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RELATIONS BETWEEN RATIONAL AND POLYNOMIAL APPROXIMATIONS IN THE BANACH SPACES

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1 Polynomial and rational approximations

Surmise that considered Banach space $(\mathbf{B};\ ||\cdot||)$ over the field of real or complex numbers has following row of properties. Let infinite system $\{g_n\}_{n=0}^{\infty}$ of linear independent elements of the space exist. Let linear envelope $\mathbf{L}$ of all these vectors have the structure of commutative algebra, where g_0 is the unit of one. Describe multiplication properties in the following terms. Name linear envelope of first $n+1$ vectors from the shared out set by subspace of not greater n degree polynomials, that is $\mathbf{P}_n = L\langle g_0, \dots, g_n\rangle$, where degree ∂P of nontrivial polynomial $P \neq 0$ is defined as the greatest value of index of vectors g_k with nonzero coefficient from linear combination representing this polynomial, $\partial(\alpha g_0) = 0$. The degree of two nontrivial polynomials product is the sum of co-factors degrees $\partial(P\cdot Q) = \partial P + \partial Q$. It is possible to divide nontrivial polynomial by one with residual

$$\forall H, P \in \mathbf{L},\ \partial P \leq \partial H,\ \exists Q, S \in \mathbf{L}:$$
$$H = P\cdot Q + S,\ \partial S \leq \partial P,\ \partial P + \partial Q = \partial H.$$

Let following property take place

$$\forall Q \in \mathbf{L},\ \exists T = T(Q) > 0:\ \forall t \in (0,T),$$
$$(g_0 - t\cdot Q)^{-1} = g_0 + t\cdot Q + \hat{o}(t) \in \mathbf{B},$$

where value $||\hat{o}(t)|| = o(t)$ if Q is fixed.

Define the union of rational functions of (n,m) degree as the set of vectors from $\mathbf{B}$, which can be represented as a product of a not greater n degree polynomial and such element from $\mathbf{B}$ that it is the converse one for a not greater m degree polynomial, that is

$$\mathbf{R}_{n,m} = \{R_{n,m} = P_n\cdot Q_m^{-1} \in \mathbf{B}:$$
$$P_n \in \mathbf{P}_n,\ Q_m \in \mathbf{P}_m,\ Q_m^{-1} \in \mathbf{B},\quad Q_m\cdot Q_m^{-1} = g_0\}.$$

Designate the values of the least deviation of element $f \in \mathbf{B}$ from subspace $\mathbf{P}_n$ and union $\mathbf{R}_{n,m}$ by corresponding equations

$$\mathbf{e}_n(f) = \inf_{P_n\in\mathbf{P}_n} ||f - P_n||,\qquad \mathbf{r}_{n,m}(f) = \inf_{R_{n,m}\in\mathbf{R}_{n,m}} ||f - R_{n,m}||.$$

In proofs the statement, that the best approaching element exists in any finite-dimentional subspace of the Banach space, is used. Symbol $\mathbf{Z}_+$ is designated the set of integer nonnegative numbers.

According to our designations, theorem 3 from the article [1] is formulated in such manner: let k be a natural number, $\mathbf{e}_{k-1}(f) > \mathbf{e}_k(f) = \mathbf{e}_n(f) > \mathbf{e}_{n+1}(f)$, then $\mathbf{e}_n(f) > \mathbf{r}_{n,n}(f)$. Following theorem amplifies this statement.

Theorem 1. *Let element $f \in \mathbf{B}$, $m, s \in \mathbf{Z}_+$, natural number $k \geq s$ be fixed. Suppose that $\mathbf{e}_{-1}(f) = +\infty$. Let $\mathbf{e}_{k-s-1}(f) > \mathbf{e}_k(f) = \mathbf{e}_{k+m}(f) > \mathbf{e}_{k+m+1}(f)$, then $\mathbf{e}_k(f) > \mathbf{r}_{k,m+s+1}(f)$.*

P R O O F . Consider the construction of auxiliary rational function

$$\begin{aligned} R &= (P_k + t\cdot S)\cdot(g_0 - t\cdot Q)^{-1} = P_k + t\cdot(S + P_k\cdot Q) + \hat{o}(t) = \\ &= (1-t)\cdot P_k + t\cdot(P_k + S + P_k\cdot Q) + \hat{o}(t), \end{aligned} \tag{1}$$

where polynomials S and Q are selected so that they satisfy following equations

$$P_{k+m+1} = P_k + S + P_k\cdot Q,$$
$$||f - P_k|| = \mathbf{e}_k(f),\quad ||f - P_{k+m+1}|| = \mathbf{e}_{k+m+1}(f).$$

It is obvious that we must deem $k-s \leq \partial P_k \leq k$, $\partial P_{k+m+1} = k+m+1$, $\partial Q \leq m+s+1$, $\partial S < k$ and, in common case, the degree of rational function R as $(k, m+s+1)$. Then, using norm properties and deeming that $t \in (0,1)$, we receive following relations

$$\begin{aligned} \mathbf{r}_{k,m+s+1}(f) &\leq ||f - R|| = \\ &= ||(1-t)\cdot(f - P_k) + t\cdot(f - P_{k+m+1}) + \hat{o}(t)|| \leq \\ &\leq (1-t)\cdot||f - P_k|| + t\cdot||f - P_{k+m+1}|| + o(t) = \\ &= (1-t)\cdot\mathbf{e}_k(f) + t\cdot\mathbf{e}_{k+m+1}(f) + o(t) = \\ &= \mathbf{e}_k(f) - t\cdot(\mathbf{e}_k(f) - \mathbf{e}_{k+m+1}(f)) + o(t). \end{aligned}$$

We have the inequality $\mathbf{e}_k(f) > \mathbf{e}_{k+m+1}(f)$ from the theorem condition, and if we supposed that $\mathbf{r}_{k,m+s+1}(f) = \mathbf{e}_k(f)$, we would received a contradiction, completing the theorem.

Corollary 1. *Let element $f \in \mathbf{B}$, $m \in \mathbf{Z}_+$, natural number k be fixed. Let $\mathbf{e}_{k-1}(f) \geq \mathbf{e}_k(f) = \mathbf{e}_{k+m}(f) > \mathbf{e}_{k+m+1}(f)$, then $\mathbf{e}_k(f) > \mathbf{r}_{k,m+1}(f)$.*

Following result generalizes first statements of the theorems 3 and 6 from the article [2].

Theorem 2. *Let element $f \in \mathbf{B}$, $s \in \mathbf{Z}_+$, natural numbers m and $k \geq s$ be fixed. Suppose that $\mathbf{e}_{-1}(f) = +\infty$. Let $\mathbf{e}_{k-s-1}(f) > \mathbf{e}_k(f) = \mathbf{r}_{k,m+s}(f)$, then $\mathbf{e}_k(f) = \mathbf{e}_{k+m}(f)$.*

P R O O F . Consider the constructions of auxiliary rational function (1), where polynomials S and Q are selected so that they satisfy following equations

$$P_{k+m} = P_k + S + P_k \cdot Q,$$

$$||f - P_k|| = \mathbf{e}_k(f), \quad ||f - P_{k+m}|| = \mathbf{e}_{k+m}(f).$$

It is obvious that we must deem $k - s \leq \partial P_k \leq k$, $\partial P_{k+m} \leq k + m$, $\partial Q \leq m + s$, $\partial S < k$ and, in common case, the degree of rational function R as $(k, m + s)$. Then, using norm properties and deeming that $t \in (0, 1)$, we receive following relations

$$\begin{aligned}
\mathbf{e}_k(f) &= \mathbf{r}_{k,m+s}(f) \leq ||f - R|| = \\
&= ||(1 - t) \cdot (f - P_k) + t \cdot (f - P_{k+m}) + \hat{o}(t)|| \leq \\
&\leq (1 - t) \cdot ||f - P_k|| + t \cdot ||f - P_{k+m}|| + o(t) = \\
&= (1 - t) \cdot \mathbf{e}_k(f) + t \cdot \mathbf{e}_{k+m}(f) + o(t) = \\
&= \mathbf{e}_k(f) - t \cdot (\mathbf{e}_k(f) - \mathbf{e}_{k+m}(f)) + o(t).
\end{aligned}$$

If we supposed $\mathbf{e}_k(f) > \mathbf{e}_{k+m}(f)$, we would received a contradiction, completing the theorem.

Corollary 2. *Let element $f \in \mathbf{B}$, natural numbers k, m be fixed. Let $\mathbf{e}_{k-1}(f) > \mathbf{e}_k(f) = \mathbf{r}_{k,m}(f)$, then $\mathbf{e}_k(f) = \mathbf{e}_{k+m}(f)$.*

Remark 1. One cannot substitute value $\mathbf{r}_{k,m+1}(f)$ by $\mathbf{r}_{k,m}(f)$ in the corollary 1 and index $k + m$ by $k + m + 1$ in the corollary 2, where $m = 1$, without complementary conditions. Actually, in the final part of this article it is shown that for any natural number k such function f from the Hardy space $H_2(\mathcal{D} = \{z : |z| < 1\})$ exists that for it the following relation takes place

$$\mathbf{e}_{k-1}(f) > \mathbf{e}_k(f) = \mathbf{r}_{k,1}(f) = \mathbf{e}_{k+1}(f) > \mathbf{e}_{k+2}(f).$$

Following result is a generalization of the corollary to the theorem 3 from the article [1].

Corollary 3. *Let element $f \in \mathbf{B}$ belong to the closing of the linear convex of all degrees polynomials $\overline{\mathbf{L}}$ too. The condition $\mathbf{e}_n(f) = \mathbf{r}_{n,n}(f)$, $\forall n \in \mathbf{Z}_+$ can be satisfy only in two following cases: either f belongs to the space $\mathbf{P}_0$ or it is such polynomial of $k > 0$ degree that*

$$\mathbf{e}_0(f) = \mathbf{e}_{k-1}(f) > \mathbf{e}_k(f) = 0. \tag{2}$$

P R O O F . Suppose that in the sequence $\{\mathbf{e}_n(f)\}_{n=0}^{\infty}$ there are such values $\mathbf{e}_s(f)$ and $\mathbf{e}_{s+m+1}(f)$, $s \geq 1$, $m \geq 0$, that following relations are fulfilled $\mathbf{e}_{s-1}(f) > \mathbf{e}_s(f) = \mathbf{e}_{s+m}(f) > \mathbf{e}_{s+m+1}(f)$. Taking into account the corollary 1, one can receive the inequation $\mathbf{e}_s(f) > \mathbf{r}_{s,m+1}(f)$. Hence $\mathbf{e}_{s+m}(f) = \mathbf{e}_s(f) > \mathbf{r}_{s,m+1}(f) \geq \mathbf{r}_{s+m,s+m}(f)$. The contradictory with the condition $\mathbf{e}_n(f) = \mathbf{r}_{n,n}(f)$, $\forall n \in \mathbf{Z}_+$ proves the corollary.

Remark 2. The form of the functions f, for which the equations $\mathbf{e}_n(f) = \mathbf{r}_{n,n}(f)$, $\forall n \in \mathbf{Z}_+$ can be fulfilled, has been studied in some concrete Banach spaces previously. In the article [1] it has been proved that in the case of analytic in $|z| < 1$ and continuous in the closed unit disk functions f with the norm

$$||f|| = \max_{|z|=1} |f(z)|,$$

the condition $\mathbf{e}_n(f) = \mathbf{r}_{n,n}(f)$, $\forall n \in \mathbf{Z}_+$ is fulfilled for and only for the functions $f(z) = a_k z^k + b_k$. In the space of functions continuous over a segment the same problem has been solved in [3]. The result is the representation these functions as $f(x) = a_k T_k(x) + b_k$, where T_k is the Chebyshew polynomial of k degree which is shifted to the considered segment. The case of $f \in L_p(\Gamma)$, $1 < p < \infty$, where Γ is a rectifiable closed Jordan curve in the complex plane has been considered in [4]. The result is that only single value $k = 1$ can be in (2). That is in the space the function f, for which the condition $\mathbf{e}_n(f) = \mathbf{r}_{n,n}(f)$, $\forall n \in \mathbf{Z}_+$ is fulfilled, can be only linear function. This restriction can be received from the following statement of the article [4]: let $n \in \mathbf{Z}_+$, f be fixed and $\mathbf{r}_{n,n}(f) > 0$, then $\mathbf{r}_{n,n}(f) > \mathbf{r}_{n+1,n+1}(f)$.

For each natural number n designate the quantity of indexes $m \in [0, n]$, $n \geq 1$, for which the values of the best polynomial of m degree and rational of (m, m) degree approximations of the given function $f \in \mathbf{B}$ are equal, by the symbol $N(n)$. Next result is an analog of the corollary to the theorem 3 from the article [2].

Corollary 4. *Let element $f \in \mathbf{B}$ be fixed. If the finite sequence $\{\mathbf{r}_{m,m}(f)\}_{m=0}^{n}$ is strictly monotone then the following inequation is held*

$$N(n) \leq 2 + \log_2 n. \tag{3}$$

Remark 3. It should be noted that the strict monotonicity of the sequence $\{\mathbf{r}_{m,m}(f)\}_{m=0}^{\infty}$ and fulfillment of the equation $\mathbf{r}_{m,m}(f) = \mathbf{e}_m(f)$ for some values of the index $m \in [1, \infty)$ are realized, for example, in the space $H_2(\mathcal{D})$. In fact, in the article [5] it has been proved that there exist entire functions f, for which $\mathbf{e}_1(f) = \mathbf{r}_{1,1}(f)$.

Remark 4. It is of interest to compare the inequation (3) with the result of Dolzhenko Ye. P., which has been proved in the article [6]. There exist such continuous over a segment functions that following equation takes place

$$\lim_{n\to\infty} \frac{N(n)}{n} = 1.$$

2 Approximation by rational functions with fixed "denominator"

Turn to considering of approaching of elements from $\mathbf{B}$ by the subspaces of rational functions with fixed "denominator" Q_j

$$\mathbf{R}_n^*(Q_j) = \{P_n \cdot Q_j^{-1} : \ P_n \in \mathbf{P}_n\}.$$

Designate the amount of the least deviation of the element f from this subspace by the symbol

$$\mathbf{r}_n^*(f; Q_j) = \inf_{P_n \in \mathbf{P}_n} \|f - P_n \cdot Q_j^{-1}\|,$$

where the top index "*" denotes that we consider the subspace of rational functions with fixed "denominator", and Q_j is this "denominator". Introduce useful designation of the best approaching vector

for f from considered subspace as $P_n^0 \cdot Q_j^{-1}$, that is

$$\mathbf{r}_n^*(f; Q_j) = \|f - P_n^0 \cdot Q_j^{-1}\|. \tag{4}$$

Next result is an analog of the theorems 1 and 2 in the considered situation.

Theorem 3. *Let $f \in \mathbf{B}$, reversible polynomial Q_j of $j \geq 1$ degree, $s \in \mathbf{Z}_+$ and natural numbers $k \geq s$ and $m \geq j$ be fixed. Suppose that $\mathbf{r}_{-1}^*(f; Q_j) = +\infty$.*

(i) *Let $\mathbf{r}_{k-s-1}^*(f; Q_j) > \mathbf{r}_k^*(f; Q_j) = \mathbf{r}_{k+m-j}^*(f; Q_j) > \mathbf{r}_{k+m-j+1}^*(f; Q_j)$, then $\mathbf{r}_k^*(f; Q_j) > \mathbf{r}_{k,m+s+1}(f)$.*

(ii) *Let $\mathbf{r}_{k-s-1}^*(f; Q_j) > \mathbf{r}_k^*(f; Q_j) = \mathbf{r}_{k,m+s}(f)$, then $\mathbf{r}_k^*(f; Q_j) = \mathbf{r}_{k+m-j}^*(f; Q_j)$.*

P R O O F . Consider the construction of auxiliary rational function

$$\begin{aligned} R &= (P_k^0 + t \cdot U) \cdot (Q_j - t \cdot V)^{-1} = \\ &= (P_k^0 + t \cdot U) \cdot (g_0 - t \cdot S)^{-1} \cdot Q_j^{-1} = \\ &= P_k^0 \cdot Q_j^{-1} + t \cdot (U + P_k^0 \cdot S) \cdot Q_j^{-1} + \hat{o}(t) = \\ &= (1 - t) \cdot P_k^0 \cdot Q_j^{-1} + t \cdot (P_k^0 + U + P_k^0 \cdot S) \cdot Q_j^{-1} + \hat{o}(t). \end{aligned} \tag{5}$$

(i) Polynomials U and V are selected so that they satisfy following equations

$$P_{k+m+1-j}^0 = P_k^0 + U + P_k^0 \cdot S, \qquad V = Q_j \cdot S.$$

It is obvious that we must deem $k - s \leq \partial P_k^0 \leq k$, $\partial P_{k+m+1-j}^0 = k + m + 1 - j$, $\partial S \leq m + s + 1 - j$, $\partial V \leq m + s + 1$ and, in common case, the degree of rational function R as $(k, m + s + 1)$. Then, using norm properties and deeming that $t \in (0, 1)$ and introduced in (4) designation, we receive following relations

$$\begin{aligned} \mathbf{r}_{k,m+s+1}(f) &\leq \|f - R\| = \\ &= \|(1 - t) \cdot (f - P_k^0 \cdot Q_j^{-1}) + \\ &\quad + t \cdot (f - P_{k+m+1-j}^0 \cdot Q_j^{-1}) + \hat{o}(t)\| \leq \\ &\leq (1 - t) \cdot \|f - P_k^0 \cdot Q_j^{-1}\| + \\ &\quad + t \cdot \|f - P_{k+m+1-j}^0 \cdot Q_j^{-1}\| + o(t) = \\ &= (1 - t) \cdot \mathbf{r}_k^*(f; Q_j) + t \cdot \mathbf{r}_{k+m+1-j}^*(f; Q_j) + o(t) = \\ &= \mathbf{r}_k^*(f; Q_j) - t \cdot (\mathbf{r}_k^*(f; Q_j) - \mathbf{r}_{k+m+1-j}^*(f; Q_j)) + o(t). \end{aligned}$$

We have the inequality $\mathbf{r}_k^*(f;Q_j) > \mathbf{r}_{k+m+1-j}^*(f;Q_j)$ from the theorem condition, and if we supposed that $\mathbf{r}_{k,m+s+1}(f) = \mathbf{r}_k^*(f;Q_j)$, we would received a contradiction, completing the point.

(ii) Consider the construction of auxiliary rational function (5), where polynomials U and V are selected so that they satisfy following equations

$$P_{k+m-j}^0 = P_k^0 + U + P_k^0 \cdot S, \qquad V = Q_j \cdot S.$$

It is obvious that we must deem $k - s \leq \partial P_k^0 \leq k$, $\partial P_{k+m-j}^0 \leq k+m-j$, $\partial S \leq m+s-j$, $\partial V \leq m+s$ and, in common case, the degree of rational function R as $(k, m+s)$. Then, using norm properties and deeming that $t \in (0,1)$ and introduced in (4) designation, we receive following relations

$$\begin{aligned}
\mathbf{r}_k^*(f;Q_j) &= \mathbf{r}_{k,m+s}(f) \leq \|f - R\| = \\
&= \|(1-t)\cdot(f - P_k^0 \cdot Q_j^{-1}) + \\
&\quad +t\cdot(f - P_{k+m-j}^0 \cdot Q_j^{-1}) + \hat{o}(t)\| \leq \\
&\leq (1-t)\cdot \mathbf{r}_k^*(f;Q_j) + t\cdot \mathbf{r}_{k+m-j}^*(f;Q_j) + o(t) = \\
&= \mathbf{r}_k^*(f;Q_j) - t\cdot(\mathbf{r}_k^*(f;Q_j) - \mathbf{r}_{k+m-j}^*(f;Q_j)) + o(t).
\end{aligned}$$

If we supposed $\mathbf{r}_k^*(f;Q_j) > \mathbf{r}_{k+m}^*(f;Q_j)$, we would received a contradiction, completing the theorem.

Corollary 5. *Let $f \in \mathbf{B}$, reversible polynomial Q_j of $j \geq 1$ degree, natural numbers k and $m \geq j$ be fixed.*
(i) *Let $\mathbf{r}_{k-1}^*(f;Q_j) > \mathbf{r}_k^*(f;Q_j) = \mathbf{r}_{k+m-j}^*(f;Q_j) > \mathbf{r}_{k+m+1-j}^*(f;Q_j)$, then $\mathbf{r}_k^*(f;Q_j) > \mathbf{r}_{k,m+1}(f)$.*
(ii) *Let $\mathbf{r}_{k-1}^*(f;Q_j) > \mathbf{r}_k^*(f;Q_j) = \mathbf{r}_{k,m}(f)$, then $\mathbf{r}_k^*(f;Q_j) = \mathbf{r}_{k+m-j}^*(f;Q_j)$.*

3 Necessary condition for the best approaching rational function

For the next theorem we need additional property of the space $\mathbf{B}$:

$$\forall Q \in \mathbf{L} \;:\; Q^{-1} \in \mathbf{B},\ \forall U \in \mathbf{L},\ \exists T = T(Q,U) > 0 :$$

$$\forall t \in (0,T), \qquad (Q - t\cdot U)^{-1} = Q^{-1} + t\cdot U\cdot Q^{-2} + \hat{o}(t) \in \mathbf{B},$$

where value $\|\hat{o}(t)\| = o(t)$ if Q and U are fixed. Designate the Banach space possessing this and all prevorious characteristics by $\mathbf{B}^+$.

If for the element $f \in \mathbf{B}^+$ the best approaching rational function of some degree exists, it realizes global minimum of rational functions of the some degree deviation from f. In the space $L_2[a,b]$ necessary conditions for local minimum have been given in [7]. In the article [8] the case of rational functions of complex argument with poles outside of fixed compact has been considered. For convex nonnegative functional on this set necessary conditions for rational function, which realizes local minimum, have been proved. In our situation of the Banach space $\mathbf{B}^+$ we receive

Theorem 4. *Let element $f \in \mathbf{B}^+$, natural numbers n and m be fixed. Suppose that there exists the rational function $P_k \cdot Q_s^{-1} \neq 0$, which realizes the best approaching in the unity of rational functions of (n,m) degree, that is*

$$\mathbf{r}_{n,m}(f) = \|f - P_k \cdot Q_s^{-1}\|,$$

where polynomials P_k and Q_s are relatively prime. Let $\partial P_k = k \leq n$, $\partial Q_s = s \leq m$, $\lambda = \max\{n+s,\ m+k\}$, then the following equation is correct

$$\mathbf{r}_{n,m}(f) = \mathbf{r}_\lambda^*(f;Q_s^2).$$

P R O O F . Designate rational function, which realizes the best approach in the subspace $\mathbf{R}_\lambda^*(Q_j^2)$ (see the definition in prevorious paragraph), by the symbol $H_\lambda \cdot Q_j^{-2}$, that is for which the following relation is fulfilled

$$\mathbf{r}_\lambda^*(f;Q_j^2) = \|f - H_\lambda \cdot Q_j^{-2}\|.$$

For enough small $t > 0$ the following rational function exists in the space $\mathbf{B}^+$.

$$\begin{aligned}
R_{n,m} &= (P_k + t\cdot V_n)\cdot(Q_s - t\cdot U_m)^{-1} = \\
&= (P_k + t\cdot V_n)\cdot(Q_s^{-1} + t\cdot U_m\cdot Q_s^{-2} + \hat{o}(t)) = \\
&= P_k\cdot Q_s^{-1} + t\cdot(V_n\cdot Q_s + P_k\cdot U_m)\cdot Q_s^{-2} + \hat{o}(t).
\end{aligned}$$

Taking into account that polynomials P_k and Q_s are relatively prime and using the reasonings which are an analog of the proof of the final lemma in the article [7], one can be convinced that there exist polynomials $V_n \in \mathbf{P}_n$ и $Q_m \in \mathbf{P}_m$ fulfilling next equality

$$H_\lambda = V_n \cdot Q_s + P_k \cdot U_m + P_k \cdot Q_s.$$

So we receive following equalities

$$\begin{aligned} R_{n,m} &= (1-t)\cdot P_k \cdot Q_s^{-1} + \\ &\quad + t\cdot (V_n \cdot Q_s + P_k \cdot U_m + P_k \cdot Q_s)\cdot Q_s^{-2} + \hat{o}(t) = \\ &= (1-t)\cdot P_k \cdot Q_s^{-1} + t \cdot H_\lambda \cdot Q_s^{-2} + \hat{o}(t). \end{aligned}$$

In this way consider next row of relations

$$\begin{aligned} \mathbf{r}_n^*(f;Q_s) &= \mathbf{r}_{n,m}(f) \leq \|f - R_{n,m}\| = \\ &= \|(1-t)\cdot(f - P_k \cdot Q_s^{-1}) + t\cdot(f - H_\lambda \cdot Q_s^{-2}) + \hat{o}(t)\| \leq \\ &\leq (1-t)\cdot \|f - P_k \cdot Q_s^{-1}\| + t \cdot \|f - H_\lambda \cdot Q_s^{-2}\| + o(t) = \\ &= \mathbf{r}_n^*(f;Q_s) - t\cdot(\mathbf{r}_n^*(f;Q_s) - \mathbf{r}_\lambda^*(f;Q_s^2)) + o(t). \end{aligned}$$

If we supposed $\mathbf{r}_n^*(f;Q_s) > \mathbf{r}_\lambda^*(f;Q_s^2)$, we would received a contradiction. Hence the theorem is completely proved.

Corollary 6. *In addition to the previous theorem condition suppose that $\partial P_k = n$ or $\partial Q_s = m$, then $\mathbf{r}_{n,m}(f) = \mathbf{r}_{n+m}^*(f;Q_s^2)$.*

Remark 5. It is not difficult to show that the degree of the best approach rational function is achieved either in "numerator" or "denominator" if $\mathbf{r}_{n-1,m-1}(f) > \mathbf{r}_{n,m}(f)$. For different spaces the situation has been proved in the articles [7], [4], [9].

In the interest of impossibility of approach improving on account of "numerator" degree increasing, we receive

Corollary 7. *In addition to the previous theorem condition suppose that $m+k > n+s$, then the following equality is correct*

$$\mathbf{r}_n^*(f;Q_s) = \mathbf{r}_{m+k-s}^*(f;Q_s).$$

4 On a possibility of the coincidence of n degree polynomial and $(n,1)$ degree rational approximations in the space $H_2(\mathcal{D})$

Remind that the Hardy space $H_2(\mathcal{D})$ are made up by analytic in the unit disk $\mathcal{D} = \{z : |z| < 1\}$ functions which have nontangential limits on the boundary $\partial\mathcal{D} = \{z : |z| = 1\}$ almost everywhere. The norm $\|\cdot\|$ is generated by scalar product

$$(f,g) = \frac{1}{2\pi}\int_{\partial\mathcal{D}} f(z)\cdot\overline{g(z)}\cdot|dz| = \frac{1}{2\pi i}\int_{\partial\mathcal{D}} \frac{f(z)\cdot\overline{g(z)}}{z}dz.$$

It is known that polynomials z^j, where j is a nonnegative integer, form the orthonormalized basis in the space, and for any element $f \in H_2(\mathcal{D})$ the following equation takes place

$$\|f\|^2 = \sum_{j=0}^{\infty} |f_j|^2, \quad \text{where} \quad f_j = (f, z^j) = \frac{f^{(j)}(0)}{j!}$$

are the coefficients of the power series $f(z) = \sum_{j=0}^{\infty} f_j z^j$, $z \in \mathcal{D}$.

Theorem 5. *For any natural number n such function $f \in H_2(\mathcal{D})\backslash\mathbf{P}_n$ exists for which the equality $\mathbf{e}_n(f) = \mathbf{r}_{n,1}(f)$ is correct.*

P R O O F . Fix natural number n and the function $f(z) = \sum_{j=n}^{\infty} f_j z^j$. For each $c \in \mathcal{D}\backslash\{0\}$ consider the subspace which generates by rational functions of $(n,1)$ degree with fixed denominator $1 - cz$. Let the element

$$r(z) = \frac{S(z)}{1-cz}$$

is the best approaching one for the function f in considered subspace. Take into account interpolation properties of the function r ([10], pages 224–225), precisely the following equations

$$r^{(k)}(0) = f^{(k)}(0), \quad k = 0,1,\ldots,n-1, \qquad r(\bar{c}) = f(\bar{c}). \tag{6}$$

Write the k degree derivative of the rational function, using known formula of the product derivative

$$r^{(k)}(z) = (S(z)\cdot\frac{1}{1-cz})^{(k)} = S^{(k)}(z)\cdot\frac{1}{1-cz} + \ldots,$$

$$r^{(k)}(0) = S^{(k)}(0) + \dots$$

Increasing the value k from 0 to $n-1$ and taking into account that $f(0) = f'(0) = \dots = f^{(n-1)}(0) = 0$, we receive relations for derivatives in zero

$$\begin{aligned} r(0) = f(0) = 0 &\implies S(0) = 0,\\ r'(0) = f'(0) = 0 &\implies S'(0) = 0,\\ \dots \quad \dots \quad \dots & \quad \dots \quad \dots \quad \dots\\ r^{(n-1)}(0) = f^{(n-1)}(0) = 0 &\implies S^{(n-1)}(0) = 0. \end{aligned}$$

Hence $S(z) = sz^n$. In this way the least equation from (6) takes the following form

$$f(\bar{c}) = \frac{S(\bar{c})}{1-|c|^2} = \frac{s\bar{c}^{\,n}}{1-|c|^2}.$$

Using orthogonality r and $f-r$, properties of the scalar product and the least relation, we receive the following row of equations

$$(\mathbf{r}_n^*(f;1-cz))^2 = ||f-r||^2 = (f, f-r) = ||f||^2 - \bar{s}\cdot\left(f(z), \frac{z^n}{1-cz}\right) =$$

$$= ||f||^2 - \bar{s}\cdot\frac{f(\bar{c})}{\bar{c}^{\,n}} = ||f||^2 - \frac{1-|c|^2}{|c|^{2n}}|f(\bar{c})|^2 =$$

$$= ||f||^2 - (1-|c|^2)\cdot\left|\sum_{j=0}^{\infty} f_{n+j}\bar{c}^{\,j}\right|^2.$$

In particular, for $g(z) = \sum_{j=1}^{\infty} g_j z^j$ we have

$$(\mathbf{r}_1^*(g;1-cz))^2 = ||g||^2 - (1-|c|^2)\cdot\left|\sum_{j=0}^{\infty} g_{1+j}\bar{c}^{\,j}\right|^2.$$

Select $g_{1+j} = f_{n+j}$ for $j \in \mathbf{Z}_+$, then $||g|| = ||f||$ and, according to the least relations, we receive the equality

$$\mathbf{r}_{n,1}(f) = \mathbf{r}_{1,1}(g).$$

Obviously $\mathbf{e}_n(f) = \mathbf{e}_1(g)$. Taking into account that in the article [5] it was proved that there exist the functions $g \in H_2(\mathcal{D})$for which $\mathbf{e}_1(g) = \mathbf{r}_{1,1}(g) > 0$, we receive the existence of $f \in H_2(\mathcal{D})\backslash\mathbf{P}_n$, for which $\mathbf{e}_n(f) = \mathbf{r}_{n,1}(f)$. Hence the proof is complete.

Corollary 8. *For any natural number n such function $f \in H_2(\mathcal{D})$ exists for which following relations are correct*

$$\mathbf{e}_{n-1}(f) > \mathbf{e}_n(f) = \mathbf{e}_{n+1}(f) = \mathbf{r}_{n,1}(f) > \mathbf{e}_{n+2}(f). \qquad (7)$$

P R O O F . Consider $g = z = \sum_{j=3}^{\infty} g_j z^j$, $f(z) = z^n + \sum_{j=n+2}^{\infty} f_j z^j$. According to the results of [5], we can receive the function looking for if the Teylor coefficients of it satisfy the inequality

$$\sum_{j=n+2}^{\infty} |f_j| \le \frac{1}{2}.$$

Following functions can be concrete examples for (7)

$$f(z) = z^n + \frac{z^{n+2}}{2}, \quad f(z) = z^{n-1}\sin z, \quad f(z) = \frac{z^n\cdot(e^z - z)}{2e}.$$

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