

СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E5-93-299

B.Z.Iliev¹

TRANSPORTS ALONG PATHS IN FIBRE BUNDLES
General Theory

¹Permanent address: Institute for Nuclear Research
and Nuclear Energy, Bulgarian Academy of Sciences,
blvd. Tzarigradsko Chaussée 72, 1784 Sofia, Bulgaria

1993

1. INTRODUCTION

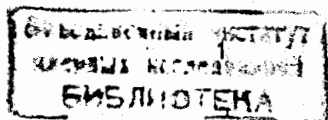
The parallel transport (translation) is a well known concept in differential geometry and fibre bundle theory, usually tied up with the connection theory [1-4]. The purpose of this paper is to propose an independent definition and the corresponding investigation of (parallel) transports (along paths) in arbitrary fibre bundles. It is written in a manner analogous to the one of [5,6], from where some ideas, results and proofs are transferred mutatis mutandis. The detailed comparison of the material presented here with the one in the literature on the same subject will be done in the next part of our series.

The transports along paths in fibre bundles are defined in Sect. 2, where also two main groups of possible restrictions on them are considered. Some simple properties and the general functional form of these transports are found in Sect. 3. Sect. 4 deals with different connections between transports along paths and liftings of paths from the base to the bundle space of the fibre bundle. Necessary and sufficient condition is obtained for a fibre bundle to admit transports along paths. In Sect. 5, a method for generating a connection by a transport along a path is considered.

2. TRANSPORTS ALONG PATHS AND POSSIBLE

RESTRICTIONS ON THEM

Let (E, π, B) be a general topological fibre bundle with a base B , total bundle space E and projection $\pi: E \rightarrow B$ [7-9]. The fibre



bundle (E, π, B) is, generally, not supposed to be locally trivial. The fibres $\pi^{-1}(x)$, $x \in B$ are supposed to be homeomorphic with each other. The set of all sections of (E, π, B) is denoted by $\text{Sec}(E, \pi, B)$, i.e., $\sigma \in \text{Sec}(E, \pi, B)$ means $\sigma: B \rightarrow E$ and $\pi \circ \sigma = \text{id}_B$ [7,9], where id_X is the identity map of the set X .

By J and $\gamma: J \rightarrow B$ are denoted arbitrary, respectively, real interval and a path in B .

The transports along paths in (E, π, B) are defined in Subsect. 2.1 where also their basic properties, describing to a certain extent the dependence on their parameters, are found. On the transports along paths one can be imposed different restrictions. In the present work two main groups of them are considered. Firstly (Subsect. 2.2), the one describing more or less the transport's dependence on the path of the transport. The second one (Subsect. 2.3) concerns the ties between some primary given structures on the fibre bundle, if any, and transports along paths in it.

2.1. DEFINITION OF TRANSPORTS ALONG PATHS IN FIBRE BUNDLES

The analysis of definition 2.1 of Ref. [6] shows that it may be generalized in such a way as the defined in it linear transport along paths in vector bundles to be a special case of more general "transports along paths" described by

Definition 2.1. A transport along paths in the fibre bundle (E, π, B) is a map \mathbf{I} which to any path $\gamma: J \rightarrow B$ puts into correspondence a map I^γ , transport along γ such that $I^\gamma: (s, t) \mapsto I_{s \rightarrow t}^\gamma$, where for every $s, t \in J$ the map

$$I_{s \rightarrow t}^\gamma: \pi^{-1}(\gamma(s)) \rightarrow \pi^{-1}(\gamma(t)), \quad (2.1)$$

a transport along γ from s to t , has the following two properties:

$$I_{t \rightarrow r}^\gamma \circ I_{s \rightarrow t}^\gamma = I_{s \rightarrow r}^\gamma, \quad r, s, t \in J, \quad (2.2)$$

$$I_{s \rightarrow s}^\gamma = \text{id}_{\pi^{-1}(\gamma(s))}, \quad s \in J. \quad (2.3)$$

The path γ and the numbers $s, t \in J$ in the map (2.1) will be called, respectively, path, initial parameter and final parameter of the transport.

The property (2.2), which may be called a group property of transports along paths, is an exact expression of the representation that the "composition of two transports along one and the same path" must be a "transport along the same path". The property (2.3) fixes a 0-ary operation in the set of "transports along paths" defining in it the "unit transport" and, besides, it is an exact expression of the naive understanding that if we "stand" at one point of a path without "moving" along it, then "nothing must happen" with a fibre over it.

The problem of existence of transports along paths, i.e., when a fibre bundle admits transports along paths, will be considered in Sect. 4 and the problem of their uniqueness or a general form will be investigated in Sect. 3.

Closely connected with definition 2.1 is the following one generalizing the concept for sections linearly transported along paths (cf. [10], Sect. 5).

Definition 2.2. Let in (E, π, B) be given a transport along paths \mathbf{I} . The section $\sigma \in \text{Sec}(E, \pi, B)$ undergoes a (an \mathbf{I}) transport or it is (\mathbf{I} -)transported (resp. along $\gamma: J \rightarrow B$), if the equality

$$\sigma(\gamma(t)) = I_{s \rightarrow t}^\gamma \sigma(\gamma(s)), \quad s, t \in J \quad (2.4)$$

holds for every (resp. the given) path $\gamma: J \rightarrow B$.

Proposition 2.1. If (2.1) is fulfilled for a fixed value $s \in J$,

then this equality is valid for every $s \in J$.

Proof. This result is a trivial corollary of (2.2). ■

Proposition 2.1 shows that in definition 2.2 it is sufficient to want (2.4) to be valid for a fixed $s = s_0 \in J$ and then (2.4) may be regarded as a necessary and sufficient condition for the section σ to be I-transported along γ .

Proposition 2.2. If the section $\sigma \in \text{Sec}(E, \pi, B)$ is I-transported along $\gamma: J \rightarrow B$, then its values $\sigma(\gamma(s))$ for every $s \in J$ are uniquely defined if the value $\sigma(\gamma(s_0))$ is given for an arbitrary fixed $s_0 \in J$.

Proof. This result follows from (2.4) for $s = s_0$. ■

Evident example of transports along paths are the linear transport along paths in vector bundles, a result following from the comparison of definitions 2.1 from this work and from [10].

Now we shall consider two examples for transports along parts which appear in fibre bundles with a certain structure.

Example 2.1. Let the fibre bundle (E, π, B) have a structure of foliation [11], i.e. on the total bundle space E , which now is supposed to be a manifold, to be given a foliation $\{K_\alpha: K_\alpha \subset E, \alpha \in A\}$ [11], which, in particular, means that $K_\alpha \cap K_\beta = \emptyset, \alpha, \beta \in A, \alpha \neq \beta$ and $\bigcup_{\alpha \in A} K_\alpha = E$. Let the foliation $\{K_\alpha\}$ be such that $\pi(K_\alpha) = B, \alpha \in A$.

Before going on we want to stress the fact that this construction is equivalent to the one when over (E, π, B) is defined a family of sections $\{\sigma_\alpha: \sigma_\alpha \in \text{Sec}(E, \pi, B), \alpha \in A\}$ such that if $\sigma_\alpha(x) = \sigma_\beta(x)$ for some $x \in B$; then $\alpha = \beta$. Actually, if such a family $\{\sigma_\alpha\}$ is given, then it is sufficient to put $K_\alpha = \sigma_\alpha(B), \alpha \in A$ and on the opposite, if a foliation $\{K_\alpha\}$ is given, then the sections $\sigma_\alpha, \alpha \in A$ are defined by the equality $\sigma_\alpha(x) := \pi^{-1}(x) \cap K_\alpha, x \in B, \alpha \in A$.

In fibre bundles with such a structure there appears a natural concept for a lifting $\bar{\gamma}_u: J \rightarrow E$ of any path $\gamma: J \rightarrow B$ in the base B through every point u over the set $\gamma(J)$, i.e. $u \in \bar{\gamma}_u(J)$ and $\pi \circ \bar{\gamma}_u = \gamma$

[12]. In fact, if we define $\alpha(u) \in A$ as the subscript of the unique fibre $K_{\alpha(u)} \in \{K_\alpha\}$ to which belongs the point u , i.e. $K_{\alpha(u)} \ni u$, then the lifting $\bar{\gamma}_u: J \rightarrow E$ of γ through u is given by

$$\bar{\gamma}_u(s) := \pi^{-1}(\gamma(s)) \cap K_{\alpha(u)}, s \in J.$$

Evidently, $\{\bar{\gamma}_u(J): u \in \pi^{-1}(\gamma(s_0))\}$ for a fixed $s_0 \in J$ is a one dimensional foliation of the $(\dim(B)+1)$ -dimensional manifold $\pi^{-1}(\gamma(J))$. Besides, it is clear that $\bar{\gamma}_u$ is the unique lifting of γ in E lying as a whole in some of the fibres of the foliation $\{K_\alpha\}$.

The so-defined lifting $\gamma \mapsto \bar{\gamma}_u$ generates a transport K along the parts in B , defined by

$$K_{s \rightarrow t}^{\bar{\gamma}}(u) := \bar{\gamma}_u(t), u \in \pi^{-1}(\gamma(s)), s \in J.$$

The following equalities, which are true for every $r, s, t \in J$ and $u \in \pi^{-1}(\gamma(s))$, show that $K_{s \rightarrow t}^{\bar{\gamma}}$ is really a transport along γ from s to t (see definition 2.1):

$$\begin{aligned} K_{s \rightarrow s}^{\bar{\gamma}}(u) &= \bar{\gamma}_u(s) = \pi^{-1}(\gamma(s)) \cap K_{\alpha(u)} = u, u \in \pi^{-1}(\gamma(s)), s \in J, \\ &= (K_{t \rightarrow r}^{\bar{\gamma}} \circ K_{s \rightarrow t}^{\bar{\gamma}})(u) = K_{t \rightarrow r}^{\bar{\gamma}}(\bar{\gamma}_u(t)) = \bar{\gamma}_{\bar{\gamma}_u(t)}(r) = \pi^{-1}(\gamma(r)) \cap K_{\alpha(\bar{\gamma}_u(t))} = \\ &= \pi^{-1}(\gamma(r)) \cap K_{\alpha(u)} = \bar{\gamma}_u(r) = K_{s \rightarrow r}^{\bar{\gamma}}(u), u \in \pi^{-1}(\gamma(s)), r, s, t \in J. \end{aligned}$$

Here, we have used that $\alpha(v) = \alpha(u)$ for every $v \in K_{\alpha(u)}$ which is a consequence of the uniqueness of the fibre of the foliation passing through an arbitrary point of the foliation.

Example 2.2. In [1], vol I, p. 174 the definitions are given for parallelism and a parallel vector field according to which "if ξ is a vector bundle over $B \times B$, whose bundle over $(x, y) \in B \times B$ is $\{f: f: T_x(B) \rightarrow T_y(B), f - \text{linear}\}$, then the parallelism on B is such a section $P \in \text{Sec} \xi$ that

$$P(z, y) \circ P(x, z) = P(x, y), P(x, x) = \text{id}_{T_x(B)}, x, y, z \in B.$$

The vector field X is parallel (with respect to P) if

$$P(x, y)X_x = X_y \text{ " .}$$

If B is parallelizable, i.e. in it exists a parallelization P , then it uniquely defines axiomatically defined (global) parallel transport in B [2]. From the cited definitions it is clear that the parallelization P on B defines a transport (along paths) in the tangent to B fibre bundle $(T(B), \pi, B)$. Besides, this transport is global in a sense that it depends only on the initial and final points of a transport and does not depend on paths connecting them. As the vector fields over B are in fact sections of $(T(B), \pi, B)$ [1,2], it is evident that the above definition for parallelism of X is a special case of the definition 2.2.

2.2. ADDITIONAL RESTRICTIONS

The considered below restrictions, which can be imposed on the transports along paths, are under the title "additional" as, nevertheless they are important from some view-points, in our opinion, they have more special character and define one or another special property of the transports along paths. One should have in mind that they describe the functional dependence of the transports along paths on the path or curve of transport and they can be imposed separately, on groups, as well as all of them together. Besides, they have a number of equivalent formulations, but we below consider only one of them.

(1) Condition for locality

If J' is a subinterval of J and $\gamma: J \rightarrow B$, then

$$I_{s \rightarrow t}^{\gamma|J'} = I_{s \rightarrow t}^{\gamma} \text{ , } s, t \in J' \quad (2.5)$$

where $\gamma|J'$ is the restriction of γ on $J' \subset J$.

(2) Condition for invariance under the parameter changes

If $\tau: J'' \rightarrow J$ is one-to-one map and $\gamma: J \rightarrow B$, then

$$I_{s \rightarrow t}^{\gamma \circ \tau} = I_{\tau(s) \rightarrow \tau(t)}^{\gamma} \text{ , } s, t \in J'' \quad (2.6)$$

(3) Conditions for smoothness

We will not formulate these conditions, but we shall only mention that there are three types of them, describing the "smoothness" respectively of:

(3a) the dependence of a transport on a path of the transport;

(3b) the dependence of a transport on its initial and final parameters;

(3c) a transport as a map between the fibres over the curve defined by the path of transport.

In connection with the locality condition the following proposition is very important.

Proposition 2.3. If I^{γ} is a transport along γ , then (2.5) is equivalent to

$$I_{s \rightarrow t}^{\gamma} = I_{s \rightarrow t}^{\gamma|[\min(s, t), \max(s, t)]} \text{ , } s, t \in J \quad (2.7)$$

Proof. (2.7) follows from (2.5) for $J' = [\min(s, t), \max(s, t)] \subset J$. On the opposite, if (2.7) holds, then

$$I_{s \rightarrow t}^{\gamma|J'} = I_{s \rightarrow t}^{\gamma|J'}|_{[\min(s, t), \max(s, t)]} = I_{s \rightarrow t}^{\gamma|[\min(s, t), \max(s, t)]} = I_{s \rightarrow t}^{\gamma} \text{ ,}$$

where $s, t \in J' \subset J$ and we have used the evident fact that if $J'' \subset J' \subset J$, then $(\gamma|J')|J'' = \gamma|J''$. ■

Proposition 2.3 shows that if (2.5) holds, then the transport along γ from s to t does not depend globally on the "whole" map γ , but only on its restriction on the interval defined from s and t .

The transports along paths satisfying the condition (2.6), in fact, depend not on the path of transport $\gamma: J \rightarrow B$, but on the curve

of transport, i.e. only on the values $\gamma(s)$, $s \in J$, or other wise stated, (2.6) leads to the dependence of a transport along γ only on the nonparametrized curve $\gamma(J)$. Namely, the set of these transports has invariant and clear geometrical sense.

It is important to note that frequently the condition (2.6) is given in a weaker form in which one wants τ to be orientation preserving homeo- or diffeomorphism.

Elsewhere it will be proved that "a transport along paths is a parallel transport iff it satisfies simultaneously (2.5) and (2.6)".

The conditions of the type (3a) in their essence coincide with the conditions for smoothness in the axiomatic approach to the parallel transport, and those of the type (3b) in the case of considered in [6] linear transports in vector bundles are reduced to the standard condition for smoothness of linear operators (between vector spaces) on scalar parameter.

2.3. CONDITIONS FOR CONSISTENCY

The conditions for consistency describe connections between transports along paths in fibre bundles and, if there exist, specific for the fibre bundles structures. In the most general case of arbitrary fibre bundles there are no such structures due to which there are no conditions for consistency. They appear in more special cases, two of which are considered below. The general case of this problem will be a subject of another work.

Let $I_{s \rightarrow t}^\gamma$ be a transport along $\gamma: J \rightarrow B$ from s to t , $s, t \in J$ in the real (resp. complex) vector bundle (E, π, B) . Then, on the transport I there can be imposed the following restriction.

(4) Condition for consistency with a vector structure

If $\lambda, \mu \in \mathbb{R}$ (resp. $\lambda, \mu \in \mathbb{C}$) and $u, v \in \pi^{-1}(\gamma(s))$, $s \in J$, then

$$I_{s \rightarrow t}^\gamma(\lambda u + \mu v) = \lambda I_{s \rightarrow t}^\gamma u + \mu I_{s \rightarrow t}^\gamma v, \quad s, t \in J. \quad (2.8)$$

Using definition 2.2 we can reformulate (2.8) in an equivalent way by saying that a real (resp. complex) linear combination of transported along γ sections is also a transported along γ section whose value at $\gamma(t)$ is obtained by transporting along γ from s to t the value of this linear combination at a point $\gamma(s)$ for an arbitrary fixed $s \in J$.

Evidently, in vector bundles the condition (2.8) reduces the transports along paths to the considered in [6] linear transports along paths.

Another important example of conditions for consistency appears in the case when in the fibre bundle (E, π, B) is given a bundle metric g , i.e. [3] when in it is fixed a family $\{g_x: \pi^{-1}(x) \times \pi^{-1}(x) \rightarrow \mathbb{R}, x \in B\}$. In particular, if (E, π, B) is a vector fibre bundle and g_x , $x \in B$ are nondegenerate bilinear forms, then g defines an usual bundle metric in the fibre bundle [2]. (To this class belong also all (pseudo-)Riemannian metrics: if B is a (pseudo-)Riemannian manifold, then one puts $E = T(B)$, $\pi^{-1}(x) = T_x(B)$, $x \in B$ and g is identified with the (pseudo-)Riemannian metric of B .) In this case the class of transports along $\gamma: J \rightarrow B$ satisfying the following condition is naturally generated.

(5) Condition for consistency with a bundle metric

If $u, v \in \pi^{-1}(\gamma(s))$, $s \in J$, then

$$g_{\gamma(s)}(u, v) = g_{\gamma(t)}(I_{s \rightarrow t}^\gamma u, I_{s \rightarrow t}^\gamma v), \quad s, t \in J. \quad (2.9)$$

In the (pseudo-)Riemannian case (2.9) simply means a conservation of the scalar product of the vectors under their (parallel) transport along an arbitrary path at any its point.

The transports along paths satisfying simultaneously (2.8) and (2.9) will be investigated elsewhere (cf. [13]).

As we shall see elsewhere the conditions for consistency of transports along paths with some structures are equivalent to the demand these structures to be "transported" with respect to properly chosen transports along paths acting in the corresponding fibre bundles. In particular, this is the case with the condition for consistency of a parallelism P (see example 2.2 at the end of Subsect. 2.1) and its torsion S, which is given in [1], vol. I, p.175 and p.235, where this condition is taken as a definition for parallelness ("parallel transport") of S; in fact, in this case, these two concepts are equivalent.

3. SOME PROPERTIES AND GENERAL FORM

Proposition 3.1. If (2.2) is valid, then (2.3) is equivalent to the invertibility of the map (2.1) when its initial and final parameters coincide, i.e. to the existence of $(I_{s \rightarrow s}^\gamma)^{-1}$, $s \in J$.

Proof. If (2.3) is valid, then, evidently, no matter whether (2.2) is true or not $(I_{s \rightarrow s}^\gamma)^{-1}$ there exists and $(I_{s \rightarrow s}^\gamma)^{-1} = id_{\pi^{-1}(\gamma(s))} = I_{s \rightarrow s}^\gamma$. On the opposite, if, for example, exists the left inverse map $(I_{s \rightarrow s}^\gamma)^{-1}$, then putting $r=s=t$ in (2.2), we get $I_{s \rightarrow s}^\gamma \circ I_{s \rightarrow s}^\gamma = I_{s \rightarrow s}^\gamma$ and multiplying from left by $(I_{s \rightarrow s}^\gamma)^{-1}$, we convince ourselves in the validity of (2.3). ■

Proposition 3.2. If (2.2) is fulfilled, then (2.3) is equivalent to the existence of the inverse map of (2.1), besides

$$(I_{s \rightarrow t}^\gamma)^{-1} = I_{t \rightarrow s}^\gamma, \quad s, t \in J. \quad (3.1)$$

Proof. If (2.3) holds, putting $r=t$ in (2.2), we get $I_{s \rightarrow t}^\gamma \circ$

$I_{t \rightarrow s}^\gamma = id_{\pi^{-1}(\gamma(t))}$, which results in (3.1). On the opposite, if (3.1) is true, then putting $r=t$ in (2.2), we get $I_{t \rightarrow t}^\gamma = I_{s \rightarrow t}^\gamma \circ I_{t \rightarrow s}^\gamma = id_{\pi^{-1}(\gamma(t))}$, i.e. (2.3) is valid. ■

In particular, (3.1) shows that the transports along a path $\gamma: J \rightarrow B$ are 1:1 invertible maps between the fibres over $\gamma(J)$.

If $\tau: J' \rightarrow J$ is an orientation changing 1:1 map between the real intervals J' and J and $\gamma: J \rightarrow B$ is a path, then the path $\gamma_\tau := \gamma \circ \tau: J' \rightarrow B$ will be called inverse to γ with respect to τ . In the canonical case, when paths of the type $\gamma: [0,1] \rightarrow B$ are considered, by definition [8,12] the inverse path to $\gamma: [0,1] \rightarrow B$ is $\gamma_- := \gamma \circ \tau_-: [0,1] \rightarrow B$, where $\tau_-: [0,1] \rightarrow [0,1]$ is defined by $\tau_-: s \mapsto 1-s$, $s \in [0,1]$.

Proposition 3.3. If (2.6) holds, then the transports along a path $\gamma: J \rightarrow B$ and its inverse path $\gamma_-: J' \rightarrow B$ are connected by

$$I_{s \rightarrow t}^{\gamma_\tau} = I_{\tau_-(s) \rightarrow \tau_-(t)}^\gamma, \quad s, t \in J'. \quad (3.2)$$

Proof. (3.2) follows from (2.6) for $\tau = \tau_-$. ■

We define the product of paths in the following way which differs from the standard one (cf. [8,12]; see below). Let us have an ordered pair (γ_1, γ_2) of two paths $\gamma_h: [a_h, b_h] \rightarrow B$, $h=1,2$ with the end of γ_1 coinciding with the beginning of γ_2 , i.e. $\gamma_1(b_1) = \gamma_2(a_2)$. Let there be given numbers $a_0, b_0, c_0 \in \mathbb{R}$, such that $a_0 \leq c_0 \leq b_0$, and one-to-one maps $\tau_1: [a_0, c_0] \rightarrow [a_1, b_1]$ and $\tau_2: [c_0, b_0] \rightarrow [a_2, b_2]$ preserving the orientation, which, in particular, means that $\tau_1(a_0) = a_1$, $\tau_1(c_0) = b_1$, $\tau_2(c_0) = a_2$ and $\tau_2(b_0) = b_2$. The product of the paths γ_1 and γ_2 is a path

$$(\gamma_1, \gamma_2)_\chi: [a_0, b_0] \rightarrow B, \quad (3.3a)$$

which depends on the parameter $\chi := (a_0, b_0, c_0; \tau_1, \tau_2)$, and it is defined by the equalities

$$(\gamma_1 \gamma_2)_\chi | [a_0, c_0] := \gamma_1 \circ \tau_1, \quad (3.3b)$$

$$(\gamma_1 \gamma_2)_\chi | [c_0, b_0] := \gamma_2 \circ \tau_2. \quad (3.3c)$$

In the canonical case (see [8,12]) the product of the paths $\gamma_h: [0,1] \rightarrow B$, $h=1,2$ is defined by a parameter $\chi^c = (0,1,1/2; \tau_1^c, \tau_2^c)$, where $\tau_1^c: s \mapsto 2s$ for $s \in [0,1/2]$ and $\tau_2^c: s \mapsto 2s-1$ for $s \in [1/2,1]$.

Proposition 3.4. Let (2.2), (2.6) and (2.7) be valid. Let $\gamma_h: [a_h, b_h] \rightarrow B$, $h=1,2$ and $(\gamma_1 \gamma_2)_\chi$ be the product of γ_1 and γ_2 defined by a parameter $\chi = (a_0, b_0, c_0; \tau_1, \tau_2)$. Then, for arbitrary $t_1 \in [a_0, c_0]$ and $t_2 \in [c_0, b_0]$ the equality is valid

$$I_{t_1 \rightarrow t_2}^{(\gamma_1 \gamma_2)_\chi} = I_{a_2 \rightarrow \tau_2(t_2)}^{\gamma_2} \circ I_{\tau_1(t_1) \rightarrow b_1}^{\gamma_1}, \quad a_0 \leq t_1 \leq c_0 \leq t_2 \leq b_0. \quad (3.4)$$

Proof. Using consequently (2.2), (2.7), (2.6) and the above definition of $(\gamma_1 \gamma_2)_\chi$, we get:

$$\begin{aligned} I_{t_1 \rightarrow t_2}^{(\gamma_1 \gamma_2)_\chi} &= I_{c_0 \rightarrow t_2}^{(\gamma_1 \gamma_2)_\chi} \circ I_{t_1 \rightarrow c_0}^{(\gamma_1 \gamma_2)_\chi} = I_{c_0 \rightarrow t_2}^{(\gamma_1 \gamma_2)_\chi} | [c_0, t_2] \circ I_{t_1 \rightarrow c_0}^{(\gamma_1 \gamma_2)_\chi} | [t_1, c_0] = \\ &= I_{c_0 \rightarrow t_2}^{\gamma_2 \circ \tau_2} \circ I_{t_1 \rightarrow c_0}^{\gamma_1 \circ \tau_1} = I_{\tau_2(c_0) \rightarrow \tau_2(t_2)}^{\gamma_2} \circ I_{\tau_1(t_1) \rightarrow \tau_1(c_0)}^{\gamma_1}, \end{aligned}$$

from where it follows (3.4), as we have $\tau_1(a_0) = a_1$, $\tau_1(c_0) = b_1$, $\tau_2(c_0) = a_2$ and $\tau_2(b_0) = b_2$ because τ_1 and τ_2 are orientation preserving. ■

Remark. If we had, for example, $t_1, t_2 \in [a_0, c_0]$, then instead of (3.4) the equality will be valid

$$I_{t_1 \rightarrow t_2}^{(\gamma_1 \gamma_2)_\chi} = I_{\tau_1(t_1) \rightarrow \tau_1(t_2)}^{\gamma_1}, \quad t_1, t_2 \in [a_0, c_0], \quad (3.5)$$

in the derivation of which only (2.6) and (2.7) can be used.

The above proposition shows some important properties of transports along paths and connections between the basic properties describing them. Let us note that when deriving (3.2), (3.4) and (3.5) the main role was played by the condition for invariance (2.6).

Once again we want to stress the fact that the basic properties describe the dependence of a transport along paths on the initial and final value of its parameters, while the additional conditions define its dependence on the path of the transport.

The general functional form of the transports along paths is given by

Theorem 3.1. Let in the base B of the fibre bundle (E, π, B) be given a path $\gamma: J \rightarrow B$ and for arbitrary $s, t \in J$ be given a map (2.1). The maps $I_{s \rightarrow t}^\gamma$, $s, t \in J$ define a transport along γ from s to t , i.e. (2.2) and (2.3) are satisfied iff there exist a set Q and a family of one-to-one maps $\{F_s^\gamma: \pi^{-1}(\gamma(s)) \rightarrow Q, s \in J\}$ such that

$$I_{s \rightarrow t}^\gamma = (F_t^\gamma)^{-1} \circ (F_s^\gamma), \quad s, t \in J. \quad (3.6)$$

Proof. The theorem is a corollary of the following lemma in which one has to put $N=J$, $Q_s = \pi^{-1}(\gamma(s))$ and $R_{s \rightarrow t} = I_{s \rightarrow t}^\gamma$, $s, t \in J$.

Lemma 3.1. Let there be given a set N , $N \neq \emptyset$ and families of equipollent sets $\{Q_s: s \in N\}$ and of maps $\{R_{s \rightarrow t}: R_{s \rightarrow t}: Q_s \rightarrow Q_t, s, t \in N\}$. Then, the maps of $\{R_{s \rightarrow t}\}$ satisfy the equalities

$$R_{s \rightarrow t} \circ R_{r \rightarrow s} = R_{r \rightarrow t}, \quad r, s, t \in N, \quad (3.7)$$

$$R_{s \rightarrow s} = id_{Q_s}, \quad s \in N \quad (3.8)$$

iff there exists an equipollent with Q_s for some $s \in N$ set Q and a family of one-to-one maps $\{F_s: F_s: Q_s \rightarrow Q, s \in N\}$, such that

$$R_{s \rightarrow t} = (F_t)^{-1} \circ (F_s), \quad s, t \in N. \quad (3.9)$$

Proof. The sufficiency is almost evident: the substitution of (3.9) into (3.7) and (3.8) converts them into identities. On the opposite, putting in (3.7) $r=t$ and using (3.8), we see that $R_{s \rightarrow t}$ has an inverse map and

$$(R_{s \rightarrow t})^{-1} = R_{t \rightarrow s}, \quad s, t \in N, \quad (3.10)$$

due to which (see and (3.7)) for any fixed $s_0 \in N$, we have $R_{s \rightarrow t} = R_{s_0 \rightarrow t} \circ R_{s_0 \rightarrow s} = (R_{t \rightarrow s_0})^{-1} \circ (R_{s \rightarrow s_0})$, i.e. (3.9) is fulfilled for $Q = Q_{s_0}$ and $F_s = R_{s \rightarrow s_0}$. ■

The arbitrariness in the choice of the set Q and a family $\{F_s\}$ in the theorem 3.1 is described by

Proposition 3.5. Let in the fibre bundle (E, π, B) be given a transport along paths I with a representation (3.6) for some set Q and a family of 1:1 maps $\{F_s^{\gamma}: \pi^{-1}(\gamma(s)) \rightarrow Q, s \in J\}$. Then, there exist a set ${}^{\circ}Q$ and a family of one-to-one maps $\{{}^{\circ}F_s^{\gamma}: \pi^{-1}(\gamma(s)) \rightarrow {}^{\circ}Q, s \in J\}$ such that

$$I_{s \rightarrow t}^{\gamma} = ({}^{\circ}F_t^{\gamma})^{-1} \circ ({}^{\circ}F_s^{\gamma}), \quad s, t \in J, \quad (3.6')$$

iff there exists an one-to-one map $D^{\gamma}: {}^{\circ}Q \rightarrow Q$ for which

$$F_s^{\gamma} = D^{\gamma} \circ ({}^{\circ}F_s^{\gamma}), \quad s \in J. \quad (3.11)$$

Proof. The proposition follows for $N=J$, $Q_s = \pi^{-1}(\gamma(s))$ and $R_{s \rightarrow t} = I_{s \rightarrow t}^{\gamma}$, $s, t \in J$ from

Lemma 3.2. Let there be given a set N , $N \neq \emptyset$, family of equipollent sets $\{Q_s: s \in N\}$, an equipollent with Q_s for some $s \in N$ set ${}^{\circ}Q$ and a family of maps $\{{}^{\circ}F_s: Q_s \rightarrow {}^{\circ}Q, s \in N\}$. If

$$R_{s \rightarrow t} = ({}^{\circ}F_t)^{-1} \circ ({}^{\circ}F_s), \quad (3.9')$$

then (3.9) is valid for some family of maps $\{F_s: Q_s \rightarrow Q, s \in N\}$, Q being some equipollent with ${}^{\circ}Q$ set, if and only if there exists a one-to-one map $D: {}^{\circ}Q \rightarrow Q$, such that

$$F_s = D \circ ({}^{\circ}F_s). \quad (3.12)$$

Proof. The sufficiency is almost evident: if (3.12) is true for some Q and $\{F_s\}$, then from it we find ${}^{\circ}F_s = D^{-1} \circ F_s$, $s \in N$ and subs-

tituting this result into (3.9'), we get (3.9). On the opposite, if (3.9) is true, then, due to (3.9'), from it results in $(F_t)^{-1} \circ (F_s) = ({}^{\circ}F_t)^{-1} \circ ({}^{\circ}F_s)$, $s, t \in N$, hereof we see that $F_s \circ ({}^{\circ}F_s)^{-1} = F_t \circ ({}^{\circ}F_t)^{-1}$ for any $s, t \in N$, but this means that the left and right hand sides of the last equality do not depend either on s or on t . Hence, fixing arbitrarily some $s_0 \in N$ and putting $D = F_{s_0} \circ ({}^{\circ}F_{s_0})^{-1}: {}^{\circ}Q \rightarrow Q$, from the last equality for $t=s_0$, we get (3.12). ■

4. TIES WITH THE LIFTING OF PATHS

The definition of a transport along paths I in a fibre bundle (E, π, B) leads to a natural lifting of every path $\gamma: J \rightarrow B$ [12].

Definition 4.1. Let $\gamma: J \rightarrow B$, I^{γ} be a transport along γ , $u \in \pi^{-1}(\gamma(J))$, $q_{\gamma}(u) := \{s: s \in J, \gamma(s) = \pi(u)\}$ and $s_0 \in q_{\gamma}(u)$. The path $\bar{\gamma}_{u, s_0}: J \rightarrow E$ defined by the equality

$$\bar{\gamma}_{u, s_0}(s) := I_{s_0 \rightarrow s}^{\gamma}(u), \quad s \in J, \quad (4.1)$$

is the lifting of γ through u generated by I^{γ} with a parameter s_0 ; the map $\gamma \mapsto \bar{\gamma}_{u, s_0}$ is the lift through u generated by I^{γ} with a parameter s_0 . If $q_{\gamma}(u)$ consists of only one element, i.e., if $\pi(u)$ is not a self-intersection point for γ , then $\bar{\gamma}_{u, s_0}$ will be denoted simply by $\bar{\gamma}_u$ and we say that $\bar{\gamma}_u$ and $\gamma \mapsto \bar{\gamma}_u$ are generated by I^{γ} .

Evidently, any transport along paths I generates through (4.1) a lift of the paths from the base B . The usage of the term "lifting" here and in definition 4.1 is correct due to

Proposition 4.1. The path $\bar{\gamma}_{u, s_0}: J \rightarrow E$, defined by (4.1), is a lifting of the path $\gamma: J \rightarrow B$ (from B into E) through u , i.e.

$$\pi \circ \bar{\gamma}_{u, s_0} = \gamma, \quad \bar{\gamma}_{u, s_0}(s_0) = u. \quad (4.2)$$

Proof. (4.2) follows from (4.1) and definition 2.1 (see also (2.1) and (2.3)). ■

From (4.1) we can make the following simple, but important, conclusion. In the general case, one transport along paths generates through the point $u \in \pi^{-1}(\gamma(J))$ as many liftings of the path $\gamma: J \rightarrow B$ as is the number of elements (the power) of $q_\gamma(u)$, equal, evidently, to one plus the number of self-intersections of γ at the point $\pi(u)$. In this connection, one naturally puts the question in what sense and when the lift of a given path through a lying above it point is unique.

Definition 4.2. The lifting generated through (4.1) by a transport along paths I is globally unique (or is unique in an absolute sense) if for every path $\gamma: J \rightarrow B$ and every $u \in \pi^{-1}(\gamma(J))$ we have

$$\bar{\gamma}_{u, r} = \bar{\gamma}_{u, s}, \quad r, s \in q_\gamma(u). \quad (4.3)$$

Proposition 4.2. The lifting generated through (4.1) by the transport I is globally unique iff

$$I_{r \rightarrow s} = \text{id}_{\pi^{-1}(\gamma(s))} \quad \text{for those } r, s \in J \text{ for which } \gamma(r) = \gamma(s), \quad (4.4)$$

or, which is equivalent, iff

$$v \in \bar{\gamma}_{u, r}(J) \iff \bar{\gamma}_{v, s} = \bar{\gamma}_{u, r}, \quad s \in q_\gamma(v), \quad r \in q_\gamma(u). \quad (4.5)$$

Proof. Let (4.3) be valid. Then, for any $t \in J$ from (4.1), we get $I_{r \rightarrow t}(u) = \bar{\gamma}_{u, r}(t) = \bar{\gamma}_{u, s}(t) = I_{s \rightarrow t}(u)$. Therefore, due to the arbitrariness of $u \in \pi^{-1}(\gamma(J))$, we find $I_{r \rightarrow t} = I_{s \rightarrow t}$, which as a consequence of (2.1), (2.2), (3.1) and the definition of $q_\gamma(u)$ (see definition 4.1), is equivalent to (4.4). Further, if $v \in \bar{\gamma}_{u, r}(J)$, $r \in q_\gamma(u)$, then by (4.1) there exists $s_1 \in J$ such that $v = \bar{\gamma}_{u, r}(s_1) = I_{r \rightarrow s_1}(u)$. So for every $t \in J$ and $s \in q_\gamma(v)$, we have $\bar{\gamma}_{v, s}(t) =$

$= I_{s \rightarrow t}(v) = I_{s \rightarrow t} \circ I_{r \rightarrow s_1}(u) = (I_{s \rightarrow t} \circ I_{s_1 \rightarrow s} \circ I_{r \rightarrow s_1})(u) = I_{r \rightarrow t}(u) = \bar{\gamma}_{u, r}(t)$. Here we used the evident fact that $s_1 \in q_\gamma(v)$ (see (4.2)), the proved equality (4.4), and, besides, we have twice applied (2.2). Hence $\bar{\gamma}_{v, s} = \bar{\gamma}_{u, r}$, i.e. the right implication in (4.5) is true. The inverse implication in (4.5) is evident and it does not depend on any additional facts: if $\bar{\gamma}_{v, s} = \bar{\gamma}_{u, r}$ for $s \in q_\gamma(v)$ and $r \in q_\gamma(u)$, then $v \in \bar{\gamma}_{v, s}(J) = \bar{\gamma}_{u, r}(J)$, i.e. $v \in \bar{\gamma}_{u, r}(J)$. So, we proved that (4.3) results in (4.4) leading to (4.5) due to which the proof of the proposition ends with the fact that from (4.5) there follows (4.3): as by definition $u \in \bar{\gamma}_{u, r}(J)$, then from (4.5) for $v = u$ there follows (4.3). ■

In a local sense, the generated from a transport along paths lifting of paths may be unique under different criteria, modifying in an appropriate way definition 4.2 and to which there correspond modified versions of proposition 4.2. For instance, the local uniqueness may be defined in the following two ways:

(a) Equality (4.3) is valid only along a given path γ (uniqueness along a given path). A necessary and sufficient condition for this is (4.4), or (4.5), to be fulfilled along the path γ . (The proof is a consequence from the one of proposition 4.2 for a fixed path γ .)

(b) Equality (4.3) to be valid only at a given point u over a path γ (uniqueness along a given path through a point above it). A necessary and sufficient condition for this is (4.4) to be fulfilled for every $r, s \in q_\gamma(u)$. (The proof is a consequence from the one of proposition 4.2 for fixed γ and u .) Due to this if the lifting is unique along γ through u , then it is unique along γ through every $v \in \pi^{-1}(\pi(u))$, so $q_\gamma(v) = q_\gamma(u)$. Now, in the general case, (4.4) and (4.5) are not equivalent.

Evidently, the uniqueness with respect to (a) is stronger: the

lifting is unique, along a given path iff it is unique along it through an arbitrary point above it.

Corollary 4.1. The generated in accordance to (4.1) from an arbitrary transport along paths lifting of any path without self-intersection through any lying above it point is unique (in any one of the above-mentioned senses) along it.

Proof. If $\gamma: J \rightarrow B$ is without intersections, then $\gamma(s) = \gamma(r)$, $s, r \in J$ is equivalent to $r = s$, as a consequence of which for every $u \in \pi^{-1}(\gamma(J))$ the set $q_\gamma(u)$ contains only one element, so the equalities (4.3)-(4.5) are identically satisfied. ■

Due to corollary 4.1 there can be uniqueness of the mentioned lifts in a "middle" between local and global sense. For instance, we can say that the liftings generated by transports along paths are unique when they act on the class of paths without self-intersections.

Concluding the discussion of the uniqueness of lifting generated by transports along paths we shall prove

Proposition 4.3. The lifting generated through (4.1) from I is unique in a sense that for every path $\gamma: J \rightarrow B$, we have

$$\bar{\gamma}_{u,s} = \bar{\gamma}_{u,s}, \quad u \in \pi^{-1}(\gamma(s)), \quad r, s \in J, \quad (4.6)$$

i.e. the lift of any path through a point belonging to its lifting, when the value of its parameter coincides with the one describing this point, coincides with the initial lift of the path.

Proof. (4.6) is equivalent to the basic property (2.2) of the transports along paths: if $t \in J$, then $\bar{\gamma}_{u,s}(t) = I_{s \rightarrow t}^\gamma(u)$ and $\bar{\gamma}_{u,s}(r) = I_{r \rightarrow t}^\gamma(\bar{\gamma}_{u,s}(r)) = I_{r \rightarrow t}^\gamma \circ I_{s \rightarrow r}^\gamma(u)$, so (4.6) and (2.2) are equivalent. ■

Corollary 4.2. If the lifting generated through (4.1) by I is globally (resp. locally) unique (resp. along γ) and $u, v \in \pi^{-1}(\gamma(J))$,

then $\bar{\gamma}_{u,r}(J) \cap \bar{\gamma}_{v,s}(J) = \emptyset$ or $\bar{\gamma}_{u,r}(J) = \bar{\gamma}_{v,s}(J)$, $r \in q_\gamma(u)$, $s \in q_\gamma(v)$ for every (resp. the given) path γ , i.e., in this case two generated from I liftings of one and the same path do not intersect with each other or coincide as sets.

Proof. Let $\bar{\gamma}_{u,r}(J) \cap \bar{\gamma}_{v,s}(J) \neq \emptyset$ and $w \in \bar{\gamma}_{u,r}(J) \cap \bar{\gamma}_{v,s}(J)$. Then, due to the global (resp. local) uniqueness (resp. along γ) (4.5) is valid because of proposition 4.2. So $\bar{\gamma}_{u,r} = \bar{\gamma}_{u,t} = \bar{\gamma}_{v,s}$ for $r \in q_\gamma(u)$, $s \in q_\gamma(v)$ and $t \in q_\gamma(w)$, i.e. $\bar{\gamma}_{u,r} = \bar{\gamma}_{v,s}$, from where we get $\bar{\gamma}_{u,r}(J) = \bar{\gamma}_{v,s}(J)$. ■

Corollary 4.3. If the generated from I lifting is unique along a path γ , $u \in \pi^{-1}(\gamma(J))$, $v \in \pi^{-1}(\pi(u))$ and $v \neq u$, then $\bar{\gamma}_{u,r}(J) \cap \bar{\gamma}_{v,s}(J) = \emptyset$, $r, s \in q_\gamma(u) = q_\gamma(v)$, i.e., in this case two lifts of one and the same path through two different points above it having equal projections do not have a common point as sets.

Proof. If we admit $\bar{\gamma}_{u,r}(J) \cap \bar{\gamma}_{v,s}(J) \neq \emptyset$ and $w \in \bar{\gamma}_{u,r}(J) \cap \bar{\gamma}_{v,s}(J)$, then by proposition 4.2 (see (4.5)) $\bar{\gamma}_{u,r} = \bar{\gamma}_{w,t} = \bar{\gamma}_{v,s}$, $r, s \in q_\gamma(u) = q_\gamma(v)$, $t \in q_\gamma(w)$. So, if we choose $s^* \in q_\gamma(u) = q_\gamma(v)$ and use (4.3), we get $u = \bar{\gamma}_{u,s^*}(s^*) = \bar{\gamma}_{u,s}(s^*) = \bar{\gamma}_{v,s}(s^*) = \bar{\gamma}_{v,s^*}(s^*) = v$, which contradicts $u \neq v$. ■

Proposition 4.4. For every $s_0 \in J$, for which the lifting generated by I is unique along γ through some $v_0 \in \pi^{-1}(\gamma(s_0))$, is fulfilled

$$\pi^{-1}(\gamma(J)) = \bigcup_{u \in \pi^{-1}(\gamma(s_0))} \bar{\gamma}_{u,s}(J) \text{ for } \gamma(s) = \gamma(s_0) \quad (4.7)$$

for every $s \in J$ and if $s = s_0$, then this equality is valid in spite of the above conditions.

Proof. From the definition of the pointed uniqueness it follows that $\bar{\gamma}_{v,s} = \bar{\gamma}_{v,s_0}$, $s, s_0 \in q_\gamma(v_0)$ for every $v \in \pi^{-1}(\gamma(s_0))$ (see above). Then $v = \bar{\gamma}_{v,s_0}(s_0) = \bar{\gamma}_{v,s}(s_0)$, from which we find $\bar{\gamma}_{v,s} = \bar{\gamma}_{v,s_0}$, so

$$v \in \bar{\gamma}_{v,s}(J) = \bar{\gamma}_{v,s_0}(J) \subset \bigcup_{u \in \pi^{-1}(\gamma(s_0))} \bar{\gamma}_{u,s}(J)$$

and consequently

$$\pi^{-1}(\gamma(J)) = \bigcup_{v \in \pi^{-1}(\gamma(J))} v \subseteq \bigcup_{u \in \pi^{-1}(\gamma(s_0))} \bar{\gamma}_{u,s}(J).$$

Evidently, for $s=s_0$ the last two formulae, as well as their proofs, are identically valid and do not have any connection with the uniqueness of the generated by I lifting.

On the other hand, due to (4.2), we have the inclusion

$$\bigcup_{u \in \pi^{-1}(\gamma(s_0))} \bar{\gamma}_{u,s}(J) \subseteq \pi^{-1}(\gamma(J)),$$

from which, due to the previous inclusion, it follows (4.7). ■

Proposition 4.5. If the generated by I lifting is unique along $\gamma: J \rightarrow B$ and E is a manifold, then for arbitrary fixed $s_0, s \in J$ such that $\gamma(s_0) = \gamma(s)$, the family $\{\bar{\gamma}_{u,s}(J): u \in \pi^{-1}(\gamma(s_0))\}$ forms (one-dimensional) foliation of the submanifold $\pi^{-1}(\gamma(J)) \subset E$.

Proof. According to corollary 4.3, the sets from the considered family do not have common points, so the proposition follows directly from the definition of a foliation [11]. ■

The above results show that the definition of a transport along paths in one fibre bundle induces a structure of foliation in the set of liftings of an arbitrary path in its base.

Now we shall consider the problem when in one (generally topological) fibre bundle there exist transports along paths.

Theorem 4.1. In the fibre bundle (E, π, B) there exist transports along paths if and only if for every path $\gamma: J \rightarrow B$, every point $u \in \pi^{-1}(\gamma(J))$ and every $\text{seq}_\gamma(u)$ there exist a lift $\bar{\gamma}_{u,s}: J \rightarrow E$ of γ through u , for which $\bar{\gamma}_{u,s}(s) = u$ and which is unique in a sense that for it (4.6) is valid.

Proof. The necessity follows directly from definition 4.1 and propositions 4.1 and 4.3. On the opposite, let in (E, π, B) exist a lift $l: (\gamma, u, s) \mapsto \bar{\gamma}_{u,s}$ of the paths in B with the described properties. For $s \in J$ and $u \in \pi^{-1}(\gamma(s))$, we define the maps

${}^1I_{s \rightarrow t}^\gamma: \pi^{-1}(\gamma(s)) \rightarrow \pi^{-1}(\gamma(t))$ by the equality

$${}^1I_{s \rightarrow t}^\gamma u := \bar{\gamma}_{u,s}(t), \quad s, t \in J, \quad u \in \pi^{-1}(\gamma(s)). \quad (4.8)$$

On the one hand, if we put here $s=t$ and use $\bar{\gamma}_{u,s}(s) = u$ and the arbitrariness of u , we get ${}^1I_{s \rightarrow s}^\gamma = \text{id}_{\pi^{-1}(\gamma(s))}$. On the other hand, from the proof of proposition 4.3 it follows that (4.6) is equivalent to ${}^1I_{t \rightarrow r}^\gamma \circ {}^1I_{s \rightarrow t}^\gamma = {}^1I_{s \rightarrow r}^\gamma$, so ${}^1I_{s \rightarrow t}^\gamma$ is a transport along γ from s to t , i.e., 1I defined by ${}^1I: \gamma \mapsto {}^1I^\gamma: (s, t) \mapsto {}^1I_{s \rightarrow t}^\gamma$, is a transport along paths in (E, π, B) . ■

Theorem 4.1 has also a local variant expressed by

Theorem 4.1'. In the fibre bundle (E, π, B) along a given path $\gamma: J \rightarrow B$ there exists a transport along it if and only if for every $u \in \pi^{-1}(\gamma(J))$ and every $\text{seq}_\gamma(u)$ there exists a lifting $\bar{\gamma}_{u,s}: J \rightarrow E$ of γ through u , for which $\bar{\gamma}_{u,s}(s) = u$ and (4.6) holds.

Proof. The proof of this theorem coincides with the proof of theorem 4.1 with the only difference that now the path γ is fixed. ■

The proof of theorem 4.1, together with proposition 4.1, gives us a reason to introduce

Definition 4.3. If for one lifting of paths equality (4.6) holds along every (resp. a given) path γ , then for the defined by (4.8) transport along paths we shall say that it is generated by this lifting (resp. along γ).

Proposition 4.6. If one transport along paths is generated through (4.8) by some lifting of paths from B to E , which is unique in a sense that (4.6) is true, then the generated from this transport, in accordance with definition 4.1, lifting of paths from B to E coincides with the initial lifting generating the considered transport along paths.

Proof. Let the transport along paths 1I be generated by the lift $l: \gamma \mapsto \bar{\gamma}_{u,s}$, $u \in \pi^{-1}(\gamma(J))$, $\bar{\gamma}_{u,s}(s) = u$, $\text{seq}_\gamma(u)$ in accordance

with (4.8). Then, due to definition 4.1 and proposition 4.1, 1I generates the lifting ${}^1l: \gamma \rightarrow \bar{\gamma}_{u,s}$, so that $\bar{\gamma}_{u,s}(t) = {}^1I_s^{\gamma} \rightarrow_t u$. Comparing the last equality with (4.8), we conclude that $\bar{\gamma}_{u,s}(t) = \bar{\gamma}_{u,s}(t)$, $t \in J$, i.e. $\bar{\gamma}_{u,s} = \bar{\gamma}_{u,s}$ and consequently ${}^1l = 1$. ■

Proposition 4.7. If one lifting of paths from B to E is generated from a transport along paths by (4.1), then this lifting generates in accordance with (4.8) a transport along paths coinciding with the transport along paths generating the considered lifting.

Proof. Let the lifting $\gamma \rightarrow \bar{\gamma}_{u,s}$, $u \in \pi^{-1}(\gamma(J))$, $\bar{\gamma}_{u,s}(s) = u$, $s \in q_{\gamma}(u)$ be generated from the transport I , i.e. $\bar{\gamma}_{u,s}(t) = I_s^{\gamma} \rightarrow_t u$. Then, this lifting generates transport 1I for which ${}^1I_s^{\gamma} \rightarrow_t u = \bar{\gamma}_{u,s}(t) = I_s^{\gamma} \rightarrow_t u$, so ${}^1I = I$. ■

5. CONNECTION GENERATED BY A TRANSPORT ALONG PATHS

In this section, the problem will be discussed of how one transport along paths, with the help of the generated by it lifting of paths, generates a connection in (part of) a given manifold.

Let (E, π, B) be a differential fibre bundle, U be k -dimensional, $1 \leq k \leq \dim$, submanifold of B and U to be covered from a congruence of paths $\gamma_{\lambda}: J \rightarrow U$ numbered with the $(k-1)$ -dimensional parameter $\lambda = (\lambda_1, \dots, \lambda_{k-1}) \in \mathbb{A}C\mathbb{R}^k$, i.e. $U = \bigcup_{\lambda \in \Lambda} (\gamma_{\lambda}(J_{\lambda}))$ and $\gamma_{\lambda}(J_{\lambda}) \cap \gamma_{\mu}(J_{\mu}) = \emptyset$ for $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. Then, from (4.7) it follows that for any fixed $x \in U$ is fulfilled

$$\pi^{-1}(U) = \bigcup_{u \in \pi^{-1}(x)} \left[\bigcup_{\lambda \in \Lambda} \overline{(\gamma_{\lambda})}_{u,s}(J_{\lambda}) \right], \quad s \in q_{\gamma}(u) \quad (5.1)$$

where the sets in square brackets, which are k -dimensional manifolds, form a k -dimensional foliation over $\pi^{-1}(U)$ [11].

If $u \in \pi^{-1}(U)$ and $T_x(M)$ is the tangent space to at $x \in M$ to the manifold M [3], then by definition [1,2,4] the vertical tangent

space to $\pi^{-1}(U)$ at u is

$$T_u^v(\pi^{-1}(U)) := T_u(\pi^{-1}(\pi(u))) \subset T_u(\pi^{-1}(U)). \quad (5.2a)$$

Let:

$$T_u^{h-v}(\pi^{-1}(U)) := T_u \left(\bigcup_{\lambda \in \Lambda} \overline{(\gamma_{\lambda})}_{u,s}(J_{\lambda}) \right) \subset T_u(\pi^{-1}(U)), \quad (5.2b)$$

$$T_u^0(\pi^{-1}(U)) := T_u^v(\pi^{-1}(U)) \cap T_u^{h-v}(\pi^{-1}(U)) \quad (5.2c)$$

and $T_u^h(\pi^{-1}(U))$ be the direct complement of $T_u^0(\pi^{-1}(U))$ in $T_u^{h-v}(\pi^{-1}(U))$, i.e.

$$T_u^{h-v}(\pi^{-1}(U)) := T_u^0(\pi^{-1}(U)) \oplus T_u^h(\pi^{-1}(U)). \quad (5.2d)$$

Lemma 5.1. For every $u \in \pi^{-1}(U)$ is fulfilled

$$T_u^v(\pi^{-1}(U)) \cap T_u^h(\pi^{-1}(U)) = \{0\}. \quad (5.3)$$

where 0 is the zero element of $T_u(\pi^{-1}(U))$.

Proof. Let $u \in \pi^{-1}(U)$ and $\Omega := T_u^v(\pi^{-1}(U)) \cap T_u^h(\pi^{-1}(U))$. If $\omega \in \Omega$, then $\omega \in T_u^h(\pi^{-1}(U))$ and according to (5.2d), we have $\omega \in T_u^{h-v}(\pi^{-1}(U))$. From here, taking into account (5.2c) and $\omega \in T_u^v(\pi^{-1}(U))$ (due to $\omega \in \Omega$), it follows $\omega \in T_u^0(\pi^{-1}(U))$. This fact, compared with $\omega \in T_u^h(\pi^{-1}(U))$ shows that $\omega \in T_u^h(\pi^{-1}(U)) \cap T_u^0(\pi^{-1}(U))$. But, in accordance with the definition for a direct sum, from (5.2d) it follows $T_u^h(\pi^{-1}(U)) \cap T_u^0(\pi^{-1}(U)) = \{0\}$ and so $\omega \in \{0\}$, i.e. $\omega = 0$. Hence, if $\omega \in \Omega$, then $\omega = 0$, due to which $\Omega = \{0\}$, i.e., (5.3) is valid. ■

Proposition 5.2. The subspaces $T_u^h(\pi^{-1}(U))$ are horizontal, i.e., they define a connection in the fibre bundle $(\pi^{-1}(U), \pi|_U, U) = (E, \pi, B)|_U$, iff

$$T_u(\pi^{-1}(U)) = T_u^v(\pi^{-1}(U)) \oplus T_u^h(\pi^{-1}(U)). \quad (5.4)$$

Remark. In the case of locally trivial fibre bundles an elementary check (see (5.2) and the dimensions of the defined by it objects) shows the equivalence of (5.4) and the condition

$$T_u^0(\pi^{-1}(U)) = \{0\}, \quad (5.4')$$

Proof. The proposition is a direct consequence of the definitions (5.2), lemma 5.1 and the definition of a connection in arbitrary fibre bundles [2-4]. ■

A priori it is clear that the validity of the condition (5.4) depends on the used transport along paths I , which are used to construct liftings of paths appearing above, as well as on the choice of the family of paths $\{\gamma_\lambda\}$.

In particular, if $U=B$, and (5.4) is fulfilled, then the $\dim(B)$ -dimensional distribution $u \mapsto T_u^h(E) = T_u^h(\pi^{-1}(B))$, $u \in E$ defines a connection in (E, π, B) and it is almost evident that the defined by it parallel transport (see [3,4]) along the paths γ_λ , $\lambda \in \Lambda$ coincides with the initial transport I along them.

6. CONCLUSION

In this work, we have defined and investigated some properties of the transports along paths in general fibre bundles. As examples of such transports we pointed out the linear transports along paths in vector bundles [6] and, in particular as their special case, the generated by derivations of tensor algebras over a manifold transports along paths [5]. Here, we were not concern with the ties of the theory developed with the ones of connections, parallel transports, bundle morphisms etc., items which will be studied elsewhere.

ACKNOWLEDGEMENTS

The author expresses his gratitude to Prof. Vl. Aleksandrov (Institute of Mathematics of Bulgarian Academy of Sciences) for constant interest in this work and stimulating discussions.

This research was partially supported by the Fund for Scientific Research of Bulgaria under contract Grant No. F 103.

REFERENCES

1. Greub W., S. Halperin, R. Vanstone, Connections, Curvature, and Cohomology, vol.1, vol.2, Academic Press, New York and London, 1972, 1973.
2. Dubrovin B.A., S.P. Novikov, A.T. Fomenko, Modern geometry, Nauka, Moscow, 1979 (In Russian).
3. Kobayashi S., K. Nomizu, Foundations of differential geometry, vol.1, Interscience publishers, New-York-London, 1963.
4. Warner F.W., Foundations of differentiable manifolds and Lie groups, Springer Verlag, New York-Berlin-Heidelberg-Tokyo, 1983.
5. Iliev B.Z., Parallel transports in tensor spaces generated by derivations of tensor algebras, Communication JINR, E5-93-1, Dubna, 1993.
6. Iliev B.Z., Linear transports along paths in vector bundles. I. General theory, Communication JINR, E5-93-239, Dubna, 1993.
7. Steenrod N., The topology of fibre bundles, 9-th ed., Princeton Univ. Press, Princeton, 1974 (1-st ed. 1951).
8. Viro O.Ya., D.B. Fuks, I. Introduction to homotopy theory, In: Reviews of science and technic, sec. Modern problems in mathematics. Fundamental directions, vol.24, Topology-2, VINITI,

Moscow, 1988, 6-121 (In Russian).

9. Husemoller D., Fibre bundles, McGraw-Hill Book Co., New York-St. Louis-San Francisco-Toronto-London-Sydney, 1966.
10. Iliev B.Z., Linear transports along paths in vector bundles. II. Some applications, Communication JINR, E5-93-260, Dubna, 1993.
11. Tamura I, Topology of foliations, Mir, Moscow, 1979 (In Russian; translation from Japanese).
12. Hu Sze-Tsen, Homotopy Theory, Academic Press, New York-London, 1959.
13. Iliev B. Z., Consistency between metrics and linear transports along curves, Communication JINR, E5-92-486, Dubna, 1992.

Received by Publishing Department
on August 11, 1993.