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INTERRELATIONS BETWEEN QUANTUM GROUPS AND REFLECTION EQUATION (BRAIDED) ALGEBRAS

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1. In the present letter we demonstrate the intimate interrelations between two famous q-algebras: quantum groups (matrix pseudo-groups) [1, 2] with the defining relations

$$\mathbf{R}\,T\,T' = T\,T'\,\mathbf{R} \tag{1}$$

and reflection equation (or braided) algebras (see [3]-[6] and references therein) with the commutation relations

$$\mathbf{R}\,u\,\mathbf{R}\,u = u\,\mathbf{R}\,u\,\mathbf{R}\,. \tag{2}$$

Here and below we use the R-matrix formalism [1] with slightly modified matrix notation [7] to simplify the appropriate calculations. Namely, we use

$$\begin{gathered} T \equiv T_1 = T^{i_1}_{j_1}\delta^{i_2}_{j_2}\,,\;\; T' \equiv T_2 = \delta^{i_1}_{j_1}T^{i_2}_{j_2}\,, \\ u \equiv u_1 = u^{i_1}_{j_1}\delta^{i_2}_{j_2}\,,\;\; v' \equiv v_2 = \delta^{i_1}_{j_1}v^{i_2}_{j_2}\,, \\ \mathbf{R} \equiv \hat{R}_{12} = P_{12}R_{12}\,,\;\; \mathbf{R}' \equiv \hat{R}_{23} = P_{23}R_{23}\,, \end{gathered} \tag{3}$$

where $T^i_j,\ u^i_j,\ v^i_j$ are quantum $N \times N$ matrices; $\mathbf{R},\ \mathbf{R}' \in Mat_N \times Mat_N$ are invertible R-matrices; P_{12} is the permutation matrix and indices $1, 2, 3, \ldots$ enumerate matrix spaces. The Yang-Baxter equation for R-matrices in this notation reads

$$R_{12}R_{13}R_{23} = R_{32}R_{13}R_{12} \Leftrightarrow \mathbf{R}\mathbf{R}'\mathbf{R} = \mathbf{R}'\mathbf{R}\mathbf{R}'.$$

First of all we recall some known facts about the algebras (1),(2) to be used below. Both algebras (1) and (2) are the Hopf [1] and braided Hopf [6] algebras respectively. The structure mappings for them are:

$$\Delta(T^i_j) = T^i_k \otimes T^k_j \equiv (T\,\tilde{T})^i_j,\ \mathcal{S}(T) = T^{-1},\ \epsilon(T^i_j) = \delta^i_j, \tag{4}$$

$$\Delta(u^i_j) = u^i_k \underline{\otimes}\, u^k_j \equiv (u\,\tilde{u})^i_j,\ \mathcal{S}(u) = u^{-1},\ \epsilon(u^i_j) = \delta^i_j\,, \tag{5}$$

where $\otimes,\ \mathcal{S}(T),\ \epsilon(T)$ are the operator tensor product, antipode and counit (see [1]) while $\underline{\otimes},\ \mathcal{S}(u),\ \epsilon(u)$ are the braided tensor product, braided antipode and braided counit (for their definition see [4]-[6]) . The braiding for the algebra (2),(5) is defined by the relations [4]-[6]:

$$\mathbf{R}^{-1}(1 \underline{\otimes}\, u)\mathbf{R}(u \underline{\otimes}\, 1) = (u \underline{\otimes}\, 1)\mathbf{R}^{-1}(1 \underline{\otimes}\, u)\mathbf{R} \Leftrightarrow \mathbf{R}^{-1}\tilde{u}\mathbf{R}u = u\mathbf{R}^{-1}\tilde{u}\mathbf{R}\,, \tag{6}$$

which specify the braided tensor product $\underline{\otimes}$ in (5). The "braiding" for the quantum groups (1),(4) is trivial $[T^i_j,\ \tilde{T}^k_l] \equiv [T,\ \tilde{T}'] = 0$.

It is well known that the algebra (2) is a covariant comodule with respect to the adjoint coaction of the quantum group (1),(4):

$$u^i_j \rightarrow \Delta_A(u^i_j) = T^i_k\mathcal{S}(T)^l_j \otimes u^k_l \equiv (TuT^{-1})^i_j\,, \tag{7}$$

where we imply again in the last equality the "trivial braiding":

$$[u,\, T'] = 0. \tag{8}$$

On the other hand, one can find that the algebra (1) is a covariant braided comodule with respect to the left braided coaction of the braided Hopf algebra (2),(5):

$$T^i_j \to \Delta_B(T^i_j) = u^i_k \underline{\otimes} T^k_j \equiv u\,T, \tag{9}$$

with the nontrivial braiding

$$Tu' = \mathbf{R} u \mathbf{R}^{-1} T. \tag{10}$$

Indeed, $\Delta_B(T)$ (9) satisfy the R-T-T relations (1) in view of eq.(10). Then, one can prove that the comodule axiom

$$(\Delta \underline{\otimes} id)\Delta_B = (id \underline{\otimes} \Delta_B)\Delta_B \tag{11}$$

and the relation (10) are consistent with the braiding (6) specific for the braided Hopf algebra (2),(5). Thus, we have demonstrated the interplay of the quantum groups (1) and the reflection equation (braided) algebras (2). Namely, in addition to the general phylosophy that the algebras (1) and (2) are related by the process of transmutation [5],[6], we have shown that these two algebras can be considered as covariant comodules with respect to the (braided) coactions of one to another.

2. The main result of this letter is that an analogous interplay is inherited for the differential extensions of the algebras (1) and (2). Here we consider the case of linear quantum groups with the $GL_q(N)$ R-matrix. In fact, we need only the Hecke condition for the R-matrix:

$$\mathbf{R}^2 = (q - q^{-1})\mathbf{R} + 1\,. \tag{12}$$

The differential Hopf algebra over $GL_q(N)$ (denoted as Ω_A) with the generators $\{T^i_j,\, dT^i_j\}$ is defined by eq.(1) and commutation relations [7]-[12]:

$$\begin{aligned} \mathbf{R}(dT)T' &= T(dT)'\mathbf{R}^{-1}\,, & (13)\\ \mathbf{R}(dT)(dT)' &= -(dT)(dT)'\mathbf{R}^{-1}\,. & (14) \end{aligned}$$

The corresponding structure maps are given by eq.(4) and [8],[12],[13]

$$\begin{gathered} \Delta(dT) = dT \otimes T + T \otimes dT \equiv dT\tilde{T} + Td\tilde{T}, \\ \mathcal{S}(dT) = -T^{-1}dTT^{-1},\ \epsilon(dT) = 0\,. \end{gathered} \tag{15}$$

To define the differential braided Hopf algebra over $BM_q(N)$ (denoted as Ω_B) we have to consider in addition to eq.(2) the following relations [14]:

$$\begin{gathered} \mathbf{R}^{-1}\, u\, \mathbf{R}\, du = du\, \mathbf{R}\, u\, \mathbf{R}, \\ \mathbf{R}\, du\, \mathbf{R}\, du = -du\, \mathbf{R}\, du\, \mathbf{R}^{-1}, \end{gathered} \tag{16}$$

and the extension of the comultiplication (5) is [15]:

$$\Delta(du) = du \underline{\otimes} u + u \underline{\otimes} du \equiv du\, \tilde{u} + u\, d\tilde{u}. \tag{17}$$

Now our propositions are:

Proposition 1. *The differential algebra Ω_B (2),(16) is a covariant comodule algebra with respect to the following coaction (homomorphism) of the algebra Ω_A (1),(13),(14):*

$$\begin{aligned} \Delta_A(u) &= T\,u\,T^{-1}\,, & (18)\\ \Delta_A(du) &= dT\,u\,T^{-1} + T\,du\,T^{-1} + T\,u\,dT^{-1}\,. & (19) \end{aligned}$$

The braiding is trivial and defined by eq.(8) and

$$[du,\, T'] = [u,\, dT'] = [du,\, dT']_+ = 0\,.$$

Proposition 2. *The differential algebra Ω_A (1),(13),(14) is a covariant braided comodule with respect to the braided coaction (homomorphism) of Ω_B (2),(16):*

$$\begin{aligned} \Delta_B(T) &= u\,T\,, & (20)\\ \Delta_B(dT) &= du\,T + u\,dT\,. & (21) \end{aligned}$$

The braiding is nontrivial and given by eq.(10) and

$$\begin{gathered} dT\,u' = \mathbf{R}\,u\,\mathbf{R}^{-1}\,dT\,,\quad T\,du' = \mathbf{R}\,du\,\mathbf{R}^{-1}\,T, \\ dT\,du' = -\mathbf{R}\,du\,\mathbf{R}^{-1}\,dT. \end{gathered} \tag{22}$$

The proofs of Propositions 1 and 2 are straightforward. For the illustration we verify the homomorphism (20),(21) using, for example, the relation (13)

$$\begin{aligned} &\mathbf{R}\,\Delta_B(dT)\,\Delta_B(T') = \mathbf{R}\,du\,\underline{T\,u'}\,T' + \mathbf{R}\,u\,\underline{dT\,u'}\,T' = \\ &= \underline{\mathbf{R}\,du\,R\,u}\;\; \underline{\mathbf{R}^{-1}T\,T'} + \mathbf{R}\,\underline{u\,R\,u\mathbf{R}^{-1}}\,\mathbf{R}^{-1}\,\underline{\mathbf{R}\,dT\,T'} = \\ &= u\,\underline{\mathbf{R}\,du\,\mathbf{R}^{-1}\,T}T'\,\mathbf{R}^{-1} + u\,\underline{\mathbf{R}\,u\,\mathbf{R}^{-1}\,T}\,dT'\,\mathbf{R}^{-1} = \\ &= u\,T(du'\,T' + u'\,dT')\,\mathbf{R}^{-1} = \Delta_B(T)\,\Delta_B(dT')\,\mathbf{R}^{-1} \end{aligned}$$

(underlining indicates the parts to which the next operation is to be applied). Analogous calculations for eq.(14) are in fact optional because their result can be foreseen from the differentiating the equality just obtained.

The comodule axiom (11) and the braiding relations (22) are consistent with the braiding relations for the differential algebra over $BM_q(N)$ (this braiding and the comultiplication (17) have been proposed by A.A.Vladimirov and published in [15])

$$\left\{ \begin{array}{l} \mathbf{R}^{-1}\,\tilde{u}\,\mathbf{R}\,u = u\,\mathbf{R}^{-1}\,\tilde{u}\,\mathbf{R}\,, \\ \mathbf{R}^{-1}\,d\tilde{u}\,\mathbf{R}\,u = u\,\mathbf{R}^{-1}\,d\tilde{u}\,\mathbf{R}\,, \\ \mathbf{R}^{-1}\,\tilde{u}\,\mathbf{R}\,du = du\,\mathbf{R}^{-1}\,\tilde{u}\,\mathbf{R}\,, \\ \mathbf{R}^{-1}\,d\tilde{u}\,\mathbf{R}\,du = -du\,\mathbf{R}^{-1}\,d\tilde{u}\,\mathbf{R}\ . \end{array} \right. \tag{23}$$

It can be verified by the substitution $T \to \tilde{u}T$ into the formulas (22).

In the papers [10],[12],[13],[16] it has been shown that the comultiplication (4),(15) for the differential algebra Ω_A (1),(13),(14) leads to the relation

$$\Delta_A(\Omega) = \tilde{T}\Omega\tilde{T}^{-1} + d\tilde{T}\tilde{T}^{-1}, \tag{24}$$

where Cartan's 1-forms $\Omega = dTT^{-1}$ satisfy

$$\mathbf{R}\Omega\mathbf{R}\Omega + \Omega\mathbf{R}\Omega\mathbf{R}^{-1} = 0 \tag{25}$$

and $[\Omega,\ \tilde{T}'] = 0$. Then, one can introduce the noncommutative 1-form connections A (transformed as in (24)) and the curvature 2-forms $F = dA - A^2$ to formulate the so-called quantum group covariant noncommutative geometry [12]. The same procedure for the algebra Ω_A, but with the braided coaction (20),(21), yields the formula (cf. with (24))

$$\Delta_B(\Omega) = u\,\Omega\,u^{-1} + du\,u^{-1}. \tag{26}$$

Here u^{-1} is a braided antipode introduced by Sh.Majid in [4]-[6] and one can deduce from (22) the corresponding braiding relations:

$$\begin{array}{c} \Omega\,\mathbf{R}\,u\,\mathbf{R}^{-1} = \mathbf{R}\,u\,\mathbf{R}^{-1}\,\Omega, \\ \Omega\,\mathbf{R}\,du\,\mathbf{R}^{-1} = -\mathbf{R}\,du\,\mathbf{R}^{-1}\,\Omega \end{array} \tag{27}$$

demonstrating the noncommutativity of the "transformation group" elements u^i_j and 1-forms Ω. Now one can again substitute, instead of Ω, the 1-form connections A transformed as in (26) and satisfying relations (25),(27). Then the curvature 2-forms $F = dA - A^2$ are transformed homogeneously

$$\Delta_B(F) = uFu^{-1}. \tag{28}$$

The braiding for the operators F and u is deduced from eqs.(27)

$$F\,\mathbf{R}\,u\,\mathbf{R}^{-1} = \mathbf{R}\,u\,\mathbf{R}^{-1}\,F, \tag{29}$$

$$F\,\mathbf{R}\,du\,\mathbf{R}^{-1} = \mathbf{R}\,du\,\mathbf{R}^{-1}\,F, \tag{30}$$

and, as it was shown in [5],[6], the relations (28),(29) (for arbitrary R-matrices) respect the commutation relations for the curvature 2-forms

$$\mathbf{R}F\mathbf{R}F = F\mathbf{R}F\mathbf{R}. \tag{31}$$

Moreover, relations (26)-(30) with substitution $\Omega \to A$ respect the following cross-commutator for A and F:

$$\mathbf{R}A\mathbf{R}F = F\mathbf{R}A\mathbf{R}. \tag{32}$$

Thus we have the following
Proposition 3. *The algebra*

$$\mathbf{R}A\mathbf{R}A\ +\ A\mathbf{R}A\mathbf{R}^{-1} = 0\,, \tag{33}$$
$$\mathbf{R}A\mathbf{R}F\ =\ F\mathbf{R}A\mathbf{R}\,, \tag{34}$$
$$\mathbf{R}F\mathbf{R}F\ =\ F\mathbf{R}F\mathbf{R}\,, \tag{35}$$

(for the Hecke type R-matrix (12)) is a covariant braided comodule algebra with respect to the braided coaction of Ω_B (2),(16) (differential extension of $BM_q(N)$):

$$\Delta_B(A^i_j) = u^i_k\,(u^{-1})^l_j \otimes A^k_l + du^i_k\,(u^{-1})^k_j \otimes 1 \equiv (u\,A\,u^{-1})^i_j + (du\,u^{-1})^i_j\,, \tag{36}$$

$$\Delta_B(F^i_j) = u^i_k\,(u^{-1})^l_j \otimes F^k_l \equiv (u\,F\,u^{-1})^i_j\,. \tag{37}$$

The braiding is nontrivial:

$$F\,\mathbf{R}\,u\,\mathbf{R}^{-1} = \mathbf{R}\,u\,\mathbf{R}^{-1}\,F,\ A\,\mathbf{R}\,u\,\mathbf{R}^{-1} = \mathbf{R}\,u\,\mathbf{R}^{-1}\,A, \tag{38}$$

$$F\,\mathbf{R}\,du\,\mathbf{R}^{-1} = \mathbf{R}\,du\,\mathbf{R}^{-1}\,F,\ A\,\mathbf{R}\,du\,\mathbf{R}^{-1} = -\mathbf{R}\,du\,\mathbf{R}^{-1}\,A, \tag{39}$$

and the comodule axiom (11) is consistent with the braiding relations (23).

Propositions 1 and 2 establish the closed relations between the differential extensions of the quantum groups Ω_A (1),(13),(14) and the reflection equation (braided) algebras Ω_B (2),(16). Proposition 3 shows that the algebra (33)-(35) is covariant not only under the coaction of Ω_A (see [12]) but also covariant under the braided coaction of Ω_B (36),(37).

3. To conclude this letter we would like to make some remarks.

A.) As it has been pointed out above, the algebra (33)-(35) considered in Proposition 3 has the geometrical interpretation when the generators A, F are associated via the relation $F = dA - A^2$. Namely, in this case, one can consider A as 1-form connections while F as curvature 2-forms. Now we introduce the braided adjoint co-invariants using the well known q-trace [1],[17]

$$C_{2k} = Tr_q(F^k) = Tr(DF^k), \tag{40}$$

where the matrix D is related to the R-matrix (see e.g. [17],[3],[5],[11]):

$$D^i{}_j = \tilde{R}^{ki}_{jk} = Tr_{(2)}\left(P_{12}((R^{t_1}_{12})^{-1})^{t_1}\right)$$

and $\tilde{R} = ((R^{t_1}_{12})^{-1})^{t_1}$. By definition, the $2k$-forms C_{2k} (40) are co-invariants not only under the adjoint coaction of the quantum groups (1) (see [12]) but also under the braided co-transformations (37). Moreover, these $2k$-forms commute with A and F and, as it has been shown in [12], they are closed:

$$dC_{2k} = Tr_q(AF^k - F^kA) = 0 \ . \tag{41}$$

To prove the last equality in (41) one has to use the commutation relations (34). Therefore, the central elements C_{2k} could be interpreted as noncommutative analogs of the Chern characters.

B.) One can generalize Proposition 2 in the following way. Let us consider the differential algebra $\overline{\Omega}_B$ generated by $\{v^i_j,\ dv^i_j\}$ (cf. with (2),(16)):

$$\begin{array}{c}\mathbf{R}^{-1}\,v'\,\mathbf{R}^{-1}\,v' = v'\,\mathbf{R}^{-1}\,v'\,\mathbf{R}^{-1},\\ \mathbf{R}^{-1}\,v'\,\mathbf{R}^{-1}\,dv' = dv'\,\mathbf{R}^{-1}\,v'\,\mathbf{R},\\ \mathbf{R}^{-1}\,dv'\,\mathbf{R}^{-1}\,dv' = -dv'\,\mathbf{R}^{-1}\,dv'\,\mathbf{R}.\end{array} \tag{42}$$

This algebra is a differential braided Hopf algebra with the comultiplication

$$\begin{array}{c}\Delta(v) = v\,\underline{\otimes}\,v \equiv v\,\tilde{v}\ ,\\ \Delta(dv) = dv\,\underline{\otimes}\,v + v\,\underline{\otimes}\,dv \equiv dv\,\tilde{v} + v\,d\tilde{v},\end{array} \tag{43}$$

and braided relations (cf. with (23))

$$\left\{\begin{array}{l}\mathbf{R}\,\tilde{v}'\,\mathbf{R}^{-1}\,v' = v'\,\mathbf{R}\,\tilde{v}'\,\mathbf{R}^{-1}\ ,\\ \mathbf{R}\,d\tilde{v}'\,\mathbf{R}^{-1}\,v' = v'\,\mathbf{R}\,d\tilde{v}'\,\mathbf{R}^{-1}\ ,\\ \mathbf{R}\,\tilde{v}'\,\mathbf{R}^{-1}\,dv' = dv'\,\mathbf{R}\,\tilde{v}'\,\mathbf{R}^{-1}\ ,\\ \mathbf{R}\,d\tilde{v}'\,\mathbf{R}^{-1}\,dv' = -dv'\,\mathbf{R}\,d\tilde{v}'\,\mathbf{R}^{-1}\ .\end{array}\right. \tag{44}$$

Then, we have the following:

Proposition 4. *The differential algebra Ω_A (1),(13),(14) is a covariant braided comodule with respect to the braided coaction (homomorphism) of two commuting algebras Ω_B (2),(16) and $\overline{\Omega}_B$ (42)* ($[\Omega_B,\ \overline{\Omega}_B]_\pm = 0$). *This coaction of $\Omega_B \otimes \overline{\Omega}_B$ can be represented in the form:*

$$\Delta_{LR}(T) \ = \ u\,T\,v\ , \tag{45}$$

$$\Delta_{LR}(dT) \ = \ du\,T\,v + u\,dT\,v + u\,T\,dv\ . \tag{46}$$

The braiding is defined by eqs.(10),(22) and

$$\begin{array}{c}v\,T' = T'\,\mathbf{R}\,v'\,\mathbf{R}^{-1}\ ,\ \ v\,dT' = dT'\,\mathbf{R}\,v'\,\mathbf{R}^{-1}\ ,\\ dv\,T' = T'\,\mathbf{R}\,dv'\,\mathbf{R}^{-1}\ ,\ \ dv\,dT' = -dT'\,\mathbf{R}\,dv'\mathbf{R}^{-1}\ .\end{array} \tag{47}$$

The proof of this Proposition is the same as the proof of Proposition 2.

Propositions 1-4 lead us to the natural conjecture that the differential algebras Ω_A (1),(13), (14) and Ω_B (2),(16) are related by the process of transmutation considered by Sh.Majid in [18],[5],[6].

C. There are some arguments that the braided quantum group covariant noncommutative geometry (briefly discussed in Proposition 3 and in the subsection A.)) could be associated with the global version of the q-gauge theories proposed by L.Castellani [19]. For example, the matrix elements u^i_j (generating "gauge transformations" (36),(37)) do not commute with the "q-gauge fields" A and F (see [19] and (38),(39)). It would be very interesting to trace these relations completely by means of formulating the "infinitesimal version" of eqs.(36),(37).

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