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LINEAR TRANSPORTS ALONG PATHS IN VECTOR BUNDLES.

I. General Theory

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1. INTRODUCTION

The corresponding to linear connections parallel transports are linear isomorphisms between the tensor spaces along the paths they act [1,2]. In the present work we appropriately generalize these transports, preserving the property that they are linear isomorphisms between vector spaces. As a ground for this we take the basic properties of the considered in [3] (parallel) transports along paths generated by derivations of tensor algebras. In some places we also follow the idea of this reference by transferring mutatis mutandis definitions and results. Part of the material of the present investigation was realized, in the case of the tangent bundle of a differentiable manifold, in [4]. That reference also contains possible applications to physics which, through some changes, have a place also with respect to this work.

In Sect. 2, we define and find the general, invariant and in components, form of the linear transports along paths in vector bundles. In Sect. 3, is proved the existence of bases in which these transports look like a usual parallel transport in an Euclidean space. In the pointed sense every linear transport along paths turns out to be locally (along any fixed path) Euclidean. (In particular, this is true for parallel transports generated by linear connections.) Sect. 4 contains investigation of two equivalent descriptions of the studied transports along paths. Firstly, the local description in terms of their local coefficients, which are analogous to the ones of a linear connection, and, secondly,

the one in terms of derivations along paths in vector bundles. At the end, in Sect. 5, some comments on the considered here problems are made.

Many problems, such as the comparison with the general connection's theory, integrability conditions (connected with the concepts curvature and torsion) and application to mathematical or physical questions, remain out of the field of this paper and will be considered elsewhere.

2. DEFINITION AND GENERAL FORM OF LINEAR TRANSPORTS ALONG PATHS IN VECTOR BUNDLES

By (E,π,B) we denote a general real vector bundle [5,6], p.44. Here the base B and the total bundle space E are topological spaces, the projection $\pi:E\longrightarrow B$ maps E onto B and the structure of E is such that the fibres $\pi^{-1}(x)\subset E$, $x\in B$ are isomorphic real vector spaces. By J and $\gamma:J\longrightarrow B$ we denote, respectively, an arbitrary real interval and a path in B.

Definition 2.1 of Ref. [3] describes the S-transports along paths in tensor bundles as maps having the mentioned in it properties. A simple overview on them reviles that part of them are specific for the considered in [3] tensor bundles, the other ones being independent of this fact. A deeper analysis of the above-cited definition shows that its straightforward generalization leads to the concept of a linear transport along paths in vector bundles which is fixed by

Definition 2.1. A linear transport (L-transport) along paths in the real vector bundle (E,π,B) is a map L which to any path $\gamma:J\longrightarrow B$ assigns a map L^{γ}, L-transport along γ, such

that $L^{\gamma}:(s,t)\longmapsto L^{\gamma}_{s\longrightarrow t}$, where for every $s,t\in J$ the map

$$L^{\gamma}_{s\longrightarrow t}:\pi^{-1}(\gamma(s))\longrightarrow\pi^{-1}(\gamma(t)), \tag{2.1}$$

called an L-transport along γ from s to t, has the following three properties:

$$L^{\gamma}_{s\longrightarrow t}(\lambda u+\mu v)=\lambda S^{\gamma}_{s\longrightarrow t}u+\mu S^{\gamma}_{s\longrightarrow t}v,\ \lambda,\mu\in\mathbb{R},\ u,v\in\pi^{-1}(\gamma(s)), \tag{2.2}$$

$$L^{\gamma}_{t\longrightarrow r}\circ L^{\gamma}_{s\longrightarrow t}=L^{\gamma}_{s\longrightarrow r},\ r,s,t\in J, \tag{2.3}$$

$$L^{\gamma}_{s\longrightarrow s}=id_{\pi^{-1}(\gamma(s))}, \tag{2.4}$$

where id_X means the identity map of the set X.

Remark. If in this definition we admit (E,π,B) to be a complex vector bundle [6] and $\lambda,\mu\in\mathbb{C}$ (instead of $\lambda,\mu\in\mathbb{R}$), we get the definition of a ($\mathbb{C}$-)linear transport along paths in complex vector bundles. Almost all further definitions and results are valid also in the complex case (and, generally, in a case of an arbitrary field $\mathbb{K}$) but, for the sake of shortness, we shall investigate only the real case.

Evident examples of L-transports along paths are the considered in [3] restrictions of S-transports on concrete tensor bundles.

Putting r=s in (2.3) and using (2.4), we get

$$\left(L^{\gamma}_{s\longrightarrow t}\right)^{-1}=L^{\gamma}_{t\longrightarrow s},\ s,t\in J. \tag{2.5}$$

So, the linear transports along a path are linear isomorphisms of the fibres over the points of that path.

The following two propositions establish the general form of the linear transports along paths.

Proposition 2.1. A map (2.1) is a linear transport along γ from s to t for every $s,t\in J$ if and only if there exist an

isomorphic with $\pi^{-1}(x)$, $x\in B$ vector space V and a family of linear isomorphisms $\{F^{\gamma}_s:\pi^{-1}(\gamma(s))\longrightarrow V,\ s\in J\}$ such that

$$L^{\gamma}_{s\longrightarrow t}=(F^{\gamma}_t)^{-1}\circ(F^{\gamma}_s),\quad s,t\in J. \tag{2.6}$$

Proof. If (2.1) is an L-transport along γ from s to t, then fixing some $s_0\in J$ and using (2.3) and (2.5), we get $L^{\gamma}_{s\longrightarrow t}=L^{\gamma}_{s_0\longrightarrow t}\circ L^{\gamma}_{s\longrightarrow s_0}=\left(L^{\gamma}_{t\longrightarrow s_0}\right)^{-1}\circ L^{\gamma}_{s\longrightarrow s_0}$, i.e. (2.6) holds for $V=\pi^{-1}(\gamma(s_0))$ and $F^{\gamma}_s=L^{\gamma}_{s\longrightarrow s_0}$. On the contrary, if (2.6) is valid for some linear isomorphisms F^{γ}_s, then a straightforward calculation shows that it converts (2.3) and (2.4) into identities and (2.2) is true due to the linearity of F^{γ}_s, i.e. $L^{\gamma}_{s\longrightarrow t}$ is an L-transport along γ from s to t for every $s,t\in J$.■

Proposition 2.2. Let in the vector bundle (E,π,B) be defined an L-transport along paths with a representation (2.6) for some vector space V and linear isomorphisms $F^{\gamma}_s:\pi^{-1}(\gamma(s))\longrightarrow V$, $s\in J$. Then for a vector space $\underline{V}$ there exist linear isomorphisms $\underline{F}^{\gamma}_s:\pi^{-1}(\gamma(s))\longrightarrow\underline{V}$, $s\in J$ for which

$$L^{\gamma}_{s\longrightarrow t}=(\underline{F}^{\gamma}_t)^{-1}\circ(\underline{F}^{\gamma}_s),\quad s,t\in J. \tag{2.7}$$

iff there exists a linear isomorphism $D^{\gamma}:V\longrightarrow\underline{V}$ such that

$$\underline{F}^{\gamma}_s=D^{\gamma}\circ F^{\gamma}_s,\quad s\in J. \tag{2.8}$$

Proof. If (2.8) holds, then substituting $F^{\gamma}_s=(D^{\gamma})^{-1}\underline{F}^{\gamma}_s$ into (2.6), we get (2.7) and vice versa, if (2.7) is valid, then from its comparison with (2.6) follows $D^{\gamma}:=(\underline{F}^{\gamma}_t)(F^{\gamma}_t)^{-1}=$ $=(\underline{F}^{\gamma}_s)(F^{\gamma}_s)^{-1}$ to be the needed (independent of s and t) isomorphism.■

So, the linear transport along γ form s to t decomposes into a composition of linear maps depending separately on s

and t (cf. proposition 2.1), the arbitrariness of these maps being described by eq. (2.8) (see proposition 2.2).

For some purposes is useful the linear transports along paths to be described by their matrix elements in corresponding bases, which we are going to do now.

Let $\{e_i\}$, where here and henceforth the Latin indices run from 1 to $\dim(\pi^{-1}(x))$, $x\in B$ and the usual summation rule from 1 to $\dim(\pi^{-1}(x))$ over repeated on different levels indices is assumed, be a field of bases along $\gamma:J\longrightarrow B$, i.e. for every $s\in J$ the set of vectors $\{e_i(s)\}$ to be a basis in the vector space $\pi^{-1}(\gamma(s))$.

Due to (2.1), we have $L^{\gamma}_{s\longrightarrow t}e_i(s)\in\pi^{-1}(\gamma(t))$, hence there exists a unique matrix $H(t,s;\gamma):=\|H^i_{\cdot j}(t,s;\gamma)\|$ such that

$$L^{\gamma}_{s\longrightarrow t}e_i(s)=H^j_{\cdot i}(t,s;\gamma)e_j(t),\quad s,t\in J. \tag{2.9}$$

From here immediately follows that $H^i_{\cdot j}(t,s;\gamma)$ are elements of a (two-point) tensor from the tensor space $\pi^{-1}(\gamma(t))\otimes$ $\otimes(\pi^{-1}(\gamma(s)))^*$, where $\otimes$ is the tensor product sign and the asterisk means the dual of the corresponding vector space.

Definition 2.2. The matrix function $H:(t,s;\gamma)\longmapsto$ $\longmapsto H(t,s;\gamma)$, $s,t\in J$ will be called a matrix of the considered L-transport along paths.

The matrix of the L-transport uniquely defines its action. In fact, if $u=u^ie_i(s)\in\pi^{-1}(\gamma(s))$, then from (2.2) and (2.9), we get

$$L^{\gamma}_{s\longrightarrow t}u=H^i_{\cdot j}(t,s;\gamma)u^je_i(t). \tag{2.10}$$

In terms of the matrix function H, due to (2.9), the basic properties (2.3) and (2.4) take, respectively, the form

$$H(r,t;\gamma)H(t,s;\gamma)=H(r,s;\gamma),\quad r,s,t\in J, \tag{2.11}$$

$$H(s,s;\gamma)=\mathbb{I},\ s\in J, \tag{2.12}$$

where $\mathbb{I}$ is the unit matrix and in the matrix multiplication as a first matrix index the superscript is understood.

On the contrary, due to (2.10), from (2.11) and (2.12) follow, respectively, (2.3) and (2.4). So, we have proved

Proposition 2.3. A linear map (2.1) is a linear transport along γ from s to t iff its matrix H, defined by (2.9), satisfies (2.11) and (2.12).

Let us introduce the matrices $F(s;\gamma):=\|F^i_{\cdot j}(s;\gamma)\|$ and $\underline{F}(s;\gamma):=\|\underline{F}^i_{\cdot j}(s;\gamma)\|$, where $F^i_{\cdot j}(s;\gamma)$ and $\underline{F}^i_{\cdot j}(s;\gamma)$ are the matrix elements, respectively, of the maps F^γ_s and $\underline{F}^\gamma_s$, i.e. $F^\gamma_s e_j(s)=:F^i_{\cdot j}(s;\gamma)1_i$ and $\underline{F}^\gamma_s e_j(s)=:\underline{F}^i_{\cdot j}(s;\gamma)\underline{1}_i$ in which $\{1_i\}$ and $\{\underline{1}_i\}$ are, respectively, bases in V and $\underline{V}$. Then, using (2.2) and (2.9), we see that in terms of matrix elements propositions 2.1 and 2.2 are, respectively, equivalent to

Proposition 2.4. The linear map (2.1) with defined by (2.9) matrix elements in some bases is an L-transport along γ from s to t iff there exist nondegenerate matrices $F(s;\gamma)$, $s\in J$ such that

$$H(t,s;\gamma)=(F(t;\gamma))^{-1}F(s;\gamma),\ s,t\in J. \tag{2.13}$$

Proposition 2.5. Let the matrix H of some L-transport along paths has the representation (2.13). Then for certain nondegenerate matrices $\underline{F}(s;\gamma)$, $s\in J$ is valid the equality

$$H(t,s;\gamma)=(\underline{F}(t;\gamma))^{-1}\underline{F}(s;\gamma),\ s,t\in J \tag{2.13'}$$

if and only if there exists a nondegenerate matrix $D(\gamma)$, depending only on γ, such that

$$\underline{F}(s;\gamma)=D(\gamma)F(s;\gamma),\ s\in J,\ \det D(\gamma)\neq 0,\infty. \tag{2.14}$$

3. SPECIAL BASES FOR LINEAR TRANSPORTS ALONG PATHS

Special bases for an L-transport along paths we call bases in which it (generally locally; see below) looks like a parallel transport in an Euclidean space. The existence of such bases is expressed by

Proposition 3.1. For every L-transport along a path γ there exist a basis in which the transport's matrix is unit. If in a given basis along a path γ the matrix of some L-transport is constant or depends only on γ, then it is unit. All bases along γ in which the matrix of an L-transport is unit are obtained from the above one, and, consequently, from one another, by a nondegenerate linear transformations with constant or depending on γ coefficients.

As from the view point of the applications this proposition is an important one, we shall give two its proofs. The first proof admits a straightforward application for different purposes, and the second one distinguishes by its shortness and clearness.

Proof 1. Let $\{e_i(s)\}$ be an arbitrary basis in $\pi^{-1}(\gamma(s))$ and $\{e^i(s)\}$ be its dual basis, $s\in J$. By proposition 2.4 there exist nondegenerate matrices $F(s;\gamma)$, $s\in J$, such that $H^i_{\cdot j}(t,s;\gamma)=((F(s;\gamma))^{-1})^i_{\cdot k}(F(s;\gamma))^k_{\cdot j}$; If the bases $\{e^{i'}(s)\}$, $s\in J$ are defined through

$$e^{i'}(s):=\delta^{i'}_i(F(s;\gamma))^i_{\cdot i}e^i(s)), \tag{3.1}$$

where δ^i_i are the Kronecker deltas ($\delta^i_i=1$ for $i=j$ and $\delta^i_i=0$ for $i\neq j$), then

$$e_{j'}=\delta^m_{j'}((F(s;\gamma))^{-1})^j_{\cdot m}e_j \tag{3.2}$$

and due to the fact that $H^i_{\cdot j}(t,s;\gamma)$ are components of a tensor from $\pi^{-1}(\gamma(t))\otimes(\pi^{-1}(\gamma(s)))^*$, we get $H^{i'}_{\cdot j'}(t,s;\gamma)=$ $=[\delta^{i'}_i(F(t;\gamma))^i_{\cdot i}][\delta^m_{j'}((F(s;\gamma))^{-1})^j_{\cdot m}]H^i_{\cdot j}(t,s;\gamma)=\delta^i_j$, i.e. in the basis $\{e_{i'}\}$ the matrix of the considered L-transport along γ is unit.

Let now in $\{e_i\}$ to be fulfilled $H(t,s;\gamma)=A(\gamma)$, $A(\gamma)$ being a nondegenerate matrix depending possibly on the path γ. According to (2.13) $A(\gamma)=(F(t;\gamma))^{-1}F(s;\gamma)$ and consequently $F(s;\gamma)=A(\gamma)F(t;\gamma)=:B(\gamma)$, where, due to the arbitrariness of s and t, $B(\gamma)$ depends only on γ. Hence $H(t,s;\gamma)=(B(\gamma))^{-1}B(\gamma)=\mathbb{I}$ and taking into account the above definition of $\{e_{i'}(s)\}$, we get $e_j=(B(\gamma))^i_{\cdot j}\delta^{i'}_i e_{i'}$, which shows that all bases in which the matrix of the L-transport is constant (unit) are obtained from $\{e_{i'}(s)\}$ by linear transformations with constant or depending on the path coefficients, so they themselves are connected with such transformations. To end this proof it remains only to be noted that due to (2.14) the dependence of $B(\gamma)$ on γ is insignificant and it may be considered only the case $B(\gamma)$=const, which is equivalent to the not changing $H(s,t;\gamma)$ redefining $F(s;\gamma)$.■

Proof 2. Let us fix $s_0\in J$ and a basis $\{f'_i\}$ in $\pi^{-1}(\gamma(s_0))$. Along $\gamma:J\longrightarrow B$ we define a basis $\{e'_i\}$ such that in $\pi^{-1}(\gamma(s))$ it has the form

$$e'_i(s):=I^\gamma_{s_0\longrightarrow s}f'_i \tag{3.3}$$

Then $\quad I^\gamma_{s\longrightarrow t}e'_i(s)=I^\gamma_{s\longrightarrow t}\circ I^\gamma_{s_0\longrightarrow s}f'_i=I^\gamma_{s_0\longrightarrow t}f'_i=e'_i(t)$,

hence, due to (2.9), the matrix of L in $\{e'_i\}$ is $H'(t,s;\gamma)=\mathbb{I}$.

Let $\{e^\gamma_i\}$ be arbitrary basis along γ in which the matrix

of L is $H(t,s;\gamma)=B(\gamma)$ for some, may be depending on γ, nondegenerate matrix $B(\gamma)$. Then there exists a nondegenerate matrix $\|A^j_i(s;\gamma)\|$ such that $e^\gamma_i(s)=A^j_i(s;\gamma)e'_j(s)$. The substitution of the last equality into $I^\gamma_{s\longrightarrow t}e^\gamma_i(s)=(B(\gamma))^j_i e^\gamma_j(t)$ (see (2.9)), due to (2.10), gives $A^j_i(s;\gamma)=(B(\gamma))^k_i A^j_k(t;\gamma)$. From here for t=s, we get $B(\gamma)=\mathbb{I}$, so the same equality reduces to $A^j_i(s;\gamma)=A^j_i(t;\gamma)$ and consequently $A^j_i(s;\gamma)$ are constants or may depend only on γ.■

In connection with the fact that generally the global version of proposition 3.1, which will be considered elsewhere, is not true it is convenient to be introduced the concept of an Euclidean case concerning L-transports along paths, i.e. when in a vector bundle (E,π,B) is given an Euclidean (L-)transport along paths defined by

Definition 3.1. An L-transport in a vector bundle (E,π,B) is Euclidean over the set $U\subset B$, or it is Euclidean if U=B, if in $\pi^{-1}(U)$ there exists a field of bases $\{e_i\}$, i.e. $\{e_i(x)\}$ is a basis in $\pi^{-1}(x)$ for $x\in U$, in which the matrix of the transport is unit along any path in U, i.e. if $\gamma:J\longrightarrow B$ and there is an $\mathbb{R}$-interval $J'\subset J$ such that $\gamma(J')\subset U$, then $H(t,s;\gamma)=\mathbb{I}$, for $s,t\in J'$.

Definition 3.2. Let $\{e_i\}$ be a basis in a vector bundle (E,π,B), i.e. $\{e_i(x)\}$ is a basis in $\pi^{-1}(x)$, $x\in B$. A generated by (associated to) $\{e_i\}$ Euclidean (L-)transport in (E,π,B) is an L-transport in this fibre bundle the matrix of which in the basis $\{e_i\}$ along any path is unit, i.e. $H^i_{\cdot j}(s,t;\gamma)=\delta^i_j$ for every path $\gamma:J\longrightarrow B$ and every $s,t\in J$.

The name "Euclidean transport" is connected with the fact that if above we put $B=\mathbb{R}^n$ and identify $T_x(\mathbb{R}^n)$ and $\mathbb{R}^n$, then in any orthogonal basis, i.e. in Cartesian coordinates,

the Euclidean transport coincides with the standard parallel transport in $\mathbb{R}^n$ under which the vector's components are left unchanged.

Proposition 3.2. Every basis in a vector bundle generates a unique Euclidean transport in it. The opposite statement being valid only locally, i.e along a given path.

Proof. The first path of the proposition is a corollary of proposition 2.4 and definition 2.2.

By proposition 3.1 for every L-transport along a fixed path there exists a basis along it in the bundle in which the transport's matrix is (locally) unit and, hence, the generated from this basis Euclidean transport along the pointed path coincides with the initial L-transport.■

The importance of the last proposition is in the fact that in the above sense <u>any L-transport is locally Euclidean</u>.

Proposition 3.3. Two (or more) bases generate one and the same Euclidean transport in a given vector bundle if and only if they can be obtained from each other through linear transformations with constant coefficients.

Proof. This result is a direct consequence from proposition 2.4 and definition 3.1.■

The above results show that any L-transport is Euclidean over any point of the base (see (2.12)), as well as over an arbitrary path in it (see proposition 3.1). For other subsets $U\subset B$ this is, generally, not true. In particular, for $U=B$ analogous result holds only for "flat" L-transports, whose curvature operator vanishes, a result which will be establish in another paper.

4. THE EQUIVALENCE OF LINEAR TRANSPORTS ALONG PATHS AND DERIVATIONS ALONG PATHS

Let a linear transport along paths L in the vector bundle $\xi=(E,\pi,B)$ be given. Let it to be smooth of class C^1 in a sense that such is its matrix H with respect to its first, and consequently to its second (see (2.13)), argument. Let $\xi|_{\gamma(J)}:=(\pi^{-1}(\gamma(J)),\pi|_{\gamma(J)},\gamma(J))$ be the restriction of ξ on the set $\gamma(J)$ defined by the path $\gamma:J\longrightarrow B$. Let $Sec(\xi)$ (resp. $Sec^k(\xi)$) be the set of all (resp. of class C^k) sections of ξ [6,1].

Definition 4.1. The derivation along paths generated by L is a map $\mathcal{D}$ such that $\mathcal{D}:\gamma\longmapsto\mathcal{D}^\gamma$, where the assigned to any path γ map

$$\mathcal{D}^\gamma:Sec^1(\xi|_{\gamma(J)})\longrightarrow Sec(\xi|_{\gamma(J)}), \tag{4.1}$$

called derivation along γ generated by L, for every $\sigma\in Sec^1(\xi|_{\gamma(J)})$ is defined by $\mathcal{D}^\gamma:\sigma\longmapsto\mathcal{D}^\gamma\sigma$ in which for $s,s+\varepsilon\in J$

$$(\mathcal{D}^\gamma\sigma)(\gamma(s)):=\mathcal{D}^\gamma_s\sigma:=\lim_{\varepsilon\longrightarrow 0}[\tfrac{1}{\varepsilon}(L^\gamma_{s+\varepsilon\longrightarrow s}\sigma(\gamma(s+\varepsilon))-\sigma(\gamma(s)))]. \tag{4.2}$$

The derivative of σ along γ with respect to L is $\mathcal{D}^\gamma\sigma$ whose value at $\gamma(s)$, i.e. $(\mathcal{D}^\gamma\sigma)(\gamma(s))$, is determined by the map

$$\mathcal{D}^\gamma_s:Sec^1(\xi|_{\gamma(J)})\longrightarrow\pi^{-1}(\gamma(s)). \tag{4.3}$$

The limit in (4.2), due to the above condition of smoothness, always exists.

Proposition 4.1. The derivation $\mathcal{D}^\gamma$ is an $\mathbb{R}$-linear map, i.e. for $\lambda_1,\lambda_2\in\mathbb{R}$ and $\sigma_1,\sigma_2\in Sec^1(\xi|_{\gamma(J)})$ is fulfilled

$$\mathcal{D}^{\gamma}(\lambda_1\sigma_1+\lambda_2\sigma_2)=\lambda_1\mathcal{D}^{\gamma}\sigma_1+\lambda_2\mathcal{D}^{\gamma}\sigma_2, \tag{4.4}$$

and its value at every $t\in J$ satisfies the identity

$$\mathcal{D}^{\gamma}_t\circ L^{\gamma}_{s\longrightarrow t}\equiv 0,\ s,t\in J. \tag{4.5}$$

Proof. (4.4) and (4.5) follow from (4.2) and, respectively, (2.2) and (2.3).■

Let along $\gamma:J\longrightarrow B$ be given a field of bases $\{e_i\}$, i.e. $\{e_i(s)\}$ to be a basis in $\pi^{-1}(\gamma(s))$, in which the L-transport along paths L to be defined by its matrix $H:(t,s;\gamma)\longmapsto$ $\longmapsto H(t,s;\gamma):=\|H^i_{\cdot j}(t,s;\gamma)\|$. Then along γ every $\sigma\in Sec(\xi|_{\gamma(J)})$ has a unique representation in the form $\sigma=\sigma^i e_i$, where $\sigma^i:\gamma(J)\longrightarrow\mathbb{R}$ are the components of σ in $\{e_i\}$. Hence $\sigma(\gamma(s))=$ $=\sigma^i(\gamma(s))e_i(s)$ and from (4.2) and (2.10), we get

$$(\mathcal{D}^{\gamma}\sigma)(\gamma(s)):=\mathcal{D}^{\gamma}_s\sigma=(\mathcal{D}^{\gamma}_s\sigma)^i e_i(s)=$$

$$=\{\frac{\partial}{\partial\varepsilon}[\sum_j H^i_{\cdot j}(s,s+\varepsilon;\gamma)\sigma^j(\gamma(s+\varepsilon))]\}|_{\varepsilon=0}e_i(s). \tag{4.6}$$

From here immediately follows

Proposition 4.2. The explicit action of $\mathcal{D}^{\gamma}$ on $\sigma\in$ $\in Sec^1\xi|_{\gamma(J)})$ is

$$(\mathcal{D}^{\gamma}\sigma)(\gamma(s))=\left[\frac{d\sigma^i(\gamma(s))}{ds}+\Gamma^i_{\cdot j}(s;\gamma)\sigma^j(\gamma(s))\right]e_i(s), \tag{4.7}$$

where

$$\Gamma^i_{\cdot j}(s;\gamma):=\frac{\partial H^i_{\cdot j}(s,t;\gamma)}{\partial t}\Big|_{t=s}=-\frac{\partial H^i_{\cdot j}(t,s;\gamma)}{\partial t}\Big|_{t=s}. \tag{4.8}$$

(The last equality follows, e.g., from (2.13).)

It is convenient in any basis to introduce the matrix function $\Gamma_{\gamma}:s\longmapsto\Gamma_{\gamma}(s):=\|\Gamma^i_{\cdot j}(s;\gamma)\|$, $s\in J$. If the matrix H of the linear transport along paths has a representation (2.13) (see proposition 2.4), then, due to (4.8), we obtain

$$\Gamma_{\gamma}(s)=\frac{H(s,t;\gamma)}{\partial t}\Big|_{t=s}=F^{-1}(s;\gamma)\frac{dF(s;\gamma)}{ds}. \tag{4.9}$$

As we shell prove (see below proposition 4.5) the quantities (4.8) give an adequate description of the transport L. They also have a sense of a "depending on the path γ coefficients of a linear connection" (cf. [6,1]) which is confirmed by

Proposition 4.3. Let the basis $\{e_i(s)\}$ in $\pi^{-1}(\gamma(s))$ be changed to $\{e_{i'}(s)=A^i_{i'}(s)e_i(s)\}$, $A(s):=\|A^i_{i'}(s)\|$ being a nondegenerate matrix and $A^{-1}(s):=\|A^{i'}_i(s)\|$. The transformation $\{e_i\}\longrightarrow\{e_{i'}=A^i_{i'}e_i\}$ leads to the transformation of H and Γ_{γ}, respectively, into

$$H'(s,t;\gamma)=A^{-1}(s)H(s,t;\gamma)A(t), \tag{4.10}$$

$$\Gamma'(s;\gamma)=A^{-1}(s)\Gamma(s;\gamma)A(s)+A(s)^{-1}\frac{dA(s)}{ds}, \tag{4.11}$$

which in component form, respectively, read

$$H^{i'}_{\cdot\cdot j'}(s,t;\gamma)=A^{i'}_i(s)A^j_{j'}(t)H^i_{\cdot j}(s,t;\gamma), \tag{4.10'}$$

$$\Gamma^{i'}_{\cdot\cdot j'}(s;\gamma)=A^{i'}_i(s)A^j_{j'}(s)\Gamma^i_{\cdot j}(s;\gamma)+A^{i'}_i(s)\frac{dA^i_{j'}(s)}{ds}. \tag{4.11'}$$

Proof. Eq. (4.10) expresses the fact that $H^i_{\cdot j}(s,t;\gamma)$ are components of a tensor from $\pi^{-1}(\gamma(s))\otimes(\pi^{-1}(\gamma(t)))^*$ and eq. (4.11) is a corollary of (4.10), (2.12) and (4.9). The equivalence of (4.10) and (4.10′) and (4.11) and (4.11′) follows from the definitions of the corresponding matrices.■

So, any (smooth) L-transport along paths generates along every path γ functions (4.8) transforming according to (4.11′). Here, naturally, arises the opposite problem. Let along every path γ in any basis $\{e_i\}$ along it be defined functions $\Gamma^i_{\cdot j}(s;\gamma)$ which, when the basis is changed, transform

through (4.11'). Does there exist an L-transport along paths generating these functions by means of eq. (4.8)? The answer of this question is given by

Proposition 4.4. Let in any basis $\{e_i\}$ along a path $\gamma: J \longrightarrow B$ be given (continues with respect to s) functions $\Gamma^i_{.j}(s;\gamma)$ which, when $\{e_i\}$ is changed, have the transformation law (4.11') and $\Gamma_\gamma(s):=\|\Gamma^i_{.j}(s;\gamma)\|$. Then there exists a unique L-transport along paths generating along γ the matrix Γ_γ by (4.9) and the matrix of which, $H:(s.t;\gamma)\longmapsto H(s.t;\gamma)=$ $=F^{-1}(s;\gamma)F(t;\gamma)$, is

$$H(s.t;\gamma)=Y(s,s_0;-\Gamma_\gamma)[Y(t,s_0;-\Gamma_\gamma)]^{-1},\quad s,t\in J, \tag{4.12}$$

where $s_0\in J$ is fixed and for any continuous matrix function $Z:s\longmapsto Z(s)$ the matrix $Y:=\|Y_{ij}\|:=Y(s.s_0;Z)$ is the unique solution of the initial-value problem

$$\frac{dY}{ds}=Z(s)Y,\quad Y=Y(s,s_0;Z),\quad s\in J, \tag{4.13a}$$

$$Y(s_0,s_0;Z)=\mathbb{I}. \tag{4.13b}$$

Proof. Let us note at the beginning that the existence and uniqueness of the solution of (4.13) is proved in [7].

At first we shall prove that (4.12) is the unique solution of (4.9) when Γ_γ is given. In fact, using $H(s.t;\gamma)=$ $=F^{-1}(s;\gamma)F(t;\gamma)$ (see proposition 2.4) and $dF^{-1}/ds=$ $=-F^{-1}(dF/ds)F^{-1}$ [8], we see eq. (4.9) with respect to H to be equivalent to the equation $dF^{-1}(s;\gamma)/ds=-\Gamma_\gamma(s;\gamma)F^{-1}(s;\gamma)$. Arbitrary fixing $s_0\in J$ and comparing this equation (with respect to F^{-1}) with (4.13), we see that its solution is $F^{-1}(s;\gamma)=Y(s,s_0;-\Gamma_\gamma)$. Substituting this expression for F^{-1} into (2.13), we get (4.12), which is independent of the concrete choice of $F(s_0;\gamma)$ and s_0 (see proposition 2.5 and the given in [7] properties of Y).

So, in any fixed basis only the matrix (4.12) generates the given Γ_γ by (4.9). Hence, defining an L-transport along paths which is such that in some basis has a matrix (4.12), we see, in conformity with (4.10) and (4.11), that in every basis along γ it generates the functions $\Gamma^i_{.j}(s;\gamma)$ through (4.8). By construction this L-transport along paths is unique.■

From propositions 4.3 and 4.4 directly follows

Proposition 4.5. Let in any basis along every path $\gamma: J\longrightarrow B$ be given a set of functions $\{\Gamma^i_{.j}(s;\gamma)\}$. Then, when the basis is changed, they transform in accordance with (4.11') iff there exists a (unique) linear transport along paths generating them through its matrix by (4.8).

Consequently, the definition of an L-transport along a path γ is equivalent to the definition in any basis along γ of functions $\Gamma^i_{.j}(s;\gamma)$ having the transformation law (4.11'). This is a reason to give

Definition 4.2. The functions $\Gamma^i_{.j}:(s;\gamma)\longmapsto\Gamma^i_{.j}(s;\gamma)$ assigned to any linear transport along paths through (4.8) or defining such a transport by means of (4.12) will be called coefficients of that L-transport along paths.

In proposition 4.2 we saw that the coefficients $\Gamma^i_{.j}$ of an L-transport along paths uniquely define the action of the operator (4.1). Besides, if $e_i\in Sec(\xi|_{\gamma(J)})$ form a basis along $\gamma: J\longrightarrow B$, i.e. if $\{e_i(s)\}$ is a basis in $\pi^{-1}(\gamma(s))$, then (4.7) results in

$$\mathcal{D}^\gamma e_j=(\Gamma_\gamma)^i_{.j}e_i, \tag{4.14}$$

which is equivalent to

$$\mathcal{D}^\gamma_s e_j=\Gamma^i_{.j}(s;\gamma)e_i(s) \tag{4.14'}$$

and can be used as an equivalent to (4.8) definition of the coefficients $\Gamma^{i}_{.j}(s;\gamma)$.

As the coefficients $\Gamma^{i}_{.j}(s;\gamma)$ give an adequate description of the L-transports along paths, the eq. (4.14) suggests that such a description can be done also in terms of operators (4.1) with an "appropriate properties". Before formulating the corresponding result we need to give one definition.

If $\sigma\in Sec^{1}(\xi|_{\gamma(J)})$ and $f:J\longrightarrow\mathbb{R}$ is a C^{1} function, then from (4.7), we find

$$\mathcal{D}^{\gamma}_{s}(f\cdot\sigma)=\frac{df(s)}{ds}\cdot\sigma(\gamma(s))+f(s)\cdot(\mathcal{D}^{\gamma}_{s}\sigma). \qquad (4.15)$$

If B is a differentiable manifold, γ is a C^{1} path and $\dot{\gamma}$ is the tangent to γ vector field, then (4.15) is equivalent to

$$\mathcal{D}^{\gamma}(f\cdot\sigma)=(\dot{\gamma}(f))\cdot\sigma+f\cdot(\mathcal{D}^{\gamma}\sigma). \qquad (4.16)$$

Definition 4.3. A derivation along a path γ (in the vector bundle $\xi=(E,\pi,B)$) is a linear map (6.1) satisfying (4.15) in which $\mathcal{D}^{\gamma}_{s}$ is defined by $\mathcal{D}^{\gamma}_{s}\sigma:=(\mathcal{D}^{\gamma}\sigma)(\gamma(s))$ for $\sigma\in Sec^{1}(\xi|_{\gamma(J)})$. A derivation along paths is a map assigning to any path a derivation along it.

Evident example of derivations along a path are the derivations of the tensor algebra (over a manifold) restricted over that path [3,1], as well as the maps (4.1) generated by L-transports along paths by means of (4.2).

Proposition 4.6. The map $\mathcal{D}^{\gamma}:Sec^{1}(\xi|_{\gamma(J)})\longrightarrow Sec(\xi|_{\gamma(J)})$ is a derivation (in a sense of definition 4.3) iff there exists an L-transport L^{γ} along γ generating it through (6.2).

Proof. If $\mathcal{D}^{\gamma}$ is generated through (6.2) by some L-transport along γ, then, as we proved, it satisfies (4.4) and (4.5), so it is a derivation along γ. Vice versa, let $\mathcal{D}^{\gamma}$ be a

derivation along γ. Defining the functions $\Gamma^{i}_{.j}(s;\gamma)$ through (4.14′) in any basis along γ, we find, due to (4.4) and (4.15), the eq. (4.7) to be valid in any basis $\{e_i\}$ along γ. Thus we see that in the basis $\{e_{i'}=A^{i}_{i'}e_i\}$ is fulfilled $\Gamma^{i'}_{..j'}(s;\gamma)e_{i'}(s)=\mathcal{D}^{\gamma}_{s}(e_{j'})=\mathcal{D}^{\gamma}_{s}(A^{j}_{j'}e_j)$ and using (4.4) and (4.15), we get that $\Gamma^{i'}_{..j'}$ and $\Gamma^{i}_{.j}$ are connected by eq. (4.11′). Hence, $\Gamma^{i}_{.j}$ are coefficients of some L-transport along paths L, which by proposition 4.5 exists, is unique, and generates them through (4.8). Besides, by proposition 4.2, the action of the assigned to it operator (4.1) is given also by (4.7), i.e. that operator, given explicitly by (4.2), coincides with $\mathcal{D}^{\gamma}$.■

In the above proof of proposition 4.6 we saw that for any derivation $\mathcal{D}^{\gamma}$ along γ its action is uniquely defined by the functions $\Gamma^{i}_{.j}$, determined uniquely by (4.14′). That is why we call them *coefficients* of $\mathcal{D}^{\gamma}$. From proposition 4.4 and the same proof immediately follows

Proposition 4.7. Every L-transport L^{γ} along γ defines through (4.2) a unique derivation along γ whose explicit action is given by (4.7) and whose coefficients coincide with those of L^{γ} and, vice versa, any derivation $\mathcal{D}^{\gamma}$ along γ defines a unique L-transports along γ who generates $\mathcal{D}^{\gamma}$ by (4.2), whose coefficients coincide with those of $\mathcal{D}^{\gamma}$ and whose matrix is given by (4.12).

Practically, the last proposition establishes the equivalence between the set of all linear transports along paths and the one of all derivations along paths.

5. CONCLUDING REMARKS

We defined the linear transports along paths in vector bundles as maps satisfying part of the basic axioms of the S-transports (parallel transports generated by derivations of tensor algebras) [3]. The ties of the L-transports along paths with the connection and parallel transports theories will be examined from a more general view-point elsewhere. Here we want only to present these connections in the special case of linear connection and the corresponding to them parallel transports.

Let the base of the vector bundle (E,π,B), i.e. B, be a differentiable manifold and $\gamma: J \longrightarrow B$ be a C^1 path in it with a tangent vector field $\dot\gamma$. Let us consider those L-transports along paths whose coefficients in some basis along γ have the form

$$\Gamma^i_{.j}(s;\gamma)=\sum_{\alpha=1}^{\dim(B)} \Gamma^i_{.j\alpha}(\gamma(s))\dot\gamma^\alpha(s), \tag{5.1}$$

where $\Gamma^i_{.j\alpha}:\gamma(J)\longrightarrow\mathbb{R}$, $i,j=1,\dots,\dim(\pi^{-1}(x))$, $x\in B$, $\alpha=1,\dots,\dim(B)$ are some real functions and $\dot\gamma^\alpha(s)$ are the local components of $\dot\gamma(s)$ in some basis in the tangent to B space at $\gamma(s)$.

Proposition 5.1. If the representation (5.1) holds in one basis $\{e_i\}$ along γ, then it is true in a basis $\{e_{i'}=A^i_{i'}e_i\}$ along γ iff under the change of the bases $\{e_i\}\longrightarrow\{e_{i'}\}$ and of the coordinates $\{x^\alpha\}\longrightarrow\{x^{\alpha'}\}$ in a neighborhood of $x=\gamma(s)\in B$ the functions $\Gamma^i_{.j\alpha}(x)$ transform into

$$\Gamma^{i'}_{.j'\alpha'}(x)=\sum_{i,j=1}^{\dim(\pi^{-1}(x))}\left(\sum_{\alpha=1}^{\dim(B)} A^{i'}_i(s)A^j_{j'}(s)\frac{\partial x^\alpha}{\partial x^{\alpha'}}\Big|_x\Gamma^i_{.j\alpha}(x)\right)+$$

$$+\sum_{i=1}^{\dim(\pi^{-1}(x))} A^{i'}_i(s)\frac{\partial A^i_{j'}(s)}{\partial x^{\alpha'}}\Big|_x,\quad x=\gamma(s)\in B \tag{5.2}$$

the right-hand-side of which depends on the map γ only through the point $x=\gamma(s)$.

Proof. The proposition is a consequence of the substitution of (5.1) (in $\{e_i\}$ and in $\{e_{i'}\}$) into (4.11′).■

The transformation law (5.2) is a straightforward generalization of the one of the coefficients of a linear connection [1,2].

In particular, if (E,π,B) is the tangent bundle to the manifold B, i.e. $E=T(B)$ and $\pi^{-1}(x)=T_x(B)$, $x\in B$, then $\dim(B)=\dim(\pi^{-1}(x))$, $x\in B$ and putting $A^{i'}_i(s)=\partial x^{i'}/\partial x^i|_{x=\gamma(s)}$ in (5.2), we see that $\Gamma^i_{.j\alpha}(x)$, $i,j,\alpha=1,\dots,\dim(B)$ are coefficients of a linear connection (covariant differentiation) ∇ [1]. In this case, due to (4.7), the assigned to the transport operator $\mathcal{D}^\gamma$ is simply $\nabla_{\dot\gamma}$, the covariant differentiation along γ, and hence the transport itself coincides with the corresponding to ∇ parallel transport [1] (cf. [4], Sect.5).

In Sect. 3 we proved that for any liner transport along a path γ in a vector bundle there is along γ a class of special bases in which the transport's matrix is unit, i.e $H(s,t;\gamma)=\mathbb{I}$. In every such basis the transport's coefficients vanish as in them (see (4.9))

$$\Gamma_\gamma(s)=\partial\mathbb{I}/\partial s=0 \tag{5.3}$$

and vice versa, any basis in which (5.3) holds belongs to this class.

So, in these bases (4.7) is equivalent to

$$(\mathcal{D}^\gamma\sigma)(\gamma(s))=\frac{d\sigma^i(\gamma(s))}{ds}e_i(s), \tag{5.4}$$

i.e. $\mathcal{D}^{\gamma}$ behaves like a derivative along γ of vectors in an Euclidean space (in Cartesian coordinates).

Thus we proved

Proposition 5.2. In a given basis along a fixed path the following three statements are equivalent:

a) The matrix of the L-transport is unit.

b) The coefficients of the L-transport vanish.

c) The action of the assigned to the L-transport derivation reduces to a simple derivation of the components of the sections with respect to the path's parameter, i.e. (5.4) holds.

If in a vector bundle over a manifold the coefficients of an L-transport along paths have the form (5.1) in some of the above special along γ bases, then (5.3) leads to

$$\sum_{\alpha=1}^{\dim(B)} \Gamma^{i}_{.j\alpha}(\gamma(s))\dot{\gamma}^{\alpha}(s)=0, \quad i,j=1,\dots,\dim(\pi^{-1}(\gamma(s))). \qquad (5.5)$$

But, nevertheless of the arbitrariness of γ, from here does not follow $\Gamma^{i}_{.j\alpha}(\gamma(s))=0$ as the basis itself, generally, depends on the path γ.

In this connection let us note that bases in which (5.3), or in some more special cases (5.5), is valid, are a far going generalization of the coordinates (or bases) in which the components of a symmetric affine connection vanish [2].

Let us also mention that to operations with vector bundles endowed with transports along paths there correspond analogous operations with these transports. In particular, to the direct sum and the tensor product of vector bundles [6] correspond, respectively, the direct sum and the tensor product of the linear transports in them, the matrices of which

are, respectively, the direct sum and the tensor product of the matrices of the initial transports along paths.

More concrete properties of the linear transports along paths in vector bundles, such as their curvature and torsion, as well as applications will be a subject of other works.

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