consists of horizontal (with respect to the connection $T^h(E)$) vectors.

Definition 2.2 (cf. [5,21,35,40,41]). The smooth (C^1) path $\bar\gamma:J\longrightarrow E$ is horizontal (with respect to the connection $T^h(E)$) if its tangent vector field $\dot{\bar\gamma}$ is horizontal, i.e. if $\dot{\bar\gamma}(s)\in T^h_{\bar\gamma(s)}(E)$.

Remark. In this definition and below we speak about smooth (or differentiable), of class C^1, paths as the corresponding generalizations for partially smooth paths are trivial.

Definition 2.3 (cf. [5,21,35,40,41]). The lift $\gamma\longmapsto\bar\gamma:J\longrightarrow E$ of $\gamma:J\longrightarrow B$, (resp. through $u\in E$) is horizontal (with respect to the connection $T^h(E)$) if $\bar\gamma$ is a horizontal path (resp. through u).

For defining the concept "parallel transport" in differentiable fibre bundles, of primary importance is the question of the <u>existence</u> of a <u>unique</u> horizontal lift of a given path from the base in the total space of the fibre bundle through any point above it. As has been pointed out in [40], p. 607, lemma 2, a sufficient condition for this is the fibre $\pi^{-1}(b)$ for some $b\in B$ to be a compact manifold. (Because of the local triviality, if this is so for some $b\in B$, the above property will be valid also for every $b\in B$.) The existence and uniqueness of the lifting mentioned are automatically fulfilled in principal fibre bundles (G-fibre bundles) [5,35,40,41] where they are assured from the additional requirement for the connection (called often a G-connection) to be invariant under the action of the structure group G of the principal fibre bundle. (More strictly, if $R_g:E\longrightarrow E$ is the right action generated from $g\in G$, then the connection $T^h(P)$ of the principal fibre bundle (P,π,B,G) is defined by the following three conditions: 1°. $T^v_u(P)\oplus T^h_u(P)=T_u(P)$, $u\in P$; 2°. $T^h_u(P)$ must depend differentiably on $u\in P$; 3°. $R_{g*}T^h_u(P)=T^h_{R_g u}(P)$, where R_{g*} is the differential of R_g [5,10,40].) Another case for the existence of a unique horizontal lift of any path from the bases through every lying above it point is when the fibres of the fibre bundle are discrete (see [17], pp. 75-76 and [38], chapter III, lemma 15.1). The above pointed problem is considered from a general point of view in chapter III of the book [38] (see e.g. sections 12, 13, 15, and 16 from it), where, in particular, are given the corresponding necessary and sufficient conditions for the existence of (maybe unique) lift of the pointed above form.

Let there be given a smooth fibre bundle (E,π,B) with connection $T^h(E)$ such that for every C^1 path $\gamma:J\longrightarrow B$ and every point $u\in\pi^{-1}(\gamma(J))$ there exists a unique horizontal lift $\gamma\longmapsto\bar\gamma_u$ of γ through u, i.e. $\bar\gamma_u:J\longrightarrow E$, $\pi\circ\bar\gamma_u=\gamma$, $u\in\bar\gamma_u(J)$ and $\dot{\bar\gamma}_u(s)\in T^h_{\bar\gamma_u(s)}(E)$, $s\in J$. (We will note that this assumption for the connection, without being mentioned, is unexplicitly used in the considerations in sections 1 and 2 of [9].) Let $J=[a,b]$, $a\le b$. Let us note that the considered below connections (and parallel transports) will be of Ehresmann's type (see [25], vol. 1, p. 314).

Definition 2.4 (cf. [5,7,15-18,21-25,35,40-42]). The parallel transport (generated by $T^h(E)$) of the fibre $\pi^{-1}(\gamma(a))$ onto the fibre $\pi^{-1}(\gamma(b))$ along the path $\gamma:[a,b]\longrightarrow B$ is a diffeomorphism

$$\varphi_\gamma:\pi^{-1}(\gamma(a))\longrightarrow\pi^{-1}(\gamma(b)), \tag{2.2}$$

such that if $u\in\pi^{-1}(\gamma(a))$, then $\varphi_\gamma:u\longmapsto\varphi_\gamma(u):=\bar\gamma_u(b)$, where

$\bar\gamma_u:[a,b]\longrightarrow E$ is the unique horizontal lift of γ in E through u.

Definition 2.5 (cf. [5,9,40]). The parallel transport defined by the connection $T^h(E)$ is a map φ from the set of C^1 paths in the base B into the group $\mathrm{Morf}(E,\pi,B)$ of the bundle morphisms of (E,π,B), such that if $\gamma:[a,b]\longrightarrow B$, then $\varphi:\gamma\longmapsto\varphi_\gamma\in\mathrm{Diff}(\pi^{-1}(\gamma(a)),\pi^{-1}(\gamma(b)))$, i.e. the image φ_γ is the defined by $T^h(E)$ parallel transport along γ, which is an element of the group $\mathrm{Diff}(\pi^{-1}(\gamma(a)),\pi^{-1}(\gamma(b)))$ of diffeomorphisms between the fibres $\pi^{-1}(\gamma(a))$ and $\pi^{-1}(\gamma(b))$.

Proposition 2.1. The parallel transport φ has the following three basic properties:

a) Invariance under orientation preserving parameter changes, i.e. if $\gamma:[a,b]\longrightarrow B$ and $\tau:[c,d]\longrightarrow[a,b]$, $c\le d$, $a\le b$ is an orientation preserving diffeomorphism, then

$$\varphi_{\gamma\circ\tau}=\varphi_\gamma. \tag{2.3}$$

b) If $\gamma_-:[0,1]\longrightarrow B$ is the (canonical) inverse to $\gamma:[0,1]\longrightarrow B$ path, i.e. $\gamma_-(s)=\gamma(1-s)$, $s\in[0,1]$, then

$$\varphi_{\gamma_-}=(\varphi_\gamma)^{-1}. \tag{2.4}$$

c) If $\gamma_1,\gamma_2:[0,1]\longrightarrow B$, $\gamma_1(1)=\gamma_2(0)$ and $\gamma_1\gamma_2$ is the (canonical) product of γ_1 and γ_2 (see Sect. 1), then

$$\varphi_{\gamma_1\gamma_2}=\varphi_{\gamma_2}\circ\varphi_{\gamma_1}. \tag{2.5}$$

Remark. Because of (2.3) it is enough to consider (2.4) and (2.5), as well as any other property of the parallel transport, only for canonically defined paths in spite of the fact that they are valid also for arbitrary ones.

Proof. The proof of this proposition can be found, for example, in [5,7-10,13,14,18,40].■

Here we shall drop the generality of the above considerations and till the end of the present section we will deal with the specific case of principal fibre bundles [5,15,21-24].

At first, let us note that in principal fibre bundles the parallel transport φ_γ along γ commutes with the right action R_g, $g\in G$ of the structure group G on the total space of the fibre bundle [8,15], i.e. $\varphi_\gamma\circ R_g=R_g\circ\varphi_\gamma$ for arbitrary path γ and every $g\in G$.

On the other hand, in these fibre bundles the parallel transport can be defined uniquely also by the right action of G (see e.g. [40], p. 632, theorem 1 and [9]). In fact, let $\gamma:[a,b]\longrightarrow B$ and

$u\in\pi^{-1}(\gamma(a))$. Then, due to the local triviality of (E,π,B) (see e.g. [43], p. 48), there exist a neighborhood U of $\gamma(a)=\pi(u)$ and a diffeomorphism $\psi:\pi^{-1}(U)\longrightarrow U\times G$, $\psi(u):=(\pi(u),\chi(u))$, where $\chi:\pi^{-1}(U)\longrightarrow G$ is right invariant, i.e. $\chi(R_g u)=\chi(u)g$, $g\in G$, and hence

$$(R_g\circ\psi^{-1})(\pi(u),\chi(u))=R_g u \;=\psi^{-1}(\pi(R_g u),\chi(R_g u)) \;=\psi^{-1}(\pi(u),\chi(u)g).$$

Denoting by e the unit of G, we find:

$$\varphi_\gamma(u) = (\varphi_\gamma\circ\psi^{-1})(\gamma(a),\chi(u)) = (\varphi_\gamma\circ\psi^{-1})(\gamma(a),e\chi(u)) = (\varphi_\gamma\circ R_{\chi(u)}\circ\psi^{-1})(\gamma(a),e) = (R_{\chi(u)}\circ\varphi_\gamma\circ\psi^{-1})(\gamma(a),e) = (R_{\chi(u)}\circ\psi^{-1})(\gamma(b),\phi_\gamma),$$

where in the last equality we have used the fact that $(\varphi_\gamma\circ\psi^{-1})(\gamma(a),e)\in\pi^{-1}(\gamma(b))$ and consequently there exists a unique $\phi_\gamma\in G$, which does not depend on u and is such that $(\varphi_\gamma\circ\psi^{-1})(\gamma(a),e)=\psi^{-1}(\gamma(b),\phi_\gamma)$. So, in principal fibre bundles the parallel transport φ_γ along γ is given by the equality $\varphi_\gamma(u)=(R_{\chi(u)}\circ\psi^{-1})(\gamma(b),\phi_\gamma)$. Hence, the definition of a parallel transport φ is equivalent to the definition of a map ϕ from the set of C^1 paths in B onto G such that $\phi:\gamma\longmapsto\phi_\gamma$.

Proposition 2.2. The map $\phi:\gamma\longmapsto\phi_\gamma$ has the properties:

$$\phi_{\gamma\circ\tau}=\phi_\gamma, \tag{2.6}$$

$$\phi_{\gamma_-}=(\phi_\gamma)^{-1}, \tag{2.7}$$

$$\phi_{\gamma_1\gamma_2}=\phi_{\gamma_1}\phi_{\gamma_2}, \tag{2.8}$$

where γ, τ, γ_-, γ_1, γ_2, and $\gamma_1\gamma_2$ are defined in proposition 2.1.

Proof. The equalities (2.6)-(2.8) follow from the definition of ϕ_γ and, respectively, the equalities (2.3)-(2.5).■

Let us note that in some works, e.g. in [11,17,19,26,36,40], the third property of a parallel transport is expressed not through the equality (2.5), but by

$$\varphi_{\gamma_1\gamma_2}=\varphi_{\gamma_1}\circ\varphi_{\gamma_2}, \tag{2.5'}$$

which, generally, is not true when using the accepted by us notions: as φ_γ acts on the <u>left</u>, then (2.5) is valid but not (2.5′). For (2.5′) to be valid, as pointed out in [17], p. 76, one has to change the orientations of γ_1, γ_2, and $\gamma_1\gamma_2$; in fact, from (2.5) and (2.4) it follows that

$$\varphi_{(\gamma_1\gamma_2)_-}=(\varphi_{\gamma_1\gamma_2})^{-1}=(\varphi_{\gamma_2}\circ\varphi_{\gamma_1})^{-1}=(\varphi_{\gamma_1})^{-1}\circ(\varphi_{\gamma_2})^{-1}=\varphi_{(\gamma_1)_-}\circ\varphi_{(\gamma_2)_-},$$

i.e.

$$\varphi_{(\gamma_1\gamma_2)_-}=\varphi_{(\gamma_1)_-}\circ\varphi_{(\gamma_2)_-}. \tag{2.9}$$

So, if we make the change $\varphi_\gamma\longmapsto\varphi_{\gamma_-}=(\varphi_\gamma)^{-1}$, (2.5′) will be valid but not (2.5). Such is the case, for instance, in the works [10, 11,19, 20] in which the parallel transport is defined as the map $\varphi':\gamma\longmapsto\varphi'_\gamma:=\varphi_{\gamma_-}:\pi^{-1}(\gamma(b))\longrightarrow\pi^{-1}(\gamma(a))$, for which, due to (2.9), (2.5′) is true.

As regards the property (2.8) (in principal fibre bundles) in its right hand side the terms are written in a needed order as ϕ_γ acts on the <u>right</u> but not on the left as φ_γ.

At the end of this section, we shall stress the fact that the properties (2.3)-(2.5) of the parallel transport φ express their dependence on the curve of transport. From this viewpoint, there naturally arises the question of the "continuity" or "differentiability" (the "smoothness") of that dependence. The author knows two approaches to that problem. First, in the set of smooth (of class C^1) paths a topology is introduced (see e.g. [14] and [38], p. 104) which, in particular, may be generated by some metric (for a case of closed paths see [13]), which is used to study the smoothness of the map $\varphi:\gamma\longmapsto\varphi_\gamma$. And second, a (generally multidimensional) smooth deformation of γ is made and the dependence of φ_γ on that deformation [9] is investigated, i.e. the class of homotopic with γ paths connecting $\gamma(a)$ and $\gamma(b)$ is considered and the dependence of a parallel transport along these paths on the parameters of the used homotopy is investigated [38].

2.2. AXIOMATIC APPROACH TO THE PARALLEL TRANSPORT IN LOCALLY TRIVIAL FIBRE BUNDLES

The axiomatic definition of a parallel transport in locally trivial fibre bundles is based on the idea of a (diffeomorphic) mapping of the fibres of a given fibre bundle one onto another. More precisely, in the known to the author literature [9-11,19,20,36,39,40] in which this question is set, it is put in the following way. Let (E,π,B) be a locally trivial fibre bundle and $x_1,x_2\in B$. To any path $\gamma:J\longrightarrow B$, where $J=[a,b]$, in the base B connecting x_1 and x_2, i.e. for which $\gamma(a)=x_1$ and $\gamma(b)=x_2$, a map (diffeomorphism) $\varphi_\gamma:\pi^{-1}(x_1)\longrightarrow\pi^{-1}(x_2)$ is put into correspondence and the dependence of φ_γ on γ is axiomatically defined. Namely, on φ_γ are imposed two kinds of restrictions. Firstly, these are conditions of a functional type defining the "change" of φ_γ when with the path γ some operation is made (e.g. changing its orientation or its representation as a product of other paths). Secondly, in an appropriate way the "smoothness" of the map $\gamma\longmapsto\varphi_\gamma$ (conditions for smoothness) is defined. We shall note that the defined in this way parallel transport is sometimes called a global or an integral connection in the fibre bundle [9,19].

A scheme for solving the stated above problem for an axiomatic definition of the parallel transport in locally trivial fibre bundles has been introduced, maybe for the first time, in the work [19], after which, with little changes (following the context or using some features in different special cases (e.g. in principal or homogeneous (associated) fibre bundles)), it is repeated in other publications of the same author [10,11,36].

The above question, but in the "infinitesimal" case (the points x_1 and x_2 are infinitely near in a coordinate sense), is investigated in the works of G.F. Laptev (see [20] and the given therein references of the printed works of G.F. Laptev).

Ref. [9] contains a more general consideration of the problem, which is analogous to the one of Subsect. 2.1, but in [9] a more general concept for connection ("infinitesimal nonlinear" connection) is used which is due to the replacement of the tangent spaces to the corresponding manifolds with the Grassmanian manifolds consisting of their one dimensional (linear) subspaces.

In [40], part II, sect. 24 the above question is described but, in fact, only a construction of a parallel transport by the method described in Subsect. 2.1 is made.

In the above sense, the defined in [39], sect. 3.2 transport along paths in an assembly of groups (a (flat) topological fibre bundle, the fibres of which are groups) is also a parallel transport.

Form here till the end of the present subsection we shall make comments on the axiomatic definition of the parallel transport in the mentioned above references and, in connection with our purposes attention will be paid mainly to the conditions of functional character.

Before going on, let us note that in the cited literature instead of an arbitrary closed interval $J=[a,b]$ the unit interval

$I=[0,1]$ is used, i.e. $I=J|_{a=0,b=1}$. This is not important because of the invariance of the parallel transport under orientation preserving changes of the parameter of the paths along which it acts (see below eq. (2.11)).

Let (E,π,B) be a locally trivial smooth fibre bundle, $J=[a,b]$, $x_1,x_2\in B$, $\gamma:J\longrightarrow B$ be a C^1 path, $\gamma(a)=x_1$ and $\gamma(b)=x_2$. The parallel transport in (E,π,B) is a map φ from the set of C^1 paths in the base B [38] onto the group $\mathrm{Morf}(E,\pi,B)$ of bundle morphisms of (E,π,B) [17,22,25], such that

$$\varphi:\gamma\longmapsto\varphi_\gamma\in\mathrm{Diff}(\pi^{-1}(\gamma(a)),\pi^{-1}(\gamma(b))). \tag{2.10}$$

The first group of restrictions imposed on φ_γ usually, contains (2.3)-(2.5), i.e. it is wanted that

$$\varphi_{\gamma\circ\tau}=\varphi_\gamma, \tag{2.11}$$

$$\varphi_{\gamma_-}=(\varphi_\gamma)^{-1}, \tag{2.12}$$

$$\varphi_{\gamma_1\gamma_2}=\varphi_{\gamma_2}\circ\varphi_{\gamma_1}, \tag{2.13}$$

where γ, τ, γ_1, γ_2 and $\gamma_1\gamma_2$ are defined in proposition 2.1.

The conditions (2.11)-(2.13), which generally are independent, are postulated, for example, in [10,11,19,36], where instead of φ_γ $\sigma^\gamma:=\Gamma(\gamma):=\varphi_{\gamma_-}$ is used, as a consequence of which (2.13) is written in the form (2.5′) (with σ^γ instead of φ_γ, and, besides, the paths γ and $\gamma':=\gamma\circ\tau$ are called equivalent, which is denoted by $\gamma\sim\gamma'$, and (2.11) is written as $\sigma^\gamma=\sigma^{\gamma'}$ for $\gamma\sim\gamma'$).

In [40] the restrictions (2.11)-(2.13) are mentioned but, in fact, they are not used for an axiomatic construction of parallel transports.

In [9], attention is paid uniquely to the condition (2.13) which taken together with the corresponding condition for smoothness defines therein φ as an integral connection of the fibre bundle. As in this work the full proofs of the stated there propositions are not given, part of which are not correct (e.g. the existence of a unique lift is supposed (see Subsect. 2.1); something which generally is not true (see e.g. [38,40])), the author of the present text was not able to re-establish them to an end, so it is not clear whether (2.11), (2.12) or some other restrictions on φ are used unexplicitly in [26].

Usually, as a consequence of other restrictions (resp. independently) (see e.g. [11,19,20]) φ satisfies (resp. on φ is imposed) the restriction

$$\varphi_{\gamma_a} = id_{\pi^{-1}(x_a)}, \quad \gamma_a:\{a\}\longrightarrow\{x_a\},\ x_a\in B,\ a\in\mathbb{R}, \tag{2.14}$$

i.e. to the degenerated into a point path there corresponds (resp. to correspond) the identity map of the fibre over that point.

For example, in [11,19] $\gamma_t=\gamma|[0,t]$, $t\in[0,1]$ is put to be the restriction of γ on $[0,t]\subset[0,1]$ and it is required that $\lim_{t\longrightarrow 0}\varphi_{\gamma_t} = id_{\pi^{-1}(\gamma(0))}$ (a functional condition) and that the principal part of the deviation of φ_{γ_t} from $id_{\pi^{-1}(\gamma(0))}$ should depend smoothly on γ_t and $\dot\gamma_t$ (condition for smoothness) from where, evidently, follows (2.14). On the contrary, if (2.14) is taken as a base, then the first of these restrictions will be a consequence from the condition for smoothness (which, in fact, needs a concrete and strict formulation (cf. [9])).

Definition 2.6. The map $\varphi:\gamma\longmapsto\varphi_\gamma$, where φ_γ satisfies (2.10)-(2.14), is called an axiomatically defined parallel transport.

Remark. In [40], p. 608 φ is called an abstract connection.

From the described here approach to the parallel transport a little aside are the investigations of G.F. Laptev (see [20] and the references in it) due to their coordinate and local (or strictly - infinitesimal) character. As a consequence of this, the functional conditions and the conditions for smoothness (differentiability) of a parallel transport are given in a unified way (see [20], p. 46-47), not sharply separately as in our text or in [9]. From the above conditions in [20], p. 47 (see therein condition c)) only (2.14) is given, but (2.11)-(2.13) therein are a consequence from the explicit coordinate and infinitesimal form of a parallel transport. Besides, in [20] φ_{γ_-} is used instead of φ_γ.

As has already been said above, the second group of restrictions imposed on the map (2.1) are the conditions for smoothness. They are defined $\varphi:\gamma\longmapsto\varphi_\gamma$ as a continuous or differentiable (from some class C^k, $k=1,\dots,\infty,\omega$) function of γ.

In the approach used in [20] these conditions are reduced to the requirement for analyticity of the principal linear part of an explicit coordinate expression for the transport from the final point of a transport (see [20], p. 47, condition d)).

In the works of U.G. Lumiste [10,11,19,36] the question of smoothness of $\varphi:\gamma\longmapsto\varphi_\gamma$ is, in fact, replaced with the requirement for continuous differentiability (smoothness) of the map $t\longmapsto\varphi_{\gamma_t}$, $t\in[0,1]$, $\gamma_t:=\gamma|[0,t]$ with, maybe, some modifications depending on the concrete case under consideration, as is, for example, in [36], p.206, condition $\sigma3$ where the concrete properties of the homogeneous fibre bundles are used. This condition for smoothness may be put in the first of the types described at the end of Subsect. 2.1 as it uses the topology of the real line (instead of the one in the set of smooth paths in B).

We shall especially mention the work [9] where the important role of the conditions for smoothness is stressed and they themselves, in the considered there cases, are formulated strictly and clearly.

At the end of this section we shall only mention that there also exist a third group of conditions which sometimes are imposed on the map (2.1) and which are connected with the concrete structure of the investigated fibre bundles. They usually define the "intercommunications" of the map (2.1) with the (structural) group of transformations acting in the fibre bundle. Typical examples of this are the conditions $\sigma2$ and $\sigma4$ from [36], p. 205-206 which concern homogeneous fibre bundles and the condition $\varphi_\gamma\circ R_g=R_g\circ\varphi_\gamma$ for commutation of φ_γ with the right action R_g, $g\in G$ of the structure group G in the case of principal fibre bundles [8,15].

3. THE AXIOMATICALLY DEFINED PARALLEL TRANSPORT AS A SPECIAL CASE OF TRANSPORTS ALONG PATHS

Before comparing a parallel transport with transport along paths we have to note the following. The axiomatically defined parallel transport is considered usually, along canonically given paths $\gamma:[0,1]\longrightarrow B$, which is significant when defining explicitly the canonically inverse path γ_- and the canonical product of two paths (see Sect. 1 and [1], Sect. 3). Because of the invariance under parameter changes of the parallel transport (see Subsect. 2.2), this restriction is not essential and it is a question of convenience and easiness in the corresponding investigations. This circumstance shows that the parallel transport must be compared not with the general transport along arbitrary paths, but with transports I^γ along the $\gamma:J\longrightarrow B$, where J is a closed interval, i.e. $J=[a,b]$. The importance of this restriction comes from the fact that, in the general case, the transports along paths are not invariant under parameter changes, i.e. they do not satisfy (1.6), so they can explicitly depend on the path of transport.

Let I be a transport along paths in the fibre bundle (E,π,B) and $\gamma:[a,b]\longrightarrow B$. To I we assign a map $\varphi:\gamma\longmapsto\varphi_\gamma$, defined by

$$\varphi_\gamma:=I^\gamma_{a\longrightarrow b}:\pi^{-1}(\gamma(a))\longrightarrow\pi^{-1}(\gamma(b)). \qquad (3.1)$$

Lemma 3.1. If I^γ is a transport along γ satisfying additional conditions (1.5) and (1.6), then the map $\varphi:\gamma\longrightarrow\varphi_\gamma$ defined by (3.1) satisfies the equalities (2.11)-(2.13) and

$$I^\gamma_{s\longrightarrow t}=\varphi_{\gamma\circ\tau^J_t}\circ(\varphi_{\gamma\circ\tau^J_s})^{-1},\ s,t\in J=[a,b], \qquad (3.2)$$

where $\tau^J_s:[a,b]\longrightarrow[a,s]$, $s\in[a,b]$ are for $s>a$ arbitrary orientation preserving diffeomorphisms depending on γ through the interval J.

Proof. Firstly, we shall prove equality (3.2). Using sequentially (1.2), (1.4), (1.5), (1.6) and (3.1), we get:

$$I^\gamma_{s\longrightarrow t}=I^\gamma_{a\longrightarrow t}\circ I^\gamma_{s\longrightarrow a}=I^\gamma_{a\longrightarrow t}\circ(I^\gamma_{a\longrightarrow s})^{-1}=I^{\gamma|[a,t]}_{a\longrightarrow t}\circ\left(I^{\gamma|[a,s]}_{a\longrightarrow s}\right)^{-1}=$$

$$=I^{\gamma|[a,t]}_{\tau^J_t(a)\longrightarrow\tau^J_t(b)}\circ\left(I^{\gamma|[a,s]}_{\tau^J_s(a)\longrightarrow\tau^J_s(b)}\right)^{-1}=I^{\gamma\circ\tau^J_t}_{a\longrightarrow b}\circ\left(I^{\gamma\circ\tau^J_s}_{a\longrightarrow b}\right)^{-1}=\varphi_{\gamma\circ\tau^J_t}\circ(\varphi_{\gamma\circ\tau^J_s})^{-1}.$$

The property (2.11) follows from the equality (1.6): if $\tau:[c,d]\longrightarrow[a,b]$ is an orientation preserving diffeomorphism, which, in particular, means $\tau(c)=a$ and $\tau(d)=b$, then from (1.6) and (3.1), we get $\varphi_{\gamma\circ\tau}=I^{\gamma\circ\tau}_{c\longrightarrow d}=I^\gamma_{\tau(c)\longrightarrow\tau(d)}=I^\gamma_{a\longrightarrow b}=\varphi_\gamma$.

The property (2.12) is a consequence of (1.6) in which, because of $\gamma_-:=\gamma\circ\tau^c_-$, $\gamma:[0,1]\longrightarrow B$, we have to put $\tau=\tau^c_-$ (see Sect. 1). Under these assumptions, from (3.1), (1.6) and (1.4) we get

$$\varphi_{\gamma_-}=I^{\gamma_-}_{0\longrightarrow 1}=I^\gamma_{\tau^c_-(0)\longrightarrow\tau^c_-(1)}=I^\gamma_{1\longrightarrow 0}=(\varphi_\gamma)^{-1}.$$

The property (2.13) is a consequence of [1], proposition 3.4 (see therein eq. (3.4)) in the case of a canonical choice of a parameter χ, i.e. (see Sect. 1 and [1]) for $\chi=\chi^c:=(0,1;1/2;\tau^c_1,\tau^c_2)$ with $\tau^c_1:s\longmapsto 2s$, $s\in[0,1/2]$ and $\tau^c_2:s\longmapsto 2s-1$, $s\in[1/2,1]$. (It should be

noted that the proof of proposition 3.4 of [1] essentially uses the condition (1.5).) Then, from (3.1) and eq. (3.4) from [1], we get

$$\varphi_{\gamma_1\gamma_2}=I^{\gamma_1\gamma_2}_{0\longrightarrow 1}=I^{\gamma_2}_{0\longrightarrow\tau_2(1)}\circ I^{\gamma_1}_{\tau_1(0)\longrightarrow 1}=I^{\gamma_2}_{0\longrightarrow 1}\circ I^{\gamma_1}_{0\longrightarrow 1}=\varphi_{\gamma_2}\circ\varphi_{\gamma_1},$$

with $\gamma_1\gamma_2:=(\gamma_1\gamma_2)_{\chi^c}$. ■

Lemma 3.2. The defined by (3.1) map $\varphi:\gamma\longmapsto\varphi_\gamma$ for any transport along paths I has the property (2.14).

Proof. If $\gamma_a:\{a\}\longrightarrow\{x_a\}$, $x_a\in B$, $a\in\mathbb{R}$, then from (3.1) and (1.3), we get $\varphi_{\gamma_a}=I^{\gamma_a}_{a\longrightarrow a}=id_{\pi^{-1}(\gamma_a(a))}=id_{\pi^{-1}(x_a)}$. ■

Theorem 3.1. If I is a smooth transport along paths, i.e.

$$I^\gamma_{a\longrightarrow b}\in\mathrm{Diff}(\pi^{-1}(\gamma(a)),\pi^{-1}(\gamma(b))),\ \gamma:J\longrightarrow B,\ J=[a,b], \qquad (3.3)$$

having the properties (1.5) and (1.6), then the defined by (3.1) map $\varphi:\gamma\longmapsto\varphi_\gamma$ is an axiomatically defined parallel transport. Vice versa, if φ is an axiomatically defined parallel transport, then the map (3.2), in which $\tau^J_s:J\longrightarrow[a,s]$, $s\in J$ are arbitrary orientation preserving diffeomorphisms and $\gamma:J\longrightarrow B$, defines a smooth transport along paths $I:\gamma\longmapsto I^\gamma$, $I^\gamma:(s,t)\longmapsto I^\gamma_{s\longrightarrow t}$ satisfying the additional conditions (1.5) and (1.6).

Remark. If φ is an axiomatically defined parallel transport, then, because of the properties of τ^J_s, $s\in J$, we can replace in (2.11) γ with $\gamma|[a,s]$ and put in it $\tau=\tau^J_s$. In this way, we obtain $\varphi_{\gamma|[a,s]}=\varphi_{(\gamma|[a,s])\circ\tau^J_s}=\varphi_{\gamma\circ\tau^J_s}$ as $(\gamma|[a,s])\circ\tau^J_s=\gamma\circ\tau^J_s$. Therefore, (3.2) is now equivalent to

$$I^\gamma_{s\longrightarrow t}=\varphi_{\gamma|[a,t]}\circ(\varphi_{\gamma|[a,s]})^{-1},\ s,t\in J=[a,b]. \qquad (3,4)$$

Proof. The first part of the theorem is a consequence of lemmas 3.1 and 3.2, definition 2.6 and the fact that now (2.10) is, due to (3.1), another form of (3.3).

On the contrary, let φ be an axiomatically defined parallel transport (see definition 2.6).

If in theorem 3.1 of [1], we put $Q=\pi^{-1}(\gamma(a))$ and $F^\gamma_s=\left(\varphi_{\gamma\circ\tau^J_s}\right)^{-1}:\pi^{-1}(\gamma(s))\longrightarrow\pi^{-1}(\gamma(a))$ $(\tau^J_s(a)=a,\ \tau^J_s(b)=s)$, we see that the map (3.2) is a transport along γ from s to t. So, $I:\gamma\longmapsto I^\gamma$, where $I^\gamma:(s,t)\longmapsto I^\gamma_{s\longrightarrow t}$, is a transport along paths.

The smoothness condition (3.3) follows from (2.10) and (3.2).

To prove the equalities (1.5) and (1.6) for the transport along paths I, we shall use the following lemma which will be proved below after this proof.

Lemma 3.3. If φ is an axiomatically defined parallel transport, then the maps (3.4) (or equivalently (3.2)) admit the representation

$$I^{\gamma}_{s\longrightarrow t}=(\varphi_{\gamma|[\min(s,t),\max(s,t)]})^{\varepsilon(s,t)}=$$

$$=\begin{cases}\varphi_{\gamma|[s,t]} & \text{, for } s\le t\\ (\varphi_{\gamma|[t,s]})^{-1} & \text{, for } s\ge t\end{cases}, \tag{3.5}$$

where $\varepsilon(s,t):=+1$ for $s\le t$ and $\varepsilon(s,t):=-1$ for $s>t$ (or $s\ge t$).

From (3.5), because of $(\gamma|J')|J'=\gamma|J'$ for any subinterval $J'\subseteq J$, it immediately follows

$$I^{\gamma}_{s\longrightarrow t}=I^{\gamma|[\min(s,t),\max(s,t)]}_{s\longrightarrow t} \tag{3.6}$$

which by [1], proposition 2.3 is equivalent to (1.5).

If $\tau:J''\longrightarrow J$ is an orientation preserving diffeomorphism, then $(\gamma\circ\tau)|[r,s]=(\gamma|[\tau(r),\tau(s)]\circ\tau$ for every $r,s\in J''$ such that $r\le s$. Combining this equality with (3.5), letting $s,t\in J''$, $\lambda:=\min(s,t)$ and $\mu:=\max(s,t)$, and using (2.11), we get:

$$I^{\gamma\circ\tau}_{s\longrightarrow t}=(\varphi_{(\gamma\circ\tau)|[\lambda,\mu]})^{\varepsilon(s,t)}=(\varphi_{(\gamma|[\tau(\lambda),\tau(\mu)])\circ\tau})^{\varepsilon(s,t)}=$$

$$=(\varphi_{(\gamma|[\tau(\lambda),\tau(\mu)])})^{\varepsilon(s,t)}=(\varphi_{(\gamma|[\tau(\lambda),\tau(\mu)])})^{\varepsilon(\tau(s),\tau(t))}=I^{\gamma}_{\tau(s)\longrightarrow\tau(t)}$$

as $r\le s$ leads to $\tau(r)\le\tau(s)$, $r,s\in J''$. ■

The proof of lemma 3.3 is based on

Lemma 3.4. If φ is an axiomatically defined parallel transport and $\gamma:J\longrightarrow B$, then

$$\varphi_{\gamma|[s,t]}\circ\varphi_{\gamma|[r,s]}=\varphi_{\gamma|[r,t]},\ \text{for } r\le s\le t,\ r,s,t\in J. \tag{3.7}$$

Proof. Let $\tau_1:[0,1]\longrightarrow[r,s]$ and $\tau_2:[0,1]\longrightarrow[s,t]$ be orientation preserving diffeomorphisms. Evidently, also such is the map $\tau:[0,1]\longrightarrow[r,t]$, defined by $\tau(\lambda):=\tau_1(2\lambda)$ for $\lambda\in[0,1/2]$ and $\tau(\lambda):=\tau_2(2\lambda-1)$ for $\lambda\in[1/2,1]$. Using (2.11), the definition of the (canonical) product of paths (see Sect. 1), and (2.13), we find:

$$\varphi_{\gamma|[s,t]}\circ\varphi_{\gamma|[r,s]}=\varphi_{\gamma\circ\tau_1}\circ\varphi_{\gamma\circ\tau_2}=\varphi_{(\gamma\circ\tau_1)(\gamma\circ\tau_2)}=\varphi_{\gamma\circ\tau}=\varphi_{\gamma|[r,t]}.\ ■$$

Proof of lemma 3.3. Combining (3.7) and (3.4) for $a\le s\le t\le b$, we

get

$$I^{\gamma}_{s\longrightarrow t}=\varphi_{\gamma|[a,t]}\circ(\varphi_{\gamma|[a,s]})^{-1}=\varphi_{\gamma|[s,t]}\circ\varphi_{\gamma|[a,s]}\circ(\varphi_{\gamma|[a,s]})^{-1}=\varphi_{\gamma|[s,t]}$$

and for $a\le t\le s\le b$, we obtain

$$I^{\gamma}_{s\longrightarrow t}=\varphi_{\gamma|[a,t]}\circ(\varphi_{\gamma|[a,s]})^{-1}=\varphi_{\gamma|[a,t]}\circ(\varphi_{\gamma|[t,s]}\circ\varphi_{\gamma|[a,t]})^{-1}=(\varphi_{\gamma|[t,s]})^{-1}.\ ■$$

Theorem 3.1 is a strict expression of the statement that the axiomatically defined parallel transport is a special case of transports along paths in fibre bundles, and that any transport along paths satisfying certain additional conditions, namely (1.5) and (1.6), defines an axiomatically defined parallel transport. This theorem also expresses a one-to-one correspondence between axiomatically defined parallel transports and transports along paths obeying the conditions (1.5) and (1.6). Speaking more freely, we can say that according to it a transport along paths is an axiomatically defined parallel transport if and only if it satisfies the additional conditions (1.5) and (1.6).

Proposition 3.1. If a transport along paths I (resp. axiomatically defined parallel transport φ) defines through (3.1) (resp. (3.2)) the axiomatically defined parallel transport φ (resp. transport along paths I), then the generated by φ (resp. I) by means of (3.2) (resp. (3.1)) transport along paths (resp. axiomatically defined parallel transport) coincides with the initial transport along paths I (resp. the axiomatically defined parallel transport φ).

Proof. Let $'I$ (resp. $'\varphi$) be the generated by φ (resp. I) transport along paths (resp. axiomatically defined parallel transport). Using (3.1) and (3.2), we find

$$'I^{\gamma}_{s\longrightarrow t}=\varphi_{\gamma\circ\tau^J_t}\circ\varphi^{-1}_{\gamma\circ\tau^J_s}=I^{\gamma\circ\tau^J_t}_{a\longrightarrow b}\circ\left(I^{\gamma\circ\tau^J_s}_{a\longrightarrow b}\right)^{-1}=I^{\gamma}_{\tau^J_t(a)\longrightarrow\tau^J_t(b)}\circ$$

$$\circ I^{\gamma}_{\tau^J_s(b)\longrightarrow\tau^J_s(a)}=I^{\gamma}_{a\longrightarrow t}\circ I^{\gamma}_{s\longrightarrow a}=I^{\gamma}_{s\longrightarrow t}$$

(resp. $'\varphi_{\gamma}=I^{\gamma}_{a\longrightarrow b}=\varphi_{\gamma\circ\tau^J_b}\circ\varphi^{-1}_{\gamma\circ\tau^J_a}=\varphi_{\gamma}\circ id_{\pi^{-1}(\gamma(a))}=\varphi_{\gamma}$). ■

4. THE GENERATED BY DERIVATIONS OF TENSOR ALGEBRAS TRANSPORTS ALONG PATHS AS PARALLEL TRANSPORTS IN TENSOR BUNDLES

In this section, by η we denote a C^1 path in the manifold M such that $\eta:[a,b]\longrightarrow M$ for a definite $a\le b$, $a,b\in\mathbb{R}$.

Let S be an S-transport along paths (in the tensor algebra over M) [2].

Definition 4.1. The S-parallel transport associated with the S-transport S is a map φ from the set of C^1 paths in M into the set of bundle morphisms of the tensor bundles over these paths such that

$$\varphi:\eta\longmapsto\varphi_\eta:=S^\eta_{a\longrightarrow b}:T_{\eta(a)}(M)\longrightarrow T_{\eta(b)}(M),\ a\le b, \tag{4.1}$$

where $T_x(M)$ is the tensor algebra at $x\in M$. The map φ_η will be called an S-parallel transport along (the path) η.

Lemma 4.1. If φ is the S-parallel transport generated by an S-transport S, $\gamma:J\longrightarrow M$ and $s,t\in J$, then

$$S^\gamma_{s\longrightarrow t}=\begin{cases}\varphi_\eta &, \eta:=\gamma|[s,t] \text{ for } s\le t\\ (\varphi_\eta)^{-1}, & \eta:=\gamma|[t,s] \text{ for } s\ge t\end{cases}. \tag{4.2}$$

Proof. (4.2) follows from (4.1) and (1.4), as any S-transport has this property (see [2], eq. (2.10) and also [3], Sect. 2).∎

Between the S-transports and S-parallel transports there exists one important difference. Namely, the S-transport along $\gamma:J\longrightarrow M$ does not use the natural order of the real numbers which defines a definite orientation on the interval J, while in the definition (4.1) of an S-parallel transport this order is used explicitly ($a\le b$). The last fact is the reason for the appearance of two different cases ($s\le t$ and $s\ge t$) in (4.2). This fact also reflects the difference between (1.6) (or (4.4)) and (4.5) (see below proposition 4.2).

Proposition 4.1. If $\eta_a:\{a\}\longrightarrow\{m_a\}$, $a\in\mathbb{R}$ and $m_a\in M$, then

$$\varphi_{\eta_a}=id_{\pi^{-1}(m_a)}. \tag{4.3}$$

Proof. (4.3) follows directly from (4.1) for b=a and (1.3) (see also [2], definition 2.1).∎

Proposition 4.2. Let $\eta:[a,b]\longrightarrow M$, $\tau:[a',b']\longrightarrow[a,b]$ be a diffeomorphism and the S-transport S^η along η be invariant under the change τ of the parameterization of η, i.e. (cf. (1.6))

$$S^{\gamma\circ\tau}_{s\longrightarrow t}=S^\gamma_{\tau(s)\longrightarrow\tau(t)},\quad s,t\in[a',b']. \tag{4.4}$$

Then, for the S-parallel transport φ, corresponding to S, there holds

$$\varphi_{\eta\circ\tau}=\varphi_\eta,\ \text{for } \tau(a')=a, \tag{4.5a}$$

$$\varphi_{\eta\circ\tau}=(\varphi_\eta)^{-1},\ \text{for } \tau(a')=b, \tag{4.5b}$$

i.e. the S-parallel transport is invariant under orientation preserving change of the parameterization (case (4.5a)), but when this change does not preserve the orientation it is replaced by its inverse map (case (4.5b)).

Proof. Using successively (4.1), (4.4) and (4.2) for $\gamma=\eta$, we find

$$\varphi_{\eta\circ\tau}=S^{\eta\circ\tau}_{a'\longrightarrow b'}=S^\eta_{\tau(a')\longrightarrow\tau(b')}=\begin{cases}\varphi_\eta & \text{for } \tau(a')\le\tau(b')\\ (\varphi_\eta)^{-1} & \text{for } \tau(a')\ge\tau(b')\end{cases},$$

which, due to that τ is a diffeomorphism, is equivalent to (4.5).∎

Proposition 4.3. Let $\eta_-:=\eta\circ\tau^c_-$ be the canonically inverse path to $\eta:[a,b]\longrightarrow M$, i.e. $\tau^c_-:[a,b]\longrightarrow[a,b]$, $\tau^c_-(s):=a+b-s$, $s\in[a,b]$ (cf. Sect. 1 and [38,39]). If (4.4) holds for changing the orientation maps τ for some S-transport, then for the corresponding to it S-parallel transport φ there holds the equality

$$\varphi_{\eta_-}=(\varphi_\eta)^{-1}. \tag{4.6}$$

Proof. This result is a corollary from (4.1), (4.4), (1.4) and the inequality $(\tau^c_-)^{-1}(a)\ge(\tau^c_-)^{-1}(b)$ as, by definition τ^c_- changes the orientations. Eq. (4.6) also follows from (4.5b) for $\tau=\tau^c_-$.∎

Proposition 4.4. Let $\eta_1\eta_2$ be the (canonical) product of the paths $\eta_h:[0,1]\longrightarrow M$, h=1,2, $\eta_1(1)=\eta_2(0)$ (see Sect. 1). If an S-transport defining the S-parallel transport φ satisfies (4.4) for preserving the orientations τ and (1.5), then

$$\varphi_{\eta_1\eta_2}=\varphi_{\eta_2}\circ\varphi_{\eta_1}. \tag{4.7}$$

Proof. Putting $\tau_1(s)=2s$, $s\in[0,1/2]$ and $\tau_2(s)=2s-1$, $s\in[1/2,1]$ and using sequentially (4.1), (1.2), (1.5) and (4.4), we get:

$$\varphi_{\eta_1\eta_2} = S^{\eta_1\eta_2}_{0\longrightarrow 1} = S^{\eta_1\eta_2}_{1/2\longrightarrow 1} \circ S^{\eta_1\eta_2}_{0\longrightarrow 1/2} = S^{(\eta_1\eta_2)|[1/2,1]}_{1/2\longrightarrow 1} \circ$$

$$\circ\, S^{(\eta_1\eta_2)|[0,1/2]}_{0\longrightarrow 1/2} = S^{\eta_2\circ\tau_2}_{1/2\longrightarrow 1} \circ S^{\eta_1\circ\tau_1}_{0\longrightarrow 1/2} = S^{\eta_2}_{\tau_2(1/2)\longrightarrow\tau_2(1)} \circ$$

$$\circ\, S^{\eta_1}_{\tau_1(0)\longrightarrow\tau_1(1/2)} = S^{\eta_2}_{0\longrightarrow 1} \circ S^{\eta_1}_{0\longrightarrow 1} = \varphi_{\eta_2} \circ \varphi_{\eta_1}. \blacksquare$$

In propositions 4.2, 4.3 and 4.4 one essentially uses the acceptance for the validity of (4.4). This is not random as the equality (4.4) expresses the invariance (under certain conditions) under the changes of parameterization of an S-transport's path, and all (parallel) transports (see Sect. 2) known to the author and used in the mathematical and physical literature possess this property.

From the above-said it is clear that under sufficiently general and "reasonable" conditions an S-parallel transport satisfies all basic (functional) conditions characterizing the parallel transport when it is axiomatically described (see Sect. 2.2). Namely, this is the reason for calling the map (4.1) an S-parallel transport: it is a "parallel transport" acting in the tensor spaces over a differentiable manifold and it is generated by derivation of the tensor algebra over the manifold. More precisely, from the above results and definition 2.6, we derive

Proposition 4.5. The S-parallel transport generated by an S-transport along paths satisfying along them (1.5) and (1.6) is the axiomatically defined parallel transport.

The next proposition expresses some properties of the S-parallel transports which are specific of them as "parallel transports" in tensor bundles.

Proposition 4.6. Any S-parallel transport φ_η along a path $\eta:[a,b]\longrightarrow M$ possesses the properties:

a) Linearity: if $\lambda',\lambda''\in\mathbb{R}$ and T' and T'' are tensors at $\eta(a)$, then:

$$\varphi_\eta(\lambda' T'+\lambda'' T'')=(\lambda')\varphi_\eta(T')+(\lambda'')\varphi_\eta(T''); \tag{4.8}$$

b) Term by term action on tensor products: if A and B are arbitrary tensors at $\eta(a)$, then

$$\varphi_\eta(A\otimes B)=(\varphi_\eta(A))\otimes(\varphi_\eta(B)); \tag{4.9}$$

c) Commutativity with the contraction operator C:

$$\varphi_\eta\circ C-C\circ\varphi_\eta=0; \tag{4.10}$$

d) An identical action on scalars: if $\lambda\in\mathbb{R}$, then

$$\varphi_\eta(\lambda)=\lambda. \tag{4.11}$$

Proof. Equalities (4.8)-(4.11) follow directly from definition 4.1 and, respectively, the properties (2.3)-(2.5) and (2.13) of [2] of the S-transports. $\blacksquare$

5. CONCLUSION

The main result of this work is that the theory of transports along paths in fibre bundles is sufficiently general and includes as its special case the theory of parallel transports, and also, consequently, the connection theory. An essential role, as we saw, in comparing these theories was played by the additional condition (1.6). The transports along paths satisfying it depend in fact not on the *path* of transport $\gamma:J\longrightarrow B$ but on the *curve* of transport, i.e. on the whole class of paths $\{\gamma\circ\tau\}$ in which τ is a 1:1 map of $\mathbb{R}$-intervals onto J. Because of the practical importance of (1.6), we shall consider it below in the most used case, the one of linear transports in vector bundles [3].

Let L be a linear transport in the vector bundle (E,π,B) [3]. If $\tau:J''\longrightarrow J$ is 1:1 map, then eq. (1.6) reads

$$L^{\gamma\circ\tau}_{s\longrightarrow t}=L^{\gamma}_{\tau(s)\longrightarrow\tau(t)},\quad s,t\in J''. \tag{5.1}$$

Let a field of bases in E be fixed along $\gamma:J\longrightarrow B$ in which $H:(s,t;\gamma)\longmapsto H(s,t;\gamma)$, $s,t\in J$ and $\Gamma_\gamma(s):=(\partial H(s,t;\gamma)/\partial t)|_{t=s}$ be, respectively, the matrix and the matrix of the coefficients of L [3]. Let $\mathcal{D}^\gamma$ be the generated by L derivation along γ and $(\mathcal{D}^\gamma\sigma)(s)=:\mathcal{D}^\gamma_s\sigma$ for a C^1 section σ of (E,π,B) (see [3], eqs. (4.2) and (4.3)).

Proposition 5.1. The condition (5.1) is equivalent to any of the following three equalities:

$$H(t,s;\gamma\circ\tau)=H(\tau(t),\tau(s);\gamma),\quad s,t\in J'', \tag{5.2}$$

$$\Gamma_{\gamma\circ\tau}(s)=\frac{d\tau(s)}{ds}\cdot\Gamma_\gamma(\tau(s)),\quad s\in J'', \tag{5.3}$$

$$\mathcal{D}^{\gamma\circ\tau}_s=\frac{d\tau(s)}{ds}\cdot\mathcal{D}^{\gamma}_{\tau(s)},\quad s\in J''. \tag{5.4}$$

Proof. The equivalence of (5.1) and (5.2) is a corollary of the definition of H (and the linearity of L; see [3], Sect. 2). Eqs. (5.3) and (5.4) are equivalent because of the connection (4.7) or (4.14′) from [3] between $\Gamma_\gamma(s)$ and $\mathcal{D}^\gamma_s$.

So, it remains to prove the equivalence between (5.2) and (5.3).

Differentiating (5.2) with respect to s and using $\Gamma_\gamma(s):=:=(\partial H(s,t;\gamma)/\partial t)|_{t=s}$, we get (5.3). On the contrary, if (5.3) holds, then using the same equality, the representation $H(t,s;\gamma)==F^{-1}(t;\gamma)F(s;\gamma)$ for some matrix function F (see [3], proposition 2.4) and $dF^{-1}/ds=-F^{-1}(dF/ds)F^{-1}$, we easily obtain:

$$\frac{d}{ds}[H(\tau(t),\tau(s);\gamma)H^{-1}(t,s;\gamma\circ\tau)] =$$

$$= H(\tau(t),\tau(s);\gamma)\left[\frac{d\tau(s)}{ds}\cdot\Gamma_\gamma(\tau(s)) - \Gamma_{\gamma\circ\tau}(s)\right]H^{-1}(t,s;\gamma\circ\tau) = 0.$$

From this, due to $H(s,s;\gamma)=\mathbb{1}$ (see [3], eq. (2.12)) and $(t=s \Leftrightarrow \tau(t)==\tau(s))$, we derive (5.2).∎

If B is a manifold, evident examples of linear transports along paths satisfying (5.3), and hence (5.1), are the ones characterized by the coefficients given by [3], eq. (5.1) and, in particular, the parallel transports generated by linear connections.

The definition of a parallel transport in principal or associated fibre bundles by the map $\gamma\longmapsto g_\gamma\in G$ (see Subsect. 2.1) is widely used in the physical literature devoted to gauge theories [13,14, 29-32,43-45]. In them, the parallel transport is given globally through an ordered (called also P-, T-, or chronological) exponent [14,40,45] along γ, i.e. $\gamma\longmapsto g_\gamma=\mathrm{Pexp}\int_\gamma A_i dx^i$, where A_i are the components of the connection form (or, in physical language, the gauge potentials). So, locally along a path γ connecting the infinitesimally near points x and x+dx it is defined by the expansion $g_\gamma|_{x,x+dx}=\mathbb{1}+A_i dx^i$ [11,43].

If γ is a closed path (a contour) passing through $x\in B$ (in the

physical literature such a path is called a loop), then the quantity $W(\gamma,x):=\mathrm{Pexp}\oint_\gamma A_i dx^i$ is called a Wilson loop [29-32] and in accordance with the above considerations it uniquely defines the parallel transport from $\pi^{-1}(x)$ onto $\pi^{-1}(x)$, i.e. of the fibre over x onto itself. The importance of Wilson's loops is in that their set $\{W(\gamma,x):\ \gamma:[a,b]\longrightarrow B,\ \gamma(a)=\gamma(b),\ x\in\gamma([a,b])\}$, which is a non-abelian group and is a representation of the group of loops, contains all the information for the considered gauge theory [13,29-32,43-45].

ACKNOWLEDGEMENT

This research was partially supported by the Fund for Scientific Research of Bulgaria under contract Grant No. F 103.

REFERENCES

1. Iliev B.Z., Transports along paths in fibre bundles. General theory, JINR Communication E5-93-299, Dubna, 1993.
2. Iliev B.Z., Parallel transports in tensor spaces generated by derivations of tensor algebras, JINR Communication JINR E5-93-1, Dubna, 1993.
3. Iliev B.Z., Linear transports along paths in vector bundles. I. General theory, JINR Communication JINR E5-93-239, Dubna, 1993.
4. Hawking S.W., G.F.R. Ellis, The large scale structure of space-time, Cambridge Univ. Press, Cambridge, 1973.
5. Kobayashi S., K. Nomizu, Foundations of differential geometry, vol.1, Interscience publishers, New-York-London, 1963.
6. Sachs R.K., H. Wu, General Relativity for Mathematicians, Springer-Verlag, New York-Heidelberg-Berlin, 1977.
7. Nash C., S.Sen, Topology and Geometry for physicists, Academic Press, London-New York, 1983.
8. Nomizu K., Lie groups and differential geometry, The mathematical Society of Japan, 1956.
9. Nikolov P., On the correspondence between infinitesimal and integral description of connections, ICTP, Internal Report, IC/81/196.
10. Lumiste U.G., Connection theory in fibre bundles, In: Science

review, sec. Mathematics: Algebra. Topology. Geometry. 1969, VINITI, Moscow, 1971, 123-168 (in Russian).
11. Lumiste U.G., Parallel transport; Connections on manifolds; Connection object; Connection form; Connections in fibre bundles, Articles in: Mathematical Encyclopedia, vol.4, Soviet encyclopedia, Moscow, 1984 (in Russian).
12. Warner F.W., Foundations of differentiable manifolds and Lie groups, Springer Verlag, New York-Berlin-Heidelberg-Tokyo, 1983.
13. Durhuus B., J.M. Leinaas, On the loop space formulation of gauge theories, CERN, preprint TH 3110, Geneva, 1981.
14. Khudaverdian O.M., A.S. Schwarz, A new comments on the string representation of gauge fields, Phys. Lett. B, 91, No.1, 1980, 107-110.
15. Bishop R.L., R.J. Crittenden, Geometry of Manifolds, Academic Press, New York-London, 1964.
16. Yano K., M. Kon, Structures on Manifolds, Series in Pure Mathematics, vol. 3, World Scientific Publ. Co., Singapore, 1984.
17. Steenrod N., The topology of fibre bundles, 9-th ed., Princeton Univ. Press, Princeton, 1974 (1-st ed. 1951).
18. Sulanke R., P. Wintgen, Differentialgeometrie und Faserbündel, VEB Deutscher Verlag der Wissenschaften, Berlin, 1972.
19. Lumiste U.G., To the foundations of global connection theory, In: Scientific writings of Tartu state university, No.150, Works on mathematics and mechanics IV, Tartu, 1964, 69-108 (in Russian).
20. Ostianu N.M. V.V. Rizhov, P.I. Shveikin, Article on the scientific investigations of German Fedorovich Laptev, In: Works of Geometrical seminar, vol.4, VINITI, Moscow, 1973, 7-70 (in Russian).
21. Choquet-Bruhat Y. et al., Analysis, manifolds and physics, North-Holland Publ.Co., Amsterdam, 1982.
22. Husemoller D., Fibre bundles, McGrow-Hill Book Co., New York-St. Louis-San Francisco-Toronto-London-Sydney, 1966.
23. Mishtenko A.S., Vector fibre bundles and their applications, Nauka, Moscow, 1984 (in Russian).
24. Hermann R., Vector bundles in mathematical physics, vol.I, Benjamin W.A., Inc., New York, 1970.

25. Greub W., S. Halperin, R. Vanstone, Connections, Curvature, and Cohomology, vol.1, vol.2, Academic Press, New York and London, 1972, 1973.
26. Atiyah M.F., K-theory, Harvard Univ., Cambridge, Mass., 1965.
27. Karoubi M., K-theory. An Introduction, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
28. Dandoloff R., W.J. Zakrzewski, Parallel transport along a space curve and related phases, J. Phys. A: Math. Gen., 22, 1989, L461-L466.
29. Azam M., Gauge-invariant objects from Wilson loops, Phys. Rev. D, 35, No.6, 1987, 2043-2046.
30. Jevicki A., B. Sakita, Collective approach to the large N limit: Euclidean field theories, Nucl. Phys. B, 185, 1981, 89-100.
31. Polyakov A.M., Gauge fields as rings of glue, Nucl. Phys. B, 164, 1979, 171-188.
32. Makeenko Yu.M., A.A. Migdal, Quantum chromodynamics as dynamics of loops, Nucl. Phys. B, 188, 1981, 269-316.
33. Tamura I, Topology of foliations, Mir, Moscow, 1979 (in Russian; translation from Japanese).
34. Hicks N.J., Notes on Differential Geometry, D. Van Nostrand Comp., Inc., Princeton, 1965.
35. Sternberg S., Lectures on differential geometry, Chelsea Publ. Co., New York, 1983.
36. Lumiste U.G., Homogeneous fibre bundles with connection and their immersions, In: Works of the Geometrical seminar, vol.1, VINITI, Moscow, 1966, 191-237 (in Russian).
37. Hartman Ph., Ordinary differential equations, John Wiley & Sons, New-York-London-Sydney, 1964.
38. Hu Sze-Tsen, Homotopy Theory, Academic Press, New York-London, 1959.
39. Viro O.Ya., D.B. Fuks, I. Introduction to homotopy theory, In: Reviews of science and technology, sec. Modern problems in mathematics. Fundamental directions, vol.24, Topology-2, VINITI, Moscow, 1988, 6-121 (in Russian).
40. Dubrovin B.A., S.P. Novikov, A.T. Fomenko, Modern geometry, Nauka, Moscow, 1979 (in Russian).
41. Lichnerowicz A., Thêorie globale des connexions et des gropes D'Holonomie, Roma, Edizione Cremónese, 1955.

42. Mangiarotti L., M. Modugno, Fibred Spaces, Jet Spaces and Connections for Field Theories, In: Proceedings of the International meeting on Geometry and Physics, Florence, October 12-15.1982, ed. M. Modugno, Pitagora Editrice, Bologna, 1983, 135-165.
43. Yang C.N., Integral formalism for gauge fields, Phys. Rev. Lett., 33, No.7, 1974, 445-447.
44. Gambini R., A. Trias, Geometrical origin of gauge theories, Phys. Rev. D, 23, No.2, 1981, 553-555.
45. Aref'eva I.Ya., On the integral formulation of gauge theories, preprint No.480, Wroclaw Univ., Wroclaw, 1979.

Received by Publishing Department
on January 21, 1994.