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DIFFERENTIATION OF PARAMETRIC EQUATIONS PRESENTED IN THE FORM OF THE RUNGE — KUTT DIFFERENCE SCHEMES

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In this paper we consider the questions on obtaining the derivatives with respect to the parameters from the solution of ordinary differential second-order equations of the type

$$x'' = \phi(p_1,\dots,p_n,x,y,z,x',y'),$$
$$y'' = f(p_1,\dots,p_n,x,y,z,x',y'), \quad (1)$$

where $x'=dx/dz$, $x''=d^2x/dz^2$, $p_1,\dots,p_n$ - parameters.
It is assumed that equations (1) can be solved by numerical integration using the Runge-Kutt difference schemes[1] with initial conditions: $z_1, x_1=x(z_1)$, $y_1=y(z_1)$, $x'_1=x'(z_1)$, $y'_1=y'(z_1)$.
At a (i+1)th step

$$z_{i+1}=z_i+h=z_1+hi,$$
$$x_{i+1}=x_i+x'_ih+1/6(m_1+m_2+m_3)h^2,$$
$$y_{i+1}=y_i+y'_ih+1/6(k_1+k_2+k_3)h^2, \quad (2)$$
$$x'_{i+1}=x'_i+1/6(m_1+2m_2+2m_3+m_4)h,$$
$$y'_{i+1}=y'_i+1/6(k_1+2k_2+2k_3+k_4)h,$$

where $h=\Delta z$- is the step of integration,

$$m_1=\phi_1(p_1,..,p_n,x_i,y_i,z_i,x'_i,y'_i),$$
$$m_2=\phi_2(p_1,..,p_n,x_i+x'_i\frac{h}{2},y_i+y'_i\frac{h}{2},z_i+\frac{h}{2},x'_i+m_1\frac{h}{2},y'_i+k_1\frac{h}{2}),$$
$$m_3=\phi_3(p_1,..,p_n,x_i+x'_i\frac{h}{2}+m_1\frac{h^2}{4},y_i+y'_i\frac{h}{2}+k_1\frac{h^2}{4},z_i+\frac{h}{2},x'_i+m_2\frac{h}{2},y'_i+k_2\frac{h}{2}),$$
$$m_4=\phi_4(p_1,..,p_n,x_i+x'_ih+m_2\frac{h^2}{2},y_i+y'_ih+k_2\frac{h^2}{2},z_i+h,x'_i+m_3h,y'_i+k_3h),$$
$$k_1=f_1(p_1,..,p_n,x_i,y_i,z_i,x'_i,y'_i), \quad (3)$$
$$k_2=f_2(p_1,..,p_n,x_i+x'_i\frac{h}{2},y_i+y'_i\frac{h}{2},z_i+\frac{h}{2},x'_i+m_1\frac{h}{2},y'_i+k_1\frac{h}{2}),$$
$$k_3=f_3(p_1,..,p_n,x_i+x'_i\frac{h}{2}+m_1\frac{h^2}{4},y_i+y'_i\frac{h}{2}+k_1\frac{h^2}{4},z_i+\frac{h}{2},x'_i+m_2\frac{h}{2},y'_i+k_2\frac{h}{2}),$$
$$k_4=f_4(p_1,..,p_n,x_i+x'_ih+m_2\frac{h^2}{2},y_i+y'_ih+k_2\frac{h^2}{2},z_i+h,x'_i+m_3h,y'_i+k_3h).$$

Performing in (2) differentiation with respect to the parameters $p_j (j=1,\dots,n)$, we get the following system of equations:

$$\frac{\partial x_{i+1}}{\partial p_j}=\frac{\partial x_i}{\partial p_j}+\frac{\partial x'_i}{\partial p_j}h+1/6\left(\frac{\partial m_1}{\partial p_j}+\frac{\partial m_2}{\partial p_j}+\frac{\partial m_3}{\partial p_j}\right)h^2,$$
$$\frac{\partial y_{i+1}}{\partial p_j}=\frac{\partial y_i}{\partial p_j}+\frac{\partial y'_i}{\partial p_j}h+1/6\left(\frac{\partial k_1}{\partial p_j}+\frac{\partial k_2}{\partial p_j}+\frac{\partial k_3}{\partial p_j}\right)h^2,$$
$$\frac{\partial x'_{i+1}}{\partial p_j}=\frac{\partial x'_i}{\partial p_j}+1/6\left(\frac{\partial m_1}{\partial p_j}+2\frac{\partial m_2}{\partial p_j}+2\frac{\partial m_3}{\partial p_j}+\frac{\partial m_4}{\partial p_j}\right)h, \qquad (4)$$
$$\frac{\partial y'_{i+1}}{\partial p_j}=\frac{\partial y'_i}{\partial p_j}+1/6\left(\frac{\partial k_1}{\partial p_j}+2\frac{\partial k_2}{\partial p_j}+2\frac{\partial k_3}{\partial p_j}+\frac{\partial k_4}{\partial p_j}\right)h.$$

Assuming that in the general case

$$x_i=u(p_j,x_i,y_i,z_i,x'_i,y'_i),$$
$$y_i=v(p_j,x_i,y_i,z_i,x'_i,y'_i),$$
$$x'_i=u'(p_j,x_i,y_i,z_i,x'_i,y'_i), \qquad (5)$$
$$y'_i=v'(p_j,x_i,y_i,z_i,x'_i,y'_i),$$

we introduce the following designations:

$$\chi_i=\chi_i(x_i,\zeta_i,x'_i)=x_i+x'_i\frac{h}{2},\ \chi'_i=\chi'_i(x'_i,\zeta_i,m_1)=x'_i+m_1\frac{h}{2},\ \zeta_i=z_i+\frac{h}{2},$$
$$\ae_i=\ae_i(y_i,\zeta_i,y'_i)=y_i+y'_i\frac{h}{2},\ \ae'_i=\ae'_i(y'_i,\zeta_i,k_1)=y'_i+k_1\frac{h}{2},\ \zeta_i=z_i+\frac{h}{2}, \qquad (6)$$
$$o_i=o_i(x_i,\zeta_i,x'_i,m_1)=x_i+x'_i\frac{h}{2}+m_1\frac{h^2}{4},\ o'_i=o'_i(x'_i,\zeta_i,m_2)=x'_i+m_2\frac{h}{2},\ \zeta_i=z_i+\frac{h}{2},$$
$$\omega_i=\omega_i(y_i,\zeta_i,y'_i,k_1)=y_i+y'_i\frac{h}{2}+k_1\frac{h^2}{4},\ \omega'_i=\omega'_i(y'_i,\zeta_i,k_2)=y'_i+k_2\frac{h}{2},\ \zeta_i=z_i+\frac{h}{2},$$
$$\gamma_i=\gamma_i(x_i,\xi_i,x'_i,m_2)=x_i+x'_ih+m_2\frac{h^2}{2},\ \gamma'_i=\gamma'_i(x'_i,\xi_i,m_3)=x'_i+m_3h,\ \xi_i=z_i+h,$$
$$\upsilon_i=\upsilon_i(y_i,\xi_i,y'_i,k_2)=y_i+y'_ih+k_2\frac{h^2}{2},\ \upsilon'_i=\upsilon'_i(y'_i,\xi_i,k_3)=y'_i+k_3h,\ \xi_i=z_i+h$$

and also define an operator

$$\frac{\delta}{\delta x_i}=\frac{\partial}{\partial u}\frac{\partial u}{\partial x_i}+\frac{\partial}{\partial v}\frac{\partial v}{\partial x_i}+\frac{\partial}{\partial u'}\frac{\partial u'}{\partial x_i}+\frac{\partial}{\partial v'}\frac{\partial v'}{\partial x_i}. \qquad (7)$$

Then taking into account (4)-(7), obtain the following representations for $\left\{\frac{\partial m_l}{\partial p_j}\right\}_{l=1}^{4}$:

$$\frac{\partial m_1}{\partial p_j}=\frac{\partial \phi_1}{\partial p_j}+\frac{\delta\phi_1}{\delta x_i}\frac{\partial x_i}{\partial p_j}+\frac{\delta\phi_1}{\delta y_i}\frac{\partial y_i}{\partial p_j}+\frac{\delta\phi_1}{\delta x'_i}\frac{\partial x'_i}{\partial p_j}+\frac{\delta\phi_1}{\delta y'_i}\frac{\partial y'_i}{\partial p_j},$$

$$\frac{\partial m_2}{\partial p_j}=\frac{\partial \phi_2}{\partial p_j}+\frac{\delta\phi_2}{\delta \chi_i}\frac{\partial x_i}{\partial p_j}+\frac{\delta\phi_2}{\delta \ae_i}\frac{\partial y_i}{\partial p_j}+\left[\frac{\delta\phi_2}{\delta \chi'_i}+\frac{h\,\delta\phi_2}{2\delta\chi_i}\right]\frac{\partial x'_i}{\partial p_j}+\left[\frac{\delta\phi_2}{\delta \ae'_i}+\frac{h\,\delta\phi_2}{2\delta\ae_i}\right]\frac{\partial y'_i}{\partial p_j}+$$
$$+\frac{h}{2}\left[\frac{\delta\phi_2}{\delta\chi'_i}\frac{\partial m_1}{\partial p_j}+\frac{\delta\phi_2}{\delta\ae'_i}\frac{\partial k_1}{\partial p_j}\right], \qquad (8)$$
$$\frac{\partial m_3}{\partial p_j}=\frac{\partial \phi_3}{\partial p_j}+\frac{\delta\phi_3}{\delta o_i}\frac{\partial x_i}{\partial p_j}+\frac{\delta\phi_3}{\delta \omega_i}\frac{\partial y_i}{\partial p_j}+\left[\frac{\delta\phi_3}{\delta o'_i}+\frac{h\,\delta\phi_3}{2\delta o_i}\right]\frac{\partial x'_i}{\partial p_j}+\left[\frac{\delta\phi_3}{\delta \omega'_i}+\frac{h\,\delta\phi_3}{2\delta\omega_i}\right]\frac{\partial y'_i}{\partial p_j}+$$
$$+\frac{h^2}{4}\left[\frac{\delta\phi_3}{\delta o_i}\frac{\partial m_1}{\partial p_j}+\frac{\delta\phi_3}{\delta\omega_i}\frac{\partial k_1}{\partial p_j}\right]+\frac{h}{2}\left[\frac{\delta\phi_3}{\delta o'_i}\frac{\partial m_2}{\partial p_j}+\frac{\delta\phi_3}{\delta\omega'_i}\frac{\partial k_2}{\partial p_j}\right],$$
$$\frac{\partial m_4}{\partial p_j}=\frac{\partial \phi_4}{\partial p_j}+\frac{\delta\phi_4}{\delta \gamma_i}\frac{\partial x_i}{\partial p_j}+\frac{\delta\phi_4}{\delta \upsilon_i}\frac{\partial y_i}{\partial p_j}+\left[\frac{\delta\phi_4}{\delta \gamma'_i}+h\frac{\delta\phi_4}{\delta\gamma_i}\right]\frac{\partial x'_i}{\partial p_j}+\left[\frac{\delta\phi_4}{\delta \upsilon'_i}+h\frac{\delta\phi_4}{\delta\upsilon_i}\right]\frac{\partial y'_i}{\partial p_j}+$$
$$+\frac{h^2}{2}\left[\frac{\delta\phi_4}{\delta\gamma_i}\frac{\partial m_2}{\partial p_j}+\frac{\delta\phi_4}{\delta\upsilon_i}\frac{\partial k_2}{\partial p_j}\right]+h\left[\frac{\delta\phi_4}{\delta\gamma'_i}\frac{\partial m_3}{\partial p_j}+\frac{\delta\phi_4}{\delta\upsilon'_i}\frac{\partial k_3}{\partial p_j}\right].$$

Representations for $\left\{\frac{\partial k_l}{\partial p_j}\right\}_{l=1}^{4}$ will have the same form (8) with one difference: function ϕ_l should be changed for f_l, $l=1,\dots,4$.

Taking into consideration (8), the final expressions for the derivatives in (4) take the following form:

$$\frac{\partial x_{i+1}}{\partial p_j}=\frac{\partial x_i}{\partial p_j}+h\frac{\partial x'_i}{\partial p_j}+1/6\left\{\sum_{l=1}^{3}\frac{\partial\phi_l}{\partial p_j}+\left[\frac{\delta\phi_1}{\delta x_i}+\frac{\delta\phi_2}{\delta\chi_i}+\frac{\delta\phi_3}{\delta o_i}\right]\frac{\partial x_i}{\partial p_j}+\right.$$
$$+\left[\frac{\delta\phi_1}{\delta y_i}+\frac{\delta\phi_2}{\delta\ae_i}+\frac{\delta\phi_3}{\delta\omega_i}\right]\frac{\partial y_i}{\partial p_j}+\left[\frac{\delta\phi_1}{\delta x'_i}+\frac{\delta\phi_2}{\delta\chi'_i}+\frac{\delta\phi_3}{\delta o'_i}+\frac{h}{2}\left(\frac{\delta\phi_2}{\delta\chi_i}+\frac{\delta\phi_3}{\delta o_i}\right)\right]\frac{\partial x'_i}{\partial p_j}+$$
$$+\left[\frac{\delta\phi_1}{\delta y'_i}+\frac{\delta\phi_2}{\delta\ae'_i}+\frac{\delta\phi_3}{\delta\omega'_i}+\frac{h}{2}\left(\frac{\delta\phi_2}{\delta\ae_i}+\frac{\delta\phi_3}{\delta\omega_i}\right)\right]\frac{\partial y'_i}{\partial p_j}+\frac{h}{2}\left[\frac{\delta\phi_2}{\delta\chi_i}+\frac{h\,\delta\phi_3}{2\delta o_i}\right]\frac{\partial m_1}{\partial p_j}+$$
$$\left.+\frac{h}{2}\left[\frac{\delta\phi_2}{\delta\ae_i}+\frac{h\,\delta\phi_3}{2\delta\omega_i}\right]\frac{\partial k_1}{\partial p_j}+\frac{h\,\delta\phi_3}{2\delta o'_i}\frac{\partial m_2}{\partial p_j}+\frac{h\,\delta\phi_3}{2\delta\omega'_i}\frac{\partial k_2}{\partial p_j}\right\}h^2,$$
$$\frac{\partial y_{i+1}}{\partial p_j}=\frac{\partial y_i}{\partial p_j}+h\frac{\partial y'_i}{\partial p_j}+1/6\left\{\sum_{l=1}^{3}\frac{\partial f_l}{\partial p_j}+\left[\frac{\delta f_1}{\delta x_i}+\frac{\delta f_2}{\delta\chi_i}+\frac{\delta f_3}{\delta o_i}\right]\frac{\partial x_i}{\partial p_j}+\right.$$
$$+\left[\frac{\delta f_1}{\delta y_i}+\frac{\delta f_2}{\delta\ae_i}+\frac{\delta f_3}{\delta\omega_i}\right]\frac{\partial y_i}{\partial p_j}+\left[\frac{\delta f_1}{\delta x'_i}+\frac{\delta f_2}{\delta\chi'_i}+\frac{\delta f_3}{\delta o'_i}+\frac{h}{2}\left(\frac{\delta f_2}{\delta\chi_i}+\frac{\delta f_3}{\delta o_i}\right)\right]\frac{\partial x'_i}{\partial p_j}+$$
$$+\left[\frac{\delta f_1}{\delta y'_i}+\frac{\delta f_2}{\delta\ae'_i}+\frac{\delta f_3}{\delta\omega'_i}+\frac{h}{2}\left(\frac{\delta f_2}{\delta\ae_i}+\frac{\delta f_3}{\delta\omega_i}\right)\right]\frac{\partial y'_i}{\partial p_j}+\frac{h}{2}\left[\frac{\delta f_2}{\delta\ae_i}+\frac{h\,\delta f_3}{2\delta\omega_i}\right]\frac{\partial m_1}{\partial p_j}+$$
$$\left.+\frac{h}{2}\left[\frac{\delta f_2}{\delta\ae_i}+\frac{h\,\delta f_3}{2\delta\omega_i}\right]\frac{\partial k_1}{\partial p_j}+\frac{h\,\delta f_3}{2\delta o'_i}\frac{\partial m_2}{\partial p_j}+\frac{h\,\delta f_3}{2\delta\omega'_i}\frac{\partial k_2}{\partial p_j}\right\}h^2, \qquad (9)$$

$$\frac{\partial x'_{i+1}}{\partial p_j} = \frac{\partial x'_i}{\partial p_j} + 1/6\left\{\frac{\partial \phi_1}{\partial p_j} + 2\frac{\partial \phi_2}{\partial p_j} + 2\frac{\partial \phi_3}{\partial p_j} + \frac{\partial \phi_4}{\partial p_j} + \left[\frac{\delta \phi_1}{\delta x_i} + 2\frac{\delta \phi_2}{\delta \chi_i} + 2\frac{\delta \phi_3}{\delta o_i} + \frac{\delta \phi_4}{\delta \gamma_i}\right]\frac{\partial x_i}{\partial p_j} + \right.$$

$$+ \left[\frac{\delta \phi_1}{\delta y_i} + 2\frac{\delta \phi_2}{\delta \text{æ}_i} + 2\frac{\delta \phi_3}{\delta \omega_i} + \frac{\delta \phi_4}{\delta \upsilon_i}\right]\frac{\partial y_i}{\partial p_j} + \left[\frac{\delta \phi_1}{\delta x'_i} + 2\frac{\delta \phi_2}{\delta \chi'_i} + 2\frac{\delta \phi_3}{\delta o'_i} + \frac{\delta \phi_4}{\delta \gamma'_i} + \right.$$

$$\left. + h\left(\frac{\delta \phi_2}{\delta \chi_i} + \frac{\delta \phi_3}{\delta o_i} + \frac{\delta \phi_4}{\delta \gamma_i}\right)\right]\frac{\partial x'_i}{\partial p_j} + \left[\frac{\delta \phi_1}{\delta y'_i} + 2\frac{\delta \phi_2}{\delta \text{æ}'_i} + 2\frac{\delta \phi_3}{\delta \omega'_i} + \frac{\delta \phi_4}{\delta \upsilon'_i} + \right.$$

$$\left. + h\left(\frac{\delta \phi_2}{\delta \text{æ}_i} + \frac{\delta \phi_3}{\delta \omega_i} + \frac{\delta \phi_4}{\delta \upsilon_i}\right)\right]\frac{\partial y'}{\partial p_j} + h\left[\frac{\delta \phi_2}{\delta \chi'} + \frac{h\,\delta \phi_3}{2\delta o_i}\right]\frac{\partial m_1}{\partial p_j} + h\left[\frac{\delta \phi_2}{\delta \text{æ}'_i} + \frac{h\,\delta \phi_3}{2\delta \omega_i}\right]\frac{\partial k_1}{\partial p_j} +$$

$$\left. + h\left[\frac{\delta \phi_3}{\delta o'_i} + \frac{h\,\delta \phi_4}{2\delta \gamma_i}\right]\frac{\partial m_2}{\partial p_j} + h\left[\frac{\delta \phi_3}{\delta \omega'_i} + \frac{h\,\delta \phi_4}{2\delta \upsilon_i}\right]\frac{\partial k_2}{\partial p_j} + h\frac{\delta \phi_4}{\delta \gamma'_i}\frac{\partial m_3}{\partial p_j} + h\frac{\delta \phi_4}{\delta \upsilon'_i}\frac{\partial k_3}{\partial p_j}\right\}h,$$

$$\frac{\partial y'_{i+1}}{\partial p_j} = \frac{\partial y'_i}{\partial p_j} + 1/6\left\{\frac{\partial f_1}{\partial p_j} + 2\frac{\partial f_2}{\partial p_j} + 2\frac{\partial f_3}{\partial p_j} + \frac{\partial f_4}{\partial p_j} + \left[\frac{\delta f_1}{\delta x_i} + 2\frac{\delta f_2}{\delta \chi_i} + 2\frac{\delta f_3}{\delta o_i} + \frac{\delta f_4}{\delta \gamma_i}\right]\frac{\partial x_i}{\partial p_j} + \right.$$

$$+ \left[\frac{\delta f_1}{\delta y_i} + 2\frac{\delta f_2}{\delta \text{æ}_i} + 2\frac{\delta f_3}{\delta \omega_i} + \frac{\delta f_4}{\delta \upsilon_i}\right]\frac{\partial y_i}{\partial p_j} + \left[\frac{\delta f_1}{\delta x'_i} + 2\frac{\delta f_2}{\delta \chi'_i} + 2\frac{\delta f_3}{\delta o'_i} + \frac{\delta f_4}{\delta \gamma'_i} + \right.$$

$$\left. + h\left(\frac{\delta f_2}{\delta \chi_i} + \frac{\delta f_3}{\delta o_i} + \frac{\delta f_4}{\delta \gamma_i}\right)\right]\frac{\partial x'_i}{\partial p_j} + \left[\frac{\delta f_1}{\delta y'_i} + 2\frac{\delta f_2}{\delta \text{æ}'_i} + 2\frac{\delta f_3}{\delta \omega'_i} + \frac{\delta f_4}{\delta \upsilon'_i} + \right.$$

$$\left. + h\left(\frac{\delta f_2}{\delta \text{æ}_i} + \frac{\delta f_3}{\delta \omega_i} + \frac{\delta f_4}{\delta \upsilon_i}\right)\right]\frac{\partial y'_i}{\partial p_j} + h\left[\frac{\delta f_2}{\delta \chi'_i} + \frac{h\,\delta f_3}{2\delta o_i}\right]\frac{\partial m_1}{\partial p_j} + h\left[\frac{\delta f_2}{\delta \text{æ}'_i} + \frac{h\,\delta f_3}{2\delta \omega_i}\right]\frac{\partial k_1}{\partial p_j} +$$

$$\left. + h\left[\frac{\delta f_3}{\delta o'_i} + \frac{h\,\delta f_4}{2\delta \gamma_i}\right]\frac{\partial m_2}{\partial p_j} + h\left[\frac{\delta f_3}{\delta \omega'_i} + \frac{h\,\delta f_4}{2\delta \upsilon_i}\right]\frac{\partial k_2}{\partial p_j} + h\frac{\delta f}{\delta \gamma'_i}\frac{\partial m_3}{\partial p_j} + h\frac{\delta f}{\delta \upsilon'_i}\frac{\partial k_3}{\partial p_j}\right\}h.$$

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