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BOUNDARY EFFECTS IN A TWO-DIMENSIONAL ABELIAN SANDPILE

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1 Introduction

Since the publication of the works by Bak, Tang and Wiesenfeld [1], critical Abelian sandpile models are in the focus of comprehensive studies, see, e.g. [2, 3, 4, 5]. The stochastic evolution of a sandpile naturally leads to a state of self-organized criticality (SOC), which is characterized by correlations with power-law decay in space and time. As the spatial aspects of the SOC are similar to those of a critical state in statistical mechanics, the program for studying the structure of SOC states essentially parallels the one for the usual statistical models like the Ising or the Potts models. Besides the evaluation of bulk critical exponents, an important item in this program is the determination of different surface exponents describing the behaviour of correlations near a boundary. In the two-dimensional case, finite-size analysis [6] and conformal field-theory [7] establish relationships between surface and bulk properties of the models, which lead to a complete description of the spatial structure at criticality. Thus, finding a correspondence between the observables of a sandpile and an appropriate model in statistical mechanics at different boundary conditions seems an attractive task.

The number of distinct recurrent configurations of the Abelian sandpile model on an arbitrary lattice has been expressed through the number of spanning trees on the same lattice [8, 9]. The statistics of spanning trees obeys the *Kirchhoff* theorem and can be, in turn, related to the $q = 0$ limit of the q-component Potts model [10, 11]. In drawing a further analogy, however, one encounters the difficulty of identifying local sandpile observables as site variables in the Potts model. The natural formulation of a sandpile model is given in terms of integer height variables z_i at each lattice site $i \in \mathcal{L}$ and toppling rules. The stable configurations in the SOC state of the Abelian model are sets of heights $\{z_i, i \in \mathcal{L}\}$, where $z_i \in \{1, 2, 3, 4\}$. Majumdar and Dhar [12] have calculated the probability $\mathcal{P}_1$ of finding the value $z_i = 1$ and the correlation function $\mathcal{P}_{11}(r)$ for two sites of unit height at a distance r apart. In the two dimensional case, the result $\mathcal{P}_{11}(r) \sim r^{-2x}$, with $x = 2$, has been obtained for large r, which provided grounds for suggesting [9] that $\mathcal{P}_{11}(r)$ is the counterpart of the energy-energy correlator in the zero-component Potts model. For the Potts model, besides the bulk exponent $x = 2$, the surface correlations decay exponent is known too, $x_{||} = 2$, in the case of free boundary conditions [13, 14]. The value of the surface exponent $x_{||}$, in turn, is related to the amplitude A of the inverse correlation length $\xi^{-1} = A/L$, which controls the exponential decay of pair correlations along an infinitely long strip of width L. Namely, the amplitude-exponent relations [15]

$$A = \begin{cases} \pi x_{||} & , \quad \text{free b.c.} \\ 2\pi x & , \quad \text{periodic b.c.} \end{cases} \tag{1}$$

follow from finite-size scaling [6] and the conformal properties of the model [14].

In this paper, we consider the behaviour of the probability $\mathcal{P}_1$ and the correlation function $\mathcal{P}_{11}(r)$ near the boundary of the Abelian two-dimensional sandpile model [8]. The toppling rules are specified by the matrix Δ with elements Δ_{ij} for sites i, j in the bulk of the lattice given by

$$\Delta_{ij} = \begin{cases} 4 & , \quad i = j \\ -1 & , \quad |i - j| = 1 \\ 0 & , \quad \text{otherwise} \end{cases} . \tag{2}$$

Here $|i-j|$ denotes the distance between sites i and j.

To formulate the toppling rules at the boundary $\partial\mathcal{L}$ of the sandpile, we use three standard boundary value problems for the *Laplacian* on a finite lattice $\mathcal{L}$.

1. Open boundary conditions (or *Dirichlet* boundary conditions), when $\Delta_{ii} = 4$ for $i \in \partial\mathcal{L}$, and, therefore, the sand particles are allowed to leave the system through the boundary.

2. Closed boundary conditions (or *Neumann* boundary conditions), when $\Delta_{ii} = 3$ for $i \in \partial\mathcal{L}$, and the sand particles cannot leave the system through $\partial\mathcal{L}$.

3. Periodic boundary conditions, when $\Delta_{ii} = 4$ as in case 1, but now the lattice is wrapped on a torus and $|i-j|$ is the distance along the surface of the torus.

Obviously, if all the boundaries are closed or periodic no steady state of the sandpile is possible. Therefore, in considering the strip geometry, we start with a rectangular shape of the lattice $\mathcal{L}$, impose open boundary conditions on the vertical edges of the rectangle and boundary conditions of one of the types 1,2 or 3 on the opposite horizontal edges. Then, we let the horizontal size of the rectangle tend to infinity.

The following results are reported here.
1. In the case of half-plane geometry, the probability $\mathcal{P}_1(0)$ of a height $z = 1$ at the boundary is obtained:
for **open** boundary conditions

$$\mathcal{P}_1(0) = \frac{9}{2} - \frac{42}{\pi} + \frac{320}{3\pi^2} - \frac{512}{9\pi^3} \approx 0.10382 \quad , \tag{3}$$

for **closed** boundary conditions

$$\mathcal{P}_1(0) = \frac{3}{4} - \frac{2}{\pi} \approx 0.11338 \quad . \tag{4}$$

On moving inside the sample, the probability $\mathcal{P}_1(l)$ at a distance l from the boundary tends to its bulk value $\mathcal{P}_1(\infty) = 2/\pi^2 - 4/\pi^3$ according to the law:

$$\mathcal{P}_1(l) = \mathcal{P}_1(\infty)(1 \pm \frac{1}{4l^2} + \cdots) \quad , \tag{5}$$

where the upper sign is related to the **open** boundary conditions and the bottom sign to the **closed** ones.

The two-point correlation function $\mathcal{P}_{11}(r)$ at the surface decays according to the law:
for **open** boundary conditions

$$\mathcal{P}_{11}(r) = -4\mathcal{P}_1^2(0)\frac{(9\pi-32)^2}{(27\pi^2-108\pi+64)^2 r^4} + \cdots \quad , \tag{6}$$

for **closed** boundary conditions

$$\mathcal{P}_{11}(r) = -\mathcal{P}_1^2(0)\frac{16}{\pi^2 r^4} + \cdots \quad . \tag{7}$$

2. In the case of an infinitely long strip of width L, the leading exponential decay of the pair correlations $\mathcal{P}_{11}(r)$ along the central line of the strip is obtained.
(a) For both **open** and **closed** boundary conditions

$$\mathcal{P}_{11}(r;L) = -\mathcal{P}_1^2(L/2)\frac{\pi^4}{L^4}e^{-2\pi r/L} + \cdots \quad , \tag{8}$$

where the finite-size corrections to the probability $\mathcal{P}_1(L/2)$ in the middle of the strip are:

$$\mathcal{P}_1(L/2) = \mathcal{P}_1(\infty)(1 \pm \frac{\pi^2}{4L^2} + \cdots) \quad . \tag{9}$$

Here the upper sign is related to the **open** boundary conditions and bottom sign to the **closed** ones.
(b) For **periodic** boundary conditions the pair correlations take the form

$$\mathcal{P}_{11}(r;L) = -\mathcal{P}_1^2(L/2)\frac{8\pi^4}{L^4}e^{-4\pi r/L} + \cdots \quad , \tag{10}$$

where

$$\mathcal{P}_1(L/2) = \mathcal{P}_1(\infty)(1 - \frac{4\pi^5}{15(\pi-2)L^4} + \cdots) \quad . \tag{11}$$

The above results can be compared with the predictions of the conformal field-theory. Thus, from (6,7) it follows that $x_{\|} = 2$, and from (8) one has $A = 2\pi$, which is in agreement with (1). For periodic boundary conditions equation (10) yields $A = 4\pi$, and the bulk exponent $x = 2$ has been obtained by Majumdar and Dhar [12], which is again in conformity with the prediction (1).

Thus our results confirm the hypothesis about the correspondence between the unit height in the Abelian sandpile model and the energy in the zero-component Potts model. As far as other observables are concerned, e.g. heights 2, 3, 4, sizes of avalanches, their duration and perimeters, we lack at present convincing evidences of their relation to observables in the Potts model. A short discussion of that problem is given in the final section.

2 Correlations and Green functions

The main tool for evaluation of height probabilities and correlations between them is a mapping of the set of sandpile configurations onto the set of spanning trees. Dhar [8] has shown that the number $\mathcal{N}_R$ of sandpile configurations in the SOC state, as well as the number of spanning trees, is given by the simple expression

$$\mathcal{N}_R = \det \Delta \quad . \tag{12}$$

The different height probabilities on a lattice $\mathcal{L}$ can be related also to the enumeration of certain types of spanning trees. In particular, the number of sandpile configurations with $z_i = 1$ at a given site i of $\mathcal{L}$ equals the number $\mathcal{N}_R^{(i)}$ of spanning trees on the same lattice $\mathcal{L}$ which have just one leaf attached to i and pointing in a fixed direction [9]. Following Majumdar and Dhar [12], one may construct a new lattice $\mathcal{L}^{(i)}$, such that the spanning trees covering $\mathcal{L}^{(i)}$ contain the fixed leaf. To this end, one just cuts off all the bonds attached to site i in $\mathcal{L}$, but the one containing the given leaf. As a result, a new toppling matrix $\Delta^{(i)}$ for the sandpile on $\mathcal{L}^{(i)}$ is obtained. Note that the defect matrix $B = \Delta^{(i)} - \Delta$ has non-zero elements only for the site i and its nearest neighbours but one. Thus the probability of finding the value $z_i = 1$ is given by the formula

$$\mathcal{P}_1 = \frac{\det \Delta^{(i)}}{\det \Delta} = \det(\mathbf{1} + GB) \quad , \tag{13}$$

where $\mathbf{1}$ is the unit matrix and $G = \Delta^{-1}$ is the lattice Green function. For example, if i is far away from the boundaries, $B_{ii} = -3$ and $B_{jj} = -1$, $B_{ij} = B_{ji} = 1$ for j being the left, right and lower neighbour of i. In this case $\mathcal{P}_1$ is independent of the choice of site i inside the bulk of the lattice $\mathcal{L}$. The pair correlation function $\mathcal{P}_{11}(r)$ can be defined in a similar way by constructing a new toppling matrix $\Delta^{(ij)}$ for the lattice, $\mathcal{L}^{(ij)}$ with defects of the above described type placed at a distance r apart;

$$\mathcal{P}_{11}(r) = \frac{\det \Delta^{(ij)}}{\det \Delta} = \det(\mathbf{1} + GB_{11}) \quad , \tag{14}$$

where B_{11} is the compound defect matrix.

Thus, the problems we have formulated in the Introduction can be reduced to the calculation of the lattice Green functions for different boundary conditions and finite-size analysis of the determinants (13) or (14) for the specified defects.

The eigenvalues and eigenfunctions of the discrete *Laplacian* Δ on a finite square lattice of the rectangular shape under the considered boundary conditions are well known. For a lattice of size $M \times L$ with open boundary conditions on the vertical edges of length L and open, closed or periodic boundary conditions on the horizontal edges, the lattice Green function $G^{(\tau)}_{LM}(n_1, m_1; n_2, m_2)$ takes the form

$$G^{(\tau)}_{LM}(n_1, m_1; n_2, m_2) = \sum_{p=1}^{L} \sum_{q=1}^{M} v^{(\tau)}_p(n_1) v^{(1)}_p(m_1) \frac{1}{\lambda^{(\tau)}_p + \lambda^{(1)}_q} \bar{v}^{(\tau)}_p(n_2) \bar{v}^{(1)}_q(m_2) \quad . \tag{15}$$

Here $n = 1, 2, \ldots, L$ labels the row and $m = 1, 2, \ldots, M$ labels the column of site $i = (n, m) \in \mathcal{L}$; $v_p(n)$ are the eigenfunctions and λ_p are the corresponding eigenvalues of the one-dimensional discrete *Laplacian* under boundary conditions $(\tau), \tau = 1$ (**open**), 2 (**closed**) and 3 (**periodic**). For the vertical direction one has the following explicit expressions $(p = 1, 2, \ldots, L; \;\; n = 1, 2, \ldots, L)$:
for **open** boundary conditions

$$\lambda^{(1)}_p = 2(1 - \cos \tfrac{\pi p}{L+1}); \quad v^{(1)}_p(n) = \sqrt{\tfrac{2}{L+1}} \sin \frac{\pi p n}{L+1}, \tag{16}$$

for **closed** boundary conditions

$$\lambda^{(2)}_p = 2(1 - \cos \tfrac{\pi p}{L}); \quad v^{(2)}_p(n) = \begin{cases} \frac{1}{\sqrt{L}} & , \quad p = 1; \\ \sqrt{\frac{2}{L}} \cos\left[\pi(p-1)(n-\frac{1}{2})/L\right] & , \quad p = 2, \cdots, L; \end{cases} \tag{17}$$

for **periodic** boundary conditions

$$\lambda^{(3)}_p = 2(1 - \cos \frac{2\pi p}{L}); \quad v^{(3)}_p(n) = \frac{1}{\sqrt{L}} \exp \frac{2\pi i p n}{L} \quad . \tag{18}$$

Passing to the limit of an infinitely long strip, $M \to \infty$, we first redefine the Green function (15) as follows.

(i) Shift the origin of the column number m and the row number n to the center, i.e. for an odd integers M, L we set $m = m' + \frac{M+1}{2}$, $n = n' + \frac{L+1}{2}$ where m_1, n_1 labels the columns and rows from the central ones (with $m_1 = 0, n_1 = 0$).

(ii) Shift the value of the Green function by a constant term to remove the divergency which appears as $M \to \infty$, even at finite L when periodic boundary conditions are imposed on the horizontal edges. Evidently, such a shift does not affect the value of the determinants (13,14).It is convenient to set the shift term equal to the value of $G^{(\tau)}_{LM}$ at coinciding sites in the center of the lattice. Then, in the limit $M \to \infty$ we obtain

$$\begin{aligned} G^{(\tau)}_{L}(n_1, m_1'; n_2, m_2') = \\ = \lim_{M\to\infty} \Big\{ G^{(\tau)}_{LM}(n_1' + \tfrac{L+1}{2}, m_1' + \tfrac{M+1}{2}; n_2' + \tfrac{L+1}{2}, m_2' + \tfrac{M+1}{2}) - \\ G^{(\tau)}_{LM}(\tfrac{L+1}{2}, \tfrac{M+1}{2}; \tfrac{L+1}{2}, \tfrac{M+1}{2}) \Big\} = \\ \sum_{p=1}^{L} \frac{1}{\pi} \int_0^{\pi} \frac{v^{(\tau)}_p(n_1) \bar{v}^{(\tau)}_p(n_2) \cos(m_1' - m_2')\alpha - v^{(\tau)}_p(\frac{L+1}{2}) \bar{v}^{(\tau)}_p(\frac{L+1}{2})}{\lambda^{(\tau)}_p + 2(1 - \cos\alpha)} d\alpha \quad , \end{aligned} \tag{19}$$

Considering correlations along the central row of the strip it is convenient to represent the resulting Green function as a sum of two parts: an even, $C^{(\tau)}_L$, and an odd, $S^{(\tau)}_L$, with respect to each of the arguments n_1' and n_2':

$$\begin{aligned} G^{(\tau)}_L(n_1' + \tfrac{L+1}{2}, m_1'; n_2' + \tfrac{L+1}{2}, m_2') = \\ = C^{(\tau)}_L(n_1', n_2'; m_1' - m_2') + S^{(\tau)}_L(n_1', n_2'; m_1' - m_2') \quad . \end{aligned} \tag{20}$$

In the remainder, referring to the strip geometry, we shall drop the prime superscript of the row and column numbers, remembering that they are counted from the center. Explicitly we have:
for **open** boundary conditions

$$\begin{aligned} C^{(1)}_L(n_1, n_2; m) = \\ \frac{1}{L+1} \sum_{l=0}^{(L-1)/2} \frac{1}{\pi} \int_0^{\pi} \frac{\cos\left[\pi(2l+1)n_1/(L+1)\right] \cos\left[\pi(2l+1)n_2/(L+1)\right] \cos m\alpha - 1}{2 - \cos\left[\pi(2l+1)n_1/(L+1)\right] - \cos\alpha} d\alpha, \end{aligned} \tag{21}$$

$$\begin{aligned} S^{(1)}_L(n_1, n_2; m) = \\ \frac{1}{L+1} \sum_{l=1}^{(L-1)/2} \frac{1}{\pi} \int_0^{\pi} \frac{\sin\left[2\pi l n_1/(L+1)\right] \sin\left[2\pi l n_2/(L+1)\right] \cos m\alpha}{2 - \cos\left[2\pi l/(L+1)\right] - \cos\alpha} d\alpha \quad , \end{aligned} \tag{22}$$

for **closed** boundary conditions

$$C_L^{(2)}(n_1,n_2;m) = \frac{1}{2\pi L}\int_0^\pi \frac{\cos m\alpha - 1}{1-\cos\alpha}d\alpha +$$
$$\frac{1}{L}\sum_{l=1}^{(L-1)/2}\frac{1}{\pi}\int_0^\pi \frac{\cos(2\pi l n_1/L)\cos(2\pi l n_2/L)\cos m\alpha - 1}{2-\cos(2\pi l/L)-\cos\alpha}d\alpha \ , \tag{23}$$

$$S_L^{(2)}(n_1,n_2;m) =$$
$$\frac{1}{L}\sum_{l=1}^{(L-1)/2}\frac{1}{\pi}\int_0^\pi \frac{\sin[\pi(2l-1)n_1/L]\sin[\pi(2l-1)n_2/L]\cos m\alpha}{2-\cos[\pi(2l-1)/L]-\cos\alpha}d\alpha \ , \tag{24}$$

for **periodic** boundary conditions

$$C_L^{(3)}(n_1,n_2;m) =$$
$$\frac{1}{2L}\sum_{l=1}^{L}\frac{1}{\pi}\int_0^\pi \frac{\cos(2\pi l n_1/L)\cos(2\pi l n_2/L)\cos m\alpha - 1}{2-\cos(2\pi l/L)-\cos\alpha}d\alpha \ , \tag{25}$$

$$S_L^{(3)}(n_1,n_2;m) =$$
$$\frac{1}{2L}\sum_{l=1}^{L}\frac{1}{\pi}\int_0^\pi \frac{\sin(2\pi l n_1/L)\sin(2\pi l n_2/L)\cos m\alpha}{2-\cos(2\pi l/L)-\cos\alpha}d\alpha \ . \tag{26}$$

In the case of a half-plane geometry, one may start with expression (19) and take the limit $L \to \infty$, thus obtaining up to unimportant constant the Green function for a half-plane with **open** boundary,

$$G^{(1)}(n_1,n_2;m) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\sin n_1\beta \sin n_2\beta \cos m\alpha}{2-\cos\beta-\cos\alpha}d\alpha d\beta \ , \tag{27}$$

and **closed** boundary

$$G^{(2)}(n_1,n_2;m) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\cos(n_1-1/2)\beta\cos(n_2-1/2)\beta\cos m\alpha - 1}{2-\cos\beta-\cos\alpha}d\alpha d\beta \ . \tag{28}$$

3 The strip geometry

We consider the correlations between two unit heights, one at the center of the strip, $n_1 = 0, m_1 = 0$, and the other at a distance r apart, at site $n_2 = 0, m_2 = r$. If we label the rows and columns of the 8×8 matrix $B_{11}(r)$, associated with the corresponding defects, in the order $(-1,0),(0,-1),(0,0),(0,1),(-1,r),(0,r-1),(0,r),(0,r+1)$, it takes the block-diagonal form

$$B_{11} = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix} \ , \tag{29}$$

where B_1 is the 4×4 matrix

$$B_1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \ . \tag{30}$$

The restriction $\tilde{G}_L(r)$ of the lattice Green function (19) (for simplicity of notation, in the remainder of this section we omit the superscript (τ) indicating the specific boundary conditions) to the corresponding set of defect sites is an 8×8 matrix with the block structure:

$$\tilde{G}_L(r) = \begin{pmatrix} A_L(0) & A_L(r) \\ A_L(-r) & A_L(0) \end{pmatrix} \ , \tag{31}$$

where, using the representation (20), $A_L(r)$ may be written as the 4×4 matrix

$$A_L(r) = \begin{pmatrix} C_L(1,1;r)+S_L(1,1;r) & C_L(1,0;r-1) & C_L(1,0;r) & C_L(1,0;r+1) \\ C_L(0,1;r+1) & C_L(0,0;r) & C_L(0,0;r+1) & C_L(0,0;r+2) \\ C_L(0,1;r) & C_L(0,0;r-1) & C_L(0,0;r) & C_L(0,0;r+1) \\ C_L(0,1;r-1) & C_L(0,0;r-2) & C_L(0,0;r-1) & C_L(0,0;r) \end{pmatrix} \tag{32}$$

and $A_L(0)$ is given by (32) at $r = 0$.

According to (14), the pair correlation function $P_{11}(r;L)$ is

$$\mathcal{P}_{11}(r;L) = \det\left[\mathbf{1} + \tilde{G}_L(r)B_{11}(r)\right] - \mathcal{P}_1^2(L/2) \ , \tag{33}$$

where

$$\mathcal{P}_1^2(L/2) = \lim_{r\to\infty}\det\left[\mathbf{1} + \tilde{G}_L(r)B_{11}(r)\right] \ , \tag{34}$$

is the square of the unit height probability at the central line of the strip. Thus the problem reduces to the evaluation of the determinant

$$\det\left[\mathbf{1} + \tilde{G}_L(r)B_{11}(r)\right] = \det\begin{pmatrix} 1 + A_L(0)B_1 & A_L(r)B_1 \\ A_L(-r)B_1 & 1 + A_L(-r)B_1 \end{pmatrix} \ , \tag{35}$$

Using the explicit form of B_1 and the symmetry properties of the even and odd components of the Green function (20), after obvious transformations, we obtain the expression

$$\det\left[\mathbf{1} + \tilde{G}_L(r)B_{11}(r)\right] = 16\left\{\left[\frac{1}{2} - S_L(q,1;0)\right]^2 - S_L^2(1,1;r)\right\}\det\begin{pmatrix} Q & R(r) \\ R^T(r) & Q \end{pmatrix} \ , \tag{36}$$

where Q and $R(r)$ are the 2×2 matrices (R^T is the transpose of R),

$$Q = \begin{pmatrix} -C_L(0,1;0) & C_L(0,0;1)-C_L(0,1;1) \\ C_L(0,0;1)-C_L(0,1;1) & -C_L(0,1;0) \end{pmatrix} , \qquad (37)$$

$$R(r) = \begin{pmatrix} C_L(0,0;r)-C_L(0,1;r) & C_L(0,0;r+1)-C_L(0,1;r+1) \\ C_L(0,0;r-1)-C_L(0,1;r-1) & CL(0,0;r)-C_L(0,1;0) \end{pmatrix} . \qquad (38)$$

The r-dependent terms in equations (36),(38) are readily evaluated as $r\to\infty$ at fixed $L \gg 1$; their leading asymptotic form depends on the boundary conditions:
1. For **open** boundary conditions

$$S_L(1,1;r) \approx \frac{2\pi}{L^2}\exp(-2\pi r/L) \qquad (39)$$

$$C_L(0,0;r) - C_L(0,1;r) \approx \frac{\pi}{2L^2}\exp(-\pi r/L) \; ; \qquad (40)$$

2. For **closed** boundary conditions

$$S_L(1,1;r) \approx \frac{\pi}{L^2}\exp(-\pi r/L) \qquad (41)$$

$$C_L(0,0;r) - C_L(0,1;r) \approx \frac{\pi}{L^2}\exp(-2\pi r/L) \; ; \qquad (42)$$

3. For **periodic** boundary conditions

$$S_L(1,1;r) \approx \frac{2\pi}{L^2}\exp(-2\pi r/L) \qquad (43)$$

$$C_L(0,0;r) - C_L(0,1;r) \approx \frac{\pi}{L^2}\exp(-2\pi r/L) \; ; \qquad (44)$$

By substitution of (40)-(44) in expression (36), see also (33), one obtains that the exponential decay of correlations is governed by the correlation length $\xi = L/2\pi$ in the cases of open and closed boundary conditions, and $\xi = L/4\pi$ in the case of periodic boundary conditions. The finite-size corrections to the unit height probability $\mathcal{P}_1(L/2)$, see (34),

$$\mathcal{P}_1(L/2) = [1 + C_L(0,0;2)][1+2C_L(0,1;1)]\Big\{\frac{1}{2} - S_L(1,1;0)\Big\} \qquad (45)$$

can be obtained by using standard techniques based on the Poisson summation formula, see e.g. [16]. All the relevant terms can be expressed through four L-dependent integrals $\varepsilon_k(L), k = 1,...,4$,

$$\varepsilon_1(L) = \frac{1}{\pi}\int_0^\pi \frac{\sin\phi}{\sqrt{1+\sin^2\phi}}\,\frac{\delta_L(\phi)}{1+2\delta_L}d\phi = \frac{\pi}{6L^2} + \mathcal{O}(L^{-4}) \qquad (46)$$

$$\varepsilon_2(L) = \frac{1}{\pi}\int_0^\pi \frac{\sin^3\phi}{\sqrt{1+\sin^2\phi}}\,\frac{\delta_L(\phi)}{1+2\delta_L}d\phi = \frac{7\pi^3}{60L^4} + \mathcal{O}(L^{-4}) \qquad (47)$$

$$\varepsilon_3(L) = \frac{1}{\pi}\int_0^\pi \frac{\sin\phi}{\sqrt{1+\sin^2\phi}}\,\delta_L(\phi)d\phi = \frac{\pi}{3L^2} + \mathcal{O}(L^{-4}) \qquad (48)$$

$$\varepsilon_4(L) = \frac{1}{\pi}\int_0^\pi \frac{\sin^3\phi}{\sqrt{1+\sin^2\phi}}\,\delta_L(\phi)d\phi = \frac{2\pi^3}{15L^4} + \mathcal{O}(L^{-4}) \; , \qquad (49)$$

where,

$$\delta_L(\phi) = \left[(\sin\phi + \sqrt{1+\sin^2\phi})^L - 1\right]^{-1} . \qquad (50)$$

Thus we obtain:
1. For **open** boundary conditions

$$C_L(0,0;2) = -1 + \frac{2}{\pi} - 4\varepsilon_1(2L+2) - 4\varepsilon_2(2L+2) \qquad (51)$$

$$C_L(0,1;1) = -\frac{1}{\pi} + 2\varepsilon_2(2L+2) \qquad (52)$$

$$S_L(1,1;0) = \frac{1}{2} - \frac{1}{\pi} - 2\varepsilon_3(2L+2) - 2\varepsilon_4(2L+2) \qquad (53)$$

2. For **closed** boundary conditions

$$C_L(0,0;2) = -1 + \frac{2}{\pi} - 4\varepsilon_3(2L) + 4\varepsilon_4(2L) \qquad (54)$$

$$C_L(0,1;1) = -\frac{1}{\pi} - 2\varepsilon_4(2L) \qquad (55)$$

$$S_L(1,1;0) = \frac{1}{2} - \frac{1}{\pi} + 2\varepsilon_1(2L) + 2\varepsilon_2(2L) \qquad (56)$$

3. For **periodic** boundary conditions

$$C_L(0,0;2) = -1 + \frac{2}{\pi} - 4\varepsilon_3(2L) + 4\varepsilon_4(2L) \qquad (57)$$

$$C_L(0,1;1) = -\frac{1}{\pi} - 2\varepsilon_4(2L) \qquad (58)$$

$$S_L(1,1;0) = \frac{1}{2} - \frac{1}{\pi} - 2\varepsilon_3(2L) - 2\varepsilon_4(2L) \; . \qquad (59)$$

The final results for the strip geometry are summarized in equations (8)-(11), see the Introduction.

4 The half-plane geometry

With the aid of expressions (27)-(28) for the Green functions, one can use the determinant formula (13) to obtain the unit height probability at a site i which belongs to the lattice boundary $\partial\mathcal{L}$. In this case, the modification of the lattice consists in cutting off the two bonds which are attached to i and belong to the boundary. If we label the defect sites in the order $(-1,0),(0,0),(1,0)$, the matrix associated with the defect is

the 3×3 matrix of the form:
for an **open** boundary

$$B_1^{(1)} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix} , \tag{60}$$

for a **closed** boundary

$$B_1^{(2)} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} . \tag{61}$$

A direct evaluation of the corresponding determinants leads to expressions (3) and (4), given in the Introduction.

Of course, one could construct a different defect matrix which leads to the same results, for example, by preserving one of the boundary bonds and cutting off, instead of it, the bond which starts from i and points inside the lattice.

To obtain a more detailed information about the spatial structure of the sandpile, we have studied also the asymptotic behaviour of the unit height probability $\mathcal{P}_1(l)$ at large distances from the boundary, $l \gg 1$. In this case, the defect matrix is the same as in the bulk, see [12]. The asymptotic behavior of the boundary Green function at large separations, for both open and closed boundary conditions, is most easily obtained with the aid of the method of reflections. Then, one needs to know only the asymptotic expansion for the bulk Green function at large separations ($r \gg 1$):

$$G_{bulk}(0,0;r) = -\frac{1}{2\pi}\ln r - \frac{\gamma}{2\pi} - \frac{3\ln 2}{4\pi} + \frac{1}{24\pi r^2} + \frac{43}{480\pi r^4} + \cdots . \tag{62}$$

where γ is the *Euler* constant.

By using the discrete *Laplace* equation to generate a sequence of recurrent relations, one may obtain the asymptotic expansion for $G_{bulk}(0,n;r)$, where $n = 1,2,\cdots$ Next, by subtracting (in the case of an **open** boundary) or adding (in the case of a **closed** boundary) the Green function for a fictitious source, placed symmetrically with respect to the boundary, one can easily find the asymptotic expansion for the Green function which corresponds to the considered boundary problem. Thus for **open** boundary conditions we obtain

$$G^{(1)}(0,0;r) = \frac{1}{\pi r^2} - \frac{1}{2\pi r^4} + \cdots , \tag{63}$$

and for **closed** ones

$$G^{(2)}(0,0;r) = -\frac{1}{\pi}\ln r - \frac{\gamma}{\pi} - \frac{3\ln 2}{2\pi} - \frac{1}{6\pi r^2} - \frac{17}{240\pi r^4} + \cdots . \tag{64}$$

The defect matrix used in the calculation of the unit height correlations at the boundary, in the representation when the defect sites are labeled in the order: $(-1,0)$, $(0,0)$, $(1,0)$, $(r-1,0)$, $(r,0)$, $(r+1,0)$, has the block-diagonal form

$$B_{11} = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix} , \tag{65}$$

where the 3×3 matrix B_1 is given by one of the equations (60),(61).

The restriction of the boundary Green function to the set of defect sites can be written in the block form

$$\tilde{G}_L(r) = \begin{pmatrix} A_L(0) & A_L(r) \\ A_L(-r) & A_L(0) \end{pmatrix} , \tag{66}$$

Finally the evaluation of the determinants (14) leads us to the results (6),(7) presented in the Introduction.

5 Conclusion

In summary, we have calculated the leading asymptotic form of the correlation functions for the unit heights, $\mathcal{P}_{11}(r)$, in the strip and half-plane geometries of the Abelian sandpile model, under three different boundary conditions. The results confirm the suggestion that the unit height behaves like the local energy operator in the q-component Potts model at $q = 0$. Unfortunately, the above conclusion cannot be directly extended to the case of heights $z_i = 2,3,4$, although correspondences between these heights and spanning trees still exist. In [17] it has been shown that $z_i = 2$ corresponds to such trees (in addition to the trees yielding $z_i = 1$), in which none of the paths, starting from three nearest neighbours of site i and ending at the open boundary, pass through i. In the case $z_i = 3$ this property must have the paths starting from two of the nearest neighbours of i, and if $z_i = 4$ - from one neighbouring site. Thus, the heights $z_i = 2,3,4$, being local sandpile observables, happen to be connected with nonlocal properties of the trees. For the enumeration of configurations with these heights one has to consider clusters of subtrees with increasing size. It has been established [12] that such cluster expansions are slowly convergent, which makes difficult the evaluation of the correlation functions $\mathcal{P}_{zz}(r)$ for $z = 2,3,4$. In principle, the above mentioned features could lead to a different from r^{-4} power law of decay.

Finding a correspondence between observables in the Potts model and characteristics of the avalanches is not an easier task either, because of the nonlocal nature of the avalanches. The solution of these problems needs further analytical and numerical studies.

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