объединенный
институт
ядерных
исследований
дубна

E5-94-119

L.S.Davtyan[1], A.N.Sissakian,
V.M.Ter-Antonyan[2]

THE HURWITZ TRANSFORMATION: NON-BILINEAR VERSION

Submitted to «Journal of Mathematical Physics»

[1]On leave of absence from Yerevan Physics Institute, Armenia
[2]On leave of absence from Yerevan State University, Armenia

1994

I. Introduction

Over 400 years ago Robert Hooke tried to explain the planet motion on the basis of forces with the linear dependence on the distance. Lately, this idea was covered by the greatness of the Isaak Newton's discovery.

Now, we can say that Hooke was not far from truth, because the transformation manifesting the Coulomb-oscillator equivalence exists. The most general form realizing this conception is known as the Hurwitz transformation [1].

The Hurwitz transformation (H) is a mapping

$$H : E^8(u_0, \ldots, u_7) \rightarrow E^5(x_0, \ldots, x_4)$$

with the following properties:

a. H is a bilinear, i.e.

$$x_i = H_{ik} u_k = W_{ikl} u_k u_l \qquad (1)$$

b. There takes place the Euler's identity

$$x^2 = u^4 ,$$

for

$$x = (x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2},$$

and

$$u = (u_0^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2)^{1/2}$$

For the u-spaces with dimension more than eight it is impossible to conserve the Euler's identity due to the Hurwitz's theorem [2]. The reduced cases of the H-transformation for the maps $E^2 \rightarrow E^2$ and $E^4 \rightarrow E^3$ were introduced by Levi-Civita [3] and Kustaanheimo [4] for regularization of the equations of celestial mechanics. The remarkable peculiarity of the Levi-Civita (LC) and Kustaanheimo-Stiefel (KS) transformations consists in the fact, that after regularization, in both cases, we arrive at the h a r m o n i c oscillator problem. This opened the way for fruitful interaction between the theory of oscillations and the methods of celestial (classical) mechanics. Particularly, in the oscillator representation the perturbations to the Kepler's motion can be calculated with a better accuracy [4].

LC- and KS-transformations were introduced in quantum mechanics too, in the context of hydrogen atom [5]. KS-transformation was applied in quantum chemistry [6], quantum field theory [7] and functional integration [8].

The general consideration of the algebraic structure for H-type transformations was made in [1], the Lie algebra under the constraints connected with these transformations was found in [9, 10]. The arising Hopf's fiber bundle [11] and corresponding analysis through the spinor representation [12] were performed too. At last, the structure of the generelized Cayley-Klein parameterization [13] and the geometric quantization procedure [14] were considered as applied to the investigation of the H-transformation structure.

II. The problem

The H-transformation may be written in the following form:

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2\,, \\
x_1 &= x_2 = x_3 = 0\,, \\
x_4 &= 2(u_0u_4 - u_1u_5 - u_2u_6 - u_3u_7)\,, \\
x_5 &= 2(u_0u_5 + u_1u_4 - u_2u_7 + u_3u_6)\,, \\
x_6 &= 2(u_0u_6 + u_1u_7 + u_2u_4 - u_3u_5)\,, \\
x_7 &= 2(u_0u_7 - u_1u_6 + u_2u_5 + u_3u_4)\,,
\end{aligned} \tag{2}$$

(with non-essential change in the notation:

$$(x_0, x_1, x_2, x_3, x_4) \to (x_0, 0, 0, 0, x_4, x_5, x_6, x_7)).$$

For the following, it is suitable to denote

$$u_L \equiv (u_0^2 + u_1^2 + u_2^2 + u_3^2)^{1/2}\,,$$

$$u_R \equiv (u_4^2 + u_5^2 + u_6^2 + u_7^2)^{1/2}\,.$$

It must be stressed that the H-transformation, in the form (2), determines the connection between C a r t e s i a n coordinates.

Now, let us suppose that coordinates u_k are expressed through the hyperspherical coordinates (u and seven angles additionally). In the general case, (2) determines x_i as functions of $r \equiv x = u^2$ and the above-mentioned seven angles. However, it is possible that the expressions for x_i, do not include s o m e angles. Angles may "shut" due to, for example, the following cause:

$$\sin\frac{\alpha+\psi}{2}\cos\frac{\alpha-\psi}{2} + \cos\frac{\alpha+\psi}{2}sin\frac{\alpha-\psi}{2} = \sin\alpha \tag{3}$$

Let us consider LC- and KS-transformations in this context:

a. In the case of the LC-transformation, (2) reduces to ($u_j = 0, e.g. j = 1, 2, 3, 5, 6, 7$)

$$\begin{aligned}
x_0 &= u_0^2 - u_4^2 \\
x_4 &= 2u_0u_4\,, \\
x_j &= 0, j = 1, 2, 3, 5, 6, 7.
\end{aligned} \tag{4}$$

Let us introduce polar coordinates in u-spaces

$$u_0 = u\cos\frac{\theta}{2}\,,\ u_4 = u\sin\frac{\theta}{2}$$

Then it is obvious that

$$x_0 = r\cos\theta\,,\ x_4 = r\sin\theta\,,$$

So, as a result, we again arrive at the polar coordinates.

b. In the case of the KS-transformation ($u_j = 0, e.g. j = 2, 3, 6, 7$):

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 - u_4^2 - u_5^2\,, \\
x_4 &= 2(u_0u_4 - u_1u_5)\,, \\
x_5 &= 2(u_0u_5 + u_1u_4)\,, \\
x_j &= 0, j = 1, 2, 3, 6, 7.
\end{aligned} \tag{5}$$

The situation here is not so simple as in the preceding case. Firstly, in E^4, there exist three possible types of the hyperspherical coordinates, instead of one. But, only o n e remains hyperspherical after the KS-transformation. Namely, if we have take the following coordinates

$$u_0 = u\cos\tfrac{\theta}{2}\cos\omega\,,\quad u_4 = u\sin\tfrac{\theta}{2}\cos\varphi\,,$$
$$u_1 = u\cos\tfrac{\theta}{2}\sin\omega\,,\quad u_5 = u\sin\tfrac{\theta}{2}\sin\varphi\,, \tag{6}$$

$(0 \le \theta \le \pi\,, 0 \le \omega \le 2\pi)$, from (4)–(6) we can be obtain

$$x_0 = r\cos\theta\,, x_4 = r\sin\theta\sin(\varphi-\omega)\,, x_5 = r\cos\theta\cos(\varphi-\omega).$$

So, we see that the angles θ and $\varphi-\omega$ form with $r = u^2$ a spherical map in E^3.

It is important that in the H-transformation case the connection between angles in E^8 and E^5 is not easy, as in examples a. and b. [13]. Furthermore, this connections define by the t r a n s c e n d e n t a l equations in general case. Practically, it is impossible to operate with these equations (except for the special case considered in [17]).

In this paper we propose an approach that is free from the transcendental connections. In brief, the idea consists in r e f u s a l from the H-transformation in the known formulation. Is it possible to derive the H-type transformation which does not generate difficulties with the hyperspherical coordinates? Below, we will prove, that two equivalent variants of the choice of the transformation exist, which conserve the two above-mentioned properties. As will be clear from the following, we have achieved the aim through the refusal from bilinearity of the H-transformation. As a result, the developed scheme allows us to determine, in sec. IV, the geometric structure of the H-transformation.

III. Left and right A-matrix

Let us substitute the following hyperspherical coordinates

$$\begin{aligned}
u_0 &= u\cos\tfrac{\theta}{2}\cos\tfrac{\beta}{2}\cos\omega & u_4 &= u\sin\tfrac{\theta}{2}\cos\tfrac{\beta'}{2}\cos\omega'\\
u_1 &= u\cos\tfrac{\theta}{2}\cos\tfrac{\beta}{2}\sin\omega & u_5 &= u\sin\tfrac{\theta}{2}\cos\tfrac{\beta'}{2}\sin\omega'\\
u_2 &= u\cos\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\cos\varphi & u_6 &= u\sin\tfrac{\theta}{2}\sin\tfrac{\beta'}{2}\cos\varphi'\\
u_3 &= u\cos\tfrac{\theta}{2}\sin\tfrac{\beta}{2}\sin\varphi & u_7 &= u\sin\tfrac{\theta}{2}\sin\tfrac{\beta'}{2}\sin\varphi'\,,
\end{aligned} \tag{7}$$

into the transformation (2), then

$$\begin{aligned}
x_4 &= r\sin\theta\left[\cos\tfrac{\beta}{2}\cos\tfrac{\beta'}{2}\cos(\omega+\omega') - \sin\tfrac{\beta}{2}\sin\tfrac{\beta'}{2}\cos(\varphi-\varphi')\right]\\
x_5 &= r\sin\theta\left[\cos\tfrac{\beta}{2}\cos\tfrac{\beta'}{2}\sin(\omega+\omega') + \sin\tfrac{\beta}{2}\sin\tfrac{\beta'}{2}\sin(\varphi-\varphi')\right]\\
x_6 &= r\sin\theta\left[\cos\tfrac{\beta}{2}\sin\tfrac{\beta'}{2}\cos(\varphi'-\omega) + \sin\tfrac{\beta}{2}\cos\tfrac{\beta'}{2}\cos(\varphi+\omega')\right]\\
x_7 &= r\sin\theta\left[\cos\tfrac{\beta}{2}\sin\tfrac{\beta'}{2}\sin(\varphi'-\omega) + \sin\tfrac{\beta}{2}\cos\tfrac{\beta'}{2}\sin(\varphi+\omega')\right]
\end{aligned} \tag{8}$$

It is evident that the relations (8) d o n o t define the hyperspherical coordinates in E^5. For another choice of hyperspherical coordinates in E^8, the expressions for $x_i (i = 4,5,6,7)$ are too more complicated.

The expressions (8)(with accounting x_0) may be rewritten in the following 8-dimensional matrix form:

$$\begin{pmatrix} x_0\\ 0\\ 0\\ 0\\ x_4\\ x_5\\ x_6\\ x_7 \end{pmatrix} = \mathbf{A_R} \begin{pmatrix} r\cos\theta\\ 0\\ 0\\ 0\\ r\sin\theta\cos\frac{\beta}{2}\cos\omega\\ r\sin\theta\cos\frac{\beta}{2}\sin\omega\\ r\sin\theta\sin\frac{\beta}{2}\cos\varphi\\ r\sin\theta\sin\frac{\beta}{2}\sin\varphi \end{pmatrix}, \tag{9}$$

where

$$\mathbf{A_R} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{Q_R} \end{pmatrix},$$

$$\mathbf{Q_R} = \begin{pmatrix} \cos\frac{\beta'}{2}\cos\omega' & -\cos\frac{\beta'}{2}\sin\omega' & -\sin\frac{\beta'}{2}\cos\varphi' & -\sin\frac{\beta'}{2}\sin\varphi' \\ \cos\frac{\beta'}{2}\sin\omega' & \cos\frac{\beta'}{2}\cos\omega' & -\sin\frac{\beta'}{2}\sin\varphi' & \sin\frac{\beta'}{2}\cos\varphi' \\ \sin\frac{\beta'}{2}\cos\varphi' & \sin\frac{\beta'}{2}\sin\varphi' & \cos\frac{\beta'}{2}\cos\omega' & -\cos\frac{\beta'}{2}\sin\omega' \\ \sin\frac{\beta'}{2}\sin\varphi' & -\sin\frac{\beta'}{2}\cos\varphi' & \cos\frac{\beta'}{2}\sin\omega' & \cos\frac{\beta'}{2}\cos\omega' \end{pmatrix}.$$

The matrix $\mathbf{Q_R}$ action is a four-dimensional rotation because

$$\mathbf{Q_R}\mathbf{Q_R}^{\mathrm{T}} = \mathbf{I}\,, Det\mathbf{Q_R} = +1$$

Obviously,

$$\begin{pmatrix} r\cos\theta \\ 0 \\ 0 \\ 0 \\ r\sin\theta\cos\frac{\beta}{2}\cos\omega \\ r\sin\theta\cos\frac{\beta}{2}\sin\omega \\ r\sin\theta\sin\frac{\beta}{2}\cos\varphi \\ r\sin\theta\sin\frac{\beta}{2}\sin\varphi \end{pmatrix} = \mathbf{A_R}^{\mathrm{T}} \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \tag{10}$$

Thus, the action of A_R^T after the H-transformation allows us to obtain hyperspherical coordinates (10) from hyperspherical coordinates (8) with simple connections between the angles. Furthermore, this can be done in two

equivalent (left and right) manners:

$$\begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \mathbf{A_L} \begin{pmatrix} r\cos\theta \\ 0 \\ 0 \\ 0 \\ r\sin\theta\cos\frac{\beta'}{2}\cos\omega' \\ r\sin\theta\cos\frac{\beta'}{2}\sin\omega' \\ r\sin\theta\sin\frac{\beta'}{2}\cos\varphi' \\ r\sin\theta\sin\frac{\beta'}{2}\sin\varphi' \end{pmatrix}, \tag{11}$$

Let us turn back to the Cartesian representation for making the consideration universal, i.e. independent of the choice of hyperspherical map:

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = 2\begin{pmatrix} u_4 & -u_5 & -u_6 & -u_7 \\ u_5 & u_4 & -u_7 & u_6 \\ u_6 & u_7 & u_4 & -u_5 \\ u_7 & -u_6 & u_5 & u_4 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = 2u_R\mathbf{Q}_R \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{12}$$

or

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = 2\begin{pmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & u_3 & -u_2 \\ u_2 & -u_3 & u_0 & u_1 \\ u_3 & u_2 & -u_1 & u_0 \end{pmatrix} \begin{pmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = 2u_L\mathbf{Q}_L \begin{pmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} \tag{13}$$

As a consequence, we can reverse (12), (13) and obtain

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{2u_R}\mathbf{Q}_R^T \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \frac{1}{2u_L}\mathbf{Q}_L^T \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \tag{15}$$

The relations (14), (15) allow us to reconstruct the complete structure of hyperspherical coordinates in E^8 from the structure of hyperspherical coordinates in E^5.

Now, let us consider the matrix form leading to (2):

$$\mathbf{H}=\begin{pmatrix} u_0 & u_1 & u_2 & u_3 & -u_4 & -u_5 & -u_6 & -u_7 \\ u_1 & -u_0 & u_3 & -u_2 & -u_5 & u_4 & u_7 & -u_6 \\ u_2 & -u_3 & -u_0 & u_1 & -u_6 & -u_7 & u_4 & u_5 \\ u_3 & u_2 & -u_1 & -u_0 & -u_7 & u_6 & -u_5 & u_4 \\ u_4 & -u_5 & -u_6 & -u_7 & u_0 & -u_1 & -u_2 & -u_3 \\ u_5 & u_4 & -u_7 & u_6 & u_1 & u_0 & u_3 & -u_2 \\ u_6 & u_7 & u_4 & -u_5 & u_2 & -u_3 & u_0 & u_1 \\ u_7 & -u_6 & u_5 & u_4 & u_3 & u_2 & -u_1 & u_0 \end{pmatrix} \tag{16}$$

It will be easy to check the orthogonality of this matrix

$$\mathbf{H}\mathbf{H}^T = u^2 \tag{17}$$

(This property validates the Euler's identity.)

Consider the matrices

$$\mathbf{H}_L = \mathbf{A}_R\mathbf{H}\,, \quad \mathbf{H}_R = \mathbf{A}_L\mathbf{H} \tag{18}$$

If the matrices $\mathbf{A}_L$ and $\mathbf{A}_R$ are orthogonal

$$\mathbf{A}_L\mathbf{A}_L^T = \mathbf{A}_R\mathbf{A}_R^T = \mathbf{I}, \tag{19}$$

then the matrices $\mathbf{H}_R$ and $\mathbf{H}_L$ will satisfy the condition (17).

In agreement with (19),

$$\mathbf{H}_L\begin{pmatrix}\mathbf{U}_L\\ \mathbf{U}_R\end{pmatrix} = \mathbf{A}_R\begin{pmatrix}\mathbf{X}_L\\ \mathbf{X}_R\end{pmatrix}, \mathbf{H}_R\begin{pmatrix}\mathbf{U}_L\\ \mathbf{U}_R\end{pmatrix} = \mathbf{A}_L\begin{pmatrix}\mathbf{X}_L\\ \mathbf{X}_R\end{pmatrix} \tag{20}$$

where $\mathbf{U}_L$, $\mathbf{U}_R$, $\mathbf{X}_L$ and $\mathbf{X}_R$ are

$$\mathbf{U}_L=\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}, \mathbf{U}_R=\begin{pmatrix}u_4\\u_5\\u_6\\u_7\end{pmatrix}, \mathbf{X}_L=\begin{pmatrix}x_0\\0\\0\\0\end{pmatrix}, \mathbf{X}_R=\begin{pmatrix}x_4\\x_5\\x_6\\x_7\end{pmatrix} \tag{21}$$

Now, it is easy to see from (20) and (21) that the matrices $\mathbf{H}_L$ and $\mathbf{H}_R$ transform E^8 into 5-dimensional Euclidean spaces, which we denote by E_L^5:

$$\begin{pmatrix}\mathbf{Y}_L\\ \mathbf{Y}_R\end{pmatrix} = \mathbf{H}_L\begin{pmatrix}\mathbf{U}_L\\ \mathbf{U}_R\end{pmatrix} \tag{22}$$

and E_R^5

$$\begin{pmatrix}\mathbf{Z}_L\\ \mathbf{Z}_R\end{pmatrix} = \mathbf{H}_R\begin{pmatrix}\mathbf{U}_L\\ \mathbf{U}_R\end{pmatrix} \tag{23}$$

where $\mathbf{Y_L}, \mathbf{Y_R}$ and $\mathbf{Z_L}, \mathbf{Z_R}$ define similiarly $\mathbf{X_L}, \mathbf{X_R}$ in (21). Instead of (2), we obtain

$$\begin{aligned} y_0 &= u_0^2+u_1^2+u_2^2+u_3^2-u_4^2-u_5^2-u_6^2-u_7^2 \\ y_1 &= y_2 = y_3 = 0 \\ y_j &= 2(u_0^2+u_1^2+u_2^2+u_3^2)^{1/2}u_j, j=4,5,6,7 \end{aligned} \tag{24}$$

and

$$\begin{aligned} z_0 &= u_0^2+u_1^2+u_2^2+u_3^2-u_4^2-u_5^2-u_6^2-u_7^2 \\ z_1 &= z_2 = z_3 = 0 \\ z_{j+4} &= 2(u_4^2+u_5^2+u_6^2+u_7^2)^{1/2}u_j, j=0,1,2,3 \end{aligned} \tag{25}$$

Particularly, for the hyperspherical coordinates (7) discussed at the begining of III, the relations (24) and (25) are written as

$$\begin{aligned} y_0 &= r\cos\theta \\ y_4 &= r\sin\theta\cos\tfrac{\beta'}{2}\cos\omega' \\ y_5 &= r\sin\theta\cos\tfrac{\beta'}{2}\sin\omega' \\ y_6 &= r\sin\theta\sin\tfrac{\beta'}{2}\cos\varphi' \\ y_7 &= r\sin\theta\sin\tfrac{\beta'}{2}\sin\varphi' \end{aligned} \tag{26}$$

or

$$\begin{array}{rcl} z_0 & = & r\cos\theta \\ z_4 & = & r\sin\theta\cos\frac{\beta}{2}\cos\omega \\ z_5 & = & r\sin\theta\cos\frac{\beta}{2}\sin\omega \\ z_6 & = & r\sin\theta\sin\frac{\beta}{2}\cos\varphi \\ z_7 & = & r\sin\theta\sin\frac{\beta}{2}\sin\varphi \end{array} \tag{27}$$

We conclude that the transformations (24) and (25) are n o n - b i l i n e a r, in distinction to the H- transformation (8). We can introduce the hyperspherical coordinates

$$(u, \theta, \alpha_L, \beta_L, \gamma_L, \alpha_R, \beta_R, \gamma_R)$$

in E^8 as follows:

$$u_j = \begin{cases} u\cos\frac{\theta}{2} f_j(\alpha_L, \beta_L, \gamma_L), j = 0,1,2,3 \\ u\sin\frac{\theta}{2} f_j(\alpha_R, \beta_R, \gamma_R), j = 4,5,6,7 \end{cases} \tag{28}$$

with the evident constraints

$$\begin{array}{c} f_0^2 + f_1^2 + f_3^2 + f_3^2 = 1, \\ f_4^2 + f_5^2 + f_6^2 + f_7^2 = 1 \end{array} \tag{29}$$

Here $(0 \leq \theta \leq \pi)$ and ranges of values for remainding angles are determined by the functional form of f_j. If we substitute (28) into (24) and (25) we obtain

$$\begin{array}{l} y_0 = u^2\cos\theta \\ y_j = u^2\sin\theta f_j(\alpha_R, \beta_R, \gamma_R), j = 4,5,6,7 \end{array} \tag{30}$$

and

$$\begin{array}{l} z_0 = u^2\cos\theta \\ z_{j+4} = u^2\sin\theta f_j(\alpha_L, \beta_L, \gamma_L), j = 0,1,2,3. \end{array} \tag{31}$$

Thus, choosing in E^8, the class of hyperspherical coordinates determined by (28) and acting with H_L and H_R on them, we obtain in E^5 two classes of the hyperspherical coordinates (30) and (31).

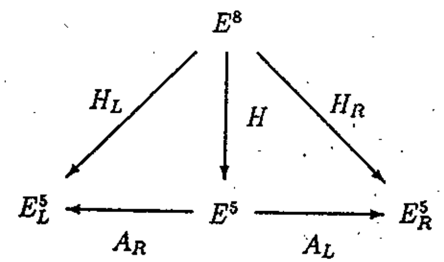
Resuming, it is to be noted, we can use on the decomposition of E^8 in agreement with the scheme

$$E^8 = E^4 \otimes E^4$$

in the approach developed here. Just the same decomposition corresponds to the hyperspherical coordinates (28). Other decompositions (for example, $E^8 = E^3 \otimes E^5$) are out of our consideration.

IV. The geometric structure of the H-transformation

We can develop connections between $\mathbf{H}$, $\mathbf{H}_L$ and $\mathbf{H}_R$ from somewhat general position. For this aim it is convenient to use the following diagram



From the latter the structures of the $\mathbf{H}_L$- and $\mathbf{H}_R$-transformations are clear. So, as the matrices $\mathbf{A}_L$ and $\mathbf{A}_R$ are orthogonal and unimodular, they realise the rotations. Therefore, the maps $E^8 \to E^5_L$ and $E^8 \to E^5_R$ are equivalent to compositions of maps $E^8 \to E^5 \to E^5_L$ ($E^8 \to E^5 \to E^5_R$).

The $\mathbf{A}_L$- and $\mathbf{A}_R$-rotations "switch off" the dependence of the coordinates in E^5 (x- space) on the angles parameterizing two corresponding subsets of variables and lead to (26) and (27).

Now, let us clarify the geometric structure of the H-transformation. It is easy to obtain that the matrix (16) is a product of three matrices

$$\mathrm{H} = \mathrm{A}_L^T \mathrm{H}_0 \mathrm{A}_L^S, \qquad (32)$$

where $\mathbf{A}_L^T$ is a transponent matrix of $\mathbf{A}_L$, $\mathbf{A}_L^S$ is an orthogonal matrix that is the "skew" transposed matrix of 4x4 - blocks of $\mathbf{A}_L$:

$$\mathbf{A}_L^S = \begin{pmatrix} \mathbf{Q}_L & 0 \\ 0 & \mathbf{I} \end{pmatrix},$$

$\mathbf{H}_0$ is a "spaced $\mathbf{H}$ - matrix" and has the following form:

$$\mathbf{H}_0 = \begin{pmatrix} u_L & 0 & 0 & 0 & -u_4 & -u_5 & -u_6 & -u_7 \\ 0 & -u_L & 0 & 0 & -u_5 & u_4 & u_7 & -u_6 \\ 0 & 0 & -u_L & 0 & -u_6 & -u_7 & u_4 & u_5 \\ 0 & 0 & 0 & -u_L & -u_7 & u_6 & -u_5 & u_4 \\ u_4 & -u_5 & -u_6 & -u_7 & u_L & 0 & 0 & 0 \\ u_5 & u_4 & -u_7 & -u_6 & 0 & u_L & 0 & 0 \\ u_6 & u_7 & u_4 & -u_5 & 0 & 0 & u_L & 0 \\ u_7 & -u_6 & u_5 & u_4 & 0 & 0 & 0 & u_L \end{pmatrix} \qquad (33)$$

$\mathbf{H}_0$, as $\mathbf{H}$, satisfies the otrhogonality condition

$$\mathrm{H}_0 \mathrm{H}_0^T = u^2$$

The sequence of the maps (32) gives

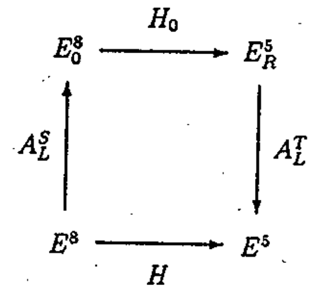
$$\mathbf{H}\begin{pmatrix} \mathbf{U}_L \\ \mathbf{U}_R \end{pmatrix} = \mathbf{A}_L^T \mathrm{H}_0 \begin{pmatrix} \mathbf{U}'_L \\ \mathbf{U}_R \end{pmatrix} = \mathbf{A}_L^T \begin{pmatrix} \mathbf{Y}_L \\ \mathbf{Y}_R \end{pmatrix} = \begin{pmatrix} \mathbf{X}_L \\ \mathbf{X}_R \end{pmatrix}$$

where

$$\mathbf{U}'_L = \begin{pmatrix} u_L \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{Y}_L = \begin{pmatrix} y_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_R = \begin{pmatrix} y_4 \\ y_5 \\ y_6 \\ y_7 \end{pmatrix}$$

and $\mathbf{U}_L$, $\mathbf{U}_R$, $\mathbf{X}_L$ and $\mathbf{X}_R$ are determed by (21).

Now, we can represent the structure of the H-transformation through the diagram



So, the H-mapping is equivalent to the following three steps:

A_L^S - rotation, in essence, coinciding with the Hopf's mapping (the so-called "quaternionic fibration" [16]),

H_0 - local scale transformation, a "straightforward generalization" of the Levi-Civita matrix (4),

A_L^T - rotation that violates the "regular" hyperspherical map of E_R^5 and leads to (8).

From this point of view, the transformation H_R is the following composition: $H_0 A_L^S$.

Resume

The versions H_L and H_R of the H-transformation violating the bilinearity are suggested. We show the following

1. H_L and H_R conserve three important properties of the H-transformation: they are orthogonal, realize the reduction $E^8 \rightarrow E^5$.

2. The relation of the H_L and H_R with the H-transformation is established.

3. The H_L and H_R transformations acting on the 8-dimensional hyperspherical coordinates with the 4x4 - structure (28) project them onto the 5-dimensional hs-coordinates (30) and (31) with the 1x4 - structure. The seven hyperspherical angles may be sorted into three groups

$$(\theta);(\alpha_L,\beta_L,\gamma_L),(\alpha_R,\beta_R,\gamma_R)$$

   $H_L(H_R)$ transforms $u \rightarrow u^2$ and $\theta/2 \rightarrow \theta$, conserving (or shutting) L and R angular triplets, respectively.

4. The geometric structure of the H-transformation is revealed.

Acknowledgments

We are thankful to L.G. Mardoyan for interesting discussions. We would like to thank also M. Kibler for presenting his (with M. Hage Hassan) paper, having resurrected our old interest to the parameterization problem of the Hurwitz transformation. The authors are grateful to V.N. Pervushin and G.S. Pogosyan for useful remarks.

References

[1] D. Lambert and M. Kibler, J.Phys.: Math.Gen. **A21**, 307-43 (1988).

[2] R.D. Schafer, *An Introduction to Nonassociative Algebras* (Academic Press, New York and London, 1966).

[3] T. Levi-Civita, *Sur la Résolution Qualitative di Probléme Restreint des Trois Corps*, Opere Mathematiche 2, 411-17 (1956).

[4] P. Kustaanheimo, *Spinor Regularization of the Kepler Motion*, Ann. Univ. Turku. Ser. **A1**,73 (1964);
E.L. Stiefel, G. Scheifele, Linear and Regular Celestial Mechanics (Springer-Verlag, Berlin, Heidelberg and New York, 1971).

[5] A. Cisneros and H.V. McIntosh, J.Math.Phys. **10**(2),277-86 (1969);
M. Kibler, A. Ronveaux, T. Negadi, J.Math.Phys., **27**(6),1541-8 (1986);
L.G. Mardoyan, G.S. Pogosyan, A.N. Sissakian and V.M. Ter-Antonyan, *Exact Solution of the Kibler-Ronveaux-Negadi Problem*, preprint JINR, P2-86-431, Dubna (in Russian) (1986).

[6] M. Kibler, T. Negadi, Croatica Chemica Acta, CCACAC, **57**, 1509 (1984).

[7] A.O. Barut, C.K.E. Schneider, R. Wilson, J.Math.Phys., **20**(11), 2244-56 (1979).

[8] A. Inomata, G. Junker, R. Wilson. Found. of Phys. **23**, 1073-90 (1993);
L. Chetouani, L. Guechi, T.F. Hammann. J.Math.Phys. **33**(10), 3410-18 (1992).

[9] L.S. Davtyan, L.G. Mardoyan, G.S. Pogosyan, A.N. Sissakian and V.M. Ter-Antonyan, J.Phys.: Math.Gen. **A20**, 6121-25 (1987).

[10] M. Kibler and P. Winternitz, J.Phys.: Math.Gen. **A21**, 1787-93 (1988).

[11] I.V. Polubarinov, *On Application of Hopf Fiber Bundles in Quantum Theory*, preprint JINR, E2-84-607, Dubna (1984).

[12] Le Van Hoang, Tony J. Viloria, and Le anh Thu, J.Phys.: Math.Gen. **A24**, 3021-30 (1991).

[13] L.S. Davtyan, J.Math.Phys., **34**(10), 4834-39 (1993).

[14] I.V. Mladenov and J. Tsanov, J.Geom. and Phys. **2**, 17 (1985).

[15] M. Hage Hassan and M. Kibler, *On Hurwitz Transformations*, preprint LYCEN 9110, Lyon (1991).

[16] B.A. Dubrovin, S.P. Novikov and A.T. Fomenko, *Modern Geometry. Methods and Applications* (in Russian), (Nauka, Moscow, 1979).

[17] L.G. Mardoyan, A.N. Sissakian and V.M. Ter-Antonyan, *The Eulerian parameterization of the Hurwitz transformation*, preprint JINR, E5-94-121,Dubna (1994).

Received by Publishing Department
on April 6, 1994.