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HOMOGENEITY OF INTEGRABILITY CONDITIONS FOR MULTIPARAMETRIC FAMILIES OF POLYNOMIAL-NONLINEAR EVOLUTION EQUATIONS

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1 Introduction

Among nonlinear evolution equations (NLEE) describing many natural phenomena, the equations integrable by the inverse spectral transform (IST) are in the center of particular attention (see, for example, collection [1] of review articles). Owing to the IST technique one can say that an integrable NLEE is an exactly solvable one. For them, in particular, one can possible to construct an important class of the locally concentrated solutions of soliton and multisoliton type. Since the discovering the ITS method about 25 yeas ago, several dozens integrable evolution equations and systems have been found and carefully investigated. Such notable NLEE as Korteveg-de Vries (KdV) and nonlinear Schrödinger equations are among others.

The basis of integrability is formed by very deep internal algebraic properties, for example,

- infinite sequence of conservation laws and evolutionary symmetries,
- nontrivial prolongation structures in the framework of Whalquist-Estabrook method,
- Painlevé property,
- Hirota bilinear representation,
- hereditary algebra.

There is a number of integrability criteria which are based on either of these properties. The use of any criterion is reduced to generation and verification or solving some either algebraic, or differential algebraic, identities or equations, respectively. Computer algebra is a very useful and often a necessary tool for doing underlying algebraic manipulations. Some software packages for different computer algebra systems are listed in [2].

Generally, each criterion taken separately can be successfully applied in practice to some quite restrictive class of NLEE. In the case of quasilinear NLEE of dimension 1+1, i.e. with one-temporal and one-spatial independent variables, the most effective as well as constructive criterion is based on the symmetry approach [3]. This criterion establishes the existence of high order evolutionary symmetries and conservation laws, and can reveal not only equations integrable by IST but also those ones can be linearized by some appropriate transformation of variables. The well-known example is

Burgers equation which is transformed to the linear diffusion equation by the Cole-Hopf differential substitution.

The symmetry approach admits effective computer algebra implementation both for one- and multi-component quasilinear NLEE [4, 5] with constant and non-degenerated coefficient matrix at the highest spatial derivatives. Moreover, in the case of the presence some arbitrary parameters in NLEE to be investigated, one can possible to generate the integrability conditions in the form of a system of polynomial equations in those parameters in a completely automatic way. Hence, provided with an appropriate software package for investigation and solving nonlinear algebraic equations, this approach would lead to an effective and automatic tool [9] for search all the integrable equations from the given multiparametric family of polynomial NLEE.

In this paper we show that the integrability conditions, being a system of polynomial equations in arbitrary parameters, in practically interesting cases of evolution equations with uniform rank, have nontrivial homogeneity properties. It follows that the efficiency of the Gröbner bases method [6] in such problems is increased to a great extent, if it is combined with the special reduction procedure for homogeneous polynomial systems.

2 NLEE with the uniform rank

Consider a quasilinear non-degenerated multi-component NLEE of the N-th order of the following form

$$\begin{aligned} u_t &= F(u, u_1, \ldots, u_N) \\ &= \Lambda\, u_N + \sum_{(i,d)} \beta_{(i,d)} \prod_{j=k\geq 0}^{K\leq N-1} (u_j^{i_j})^{d_j}, \end{aligned} \qquad (1)$$

where

$$\begin{aligned} &u = u(x,t) = (u^1, \ldots, u^M), \quad u_i = D^i(u) \\ &D = d/dx, \ \Lambda = diag(\lambda_1, \ldots, \lambda_M), \ \lambda_i, \beta_{(i,d)} \in \mathbf{C} \\ &\lambda_i \neq 0, \quad (i,d) = (i_1, \ldots, i_K; d_1 \ldots, d_K). \end{aligned}$$

Definition 2.1. A rank of differential monomial

$$(u^{i_1})^{d_0}(u_1^{i_2})^{d_1} \cdots (u_K^{i_K})^{d_K}$$
$$d_j \geq 0, \quad 1 \leq i_k \leq M \quad (K \leq N-1), \qquad (2)$$

is a number r [7]

$$r = w_0 \bar{D} + w_1 \bar{I}, \qquad (3)$$

where $w_0, w_1 \in \mathbf{IN}$ are integer weights assigned to differential indeterminate u and to derivative operator d/dx, respectively; $\bar{D}$ is the total degree of monomial (2) and $\bar{I}$ is the total number of differentiations in it

$$\bar{D} = \sum_{j=0}^{K} d_j, \quad \bar{I} = \sum_{j=1}^{K} j\, i_j. \qquad (4)$$

Definition 2.2. A differential polynomial F has **uniform rank** if one can choose the weights w_0 and w_1 such that all its monomials have the same rank.

Definition 2.3. NLEE (1) is **equation with uniform rank** if F is a uniform rank differential polynomial.

Note that the above definition of rank is the special case of permissible gradings in differential algebra [8].

Uniformity of rank of integrable NLEE is their important intrinsic characteristic. It may be drawn from the fact that integrability is preserved under the scale transformations $u \to \mu u$, $x \to \nu x$. Therefore, for investigating the integrability of multiparametric family (1) it is reasonable to impose the rank uniformity condition a right from the start.

3 Symmetries and Conservation Laws

In accordance to the symmetry approach [3] to be integrable, NLEE (1) must possess infinitely many time-independent high order evolutionary symmetries.

Definition 3.1. A vector-function $H = (H^1, \ldots, H^M)$ in a finite number of differential variables $(u, u_1, \ldots, u_n)$ $(n > N)$ is a **local evolutionary symmetry** of the order n of (1) if NLEE

$$u_\tau = H(u, u_1, \ldots, u_n)$$

is compatible with (1).

It means that H must satisfy the following differential equation

$$\frac{dH}{dt} = F_* H, \qquad (5)$$

where

$$F_* = \sum_{i=0}^{N} F_i D^i, \ [F_i]_{kj} = \partial F^k / \partial u_i^j.$$

Theorem 3.2 [3]. Solubility (5) for as high order of H as desired, is equivalent to solubility of the operator equation

$$L_t - [\Phi_*, L] = 0 \qquad (6)$$

in terms of formal series

$$L = \sum_{k=-\infty}^{m>0} A_k D^k, \ deg(L) \equiv m, \ res(L) \equiv A_{-1} \qquad (7)$$

with the matrix $M \times M$ coefficients A_k being differential monomials, i.e. depending on a finite number of differential variables $u, u_1, \ldots$, and such that $det[A_m] \neq 0$.

The integrability conditions, the necessary ones for solubility of (6), have the form of conservation laws

$$\frac{d}{dt}(R(i,j)) \in ImD, \quad i \geq 0, \; j = 1, \ldots, M, \tag{8}$$

where $q \in Im\,D$ means that $q = D(\sigma(x, u, \ldots, u_k))$. The conservation law densities $R(i,j)$ are determined by formulas

$$R(i,j) = \begin{cases} \partial F^j / \partial u^j_{N-1}, & i = 0, \\ \partial / \partial \mu_j [trace\,(res(L_i))], & i > 0. \end{cases} \tag{9}$$

Here L_i is series (7) of degree $m = i$ and μ_j are integration constants which arise at the step of the D operator inversion in the recurrent formulas for sequential computation $A_m, A_{m-1}, \ldots, A_{-1}$ in terms of F in (1) [4, 9].

4 Integrability Conditions

In the framework of the symmetry approach integrability means just invertibility of operator D in the class of differential polynomials, both at the step of verification (8) and computation of A_i. Such an inversion is generally possible only under the certain restrictions on the structure of a given NLEE. If we consider coefficients $\beta_{(i,d)}$ in (1) as numerical parameters, those restrictions have the form of polynomial equations in the parameters.

In addition to the necessary integrability conditions which provide the existence of the local conservation laws (8,9), there are "sufficient" ones providing the existence of a higher local evolutionary symmetry of the given order as a solution of (5). Sufficiency of these conditions is justified by the well-known heuristic observation that the existence of M different higher symmetries of the order $n > N$ is sufficient for integrability, i.e. for the existence of infinitely many symmetries of the M - component NLEE of the order N [10]. These sufficient integrability conditions for multiparametric families of NLEE (1) have the form of nonlinear algebraic equations in $\beta_{i,d}$. As well as the necessary conditions, they are also produced from the invertibility ones for a differential operator. The latter is a slightly modified D operator determining the structure of higher order evolutionary symmetries [5, 9].

The algorithms for generation of both types integrability conditions have been implemented in the form of a software package [4, 9] written in the Reduce computer algebra language [11]. That package allows one to generate the integrability conditions in the form of polynomial equations as well as to find the explicit form of higher order evolutionary symmetries as solutions of (5).

One should note that for NLEE of the form (1) the given approach provides much more efficient methods and algorithms for construction local, time-independent higher

or Lie-Bäcklund symmetries of evolution type, than the general one, based on generation and integration the determining equations for infinitesimal Lie-Bäcklund symmetry generators [12].

From the above discussion it follows that for computation and verification the integrability conditions one can use two alternative computational strategies:

Strategy I.

1. Compute the densities $R(i,j)$ for the first several necessary integrability conditions (8) with $i = 0, 1 \ldots$ and $j = 1, \ldots, M$, and verify these conditions.

2. In the presence of arbitrary parameters $\beta_{(i,d)}$ and/or λ_i in (1), to solve the corresponding polynomial system of Step 1. If there is a nontrivial solution with some $\beta_{(i,d)} \neq 0$, generate and verify a few next conditions (8). Three different situations may occur at this step:

   - The next conditions are fulfilled identically without any additional restrictions.
   - New conditions produce additional algebraic equations in arbitrary parameters, if they present in (1).
   - The next conditions are not satisfied.

   In the last case the integrability analysis terminates.

3. Generate and verify the sufficient integrability conditions which provide the existence of higher evolutionary symmetries for (1), specified at Steps 1 and 2. In practice, the first several conditions (8) usually provide the integrability. This step, however, may also lead to new restrictions.

4. Among all the integrable cases select essentially different NLEE which are not reduced one to another by reparametrization.

Strategy II.

1. Generate the sufficient integrability conditions for (1) providing the existence of M different higher evolutionary symmetries of some fixed orders $n > N$, and verify their fulfillment.

2. In the case of presence arbitrary parameters in (1), to solve the resulting polynomial system of Step 2.

3. Select essentially different solutions of Step 2.

At first sight the straightforward Strategy II seems to be more attractive. However, except very simple problems, Strategy I is more optimal in practice by the following reasons:

- Restrictions which arise as sufficient integrability conditions are usually much more complicated then those ones resulting from the necessary conditions. For their generating, simplifying and solving one needs much more amount of computing time and space.
- A symmetry of the given order, say $N+1$, may not exist at all even for integrable NLEE. In this case appropriate computations are vain, unlike analysis of (8). The latter allows one to precise the structure of integrable equations step by step.
- As mentioned above, the necessary conditions are often the sufficient ones as well.

5 Homogeneity

Let

$$f_m = \sum_{(i)} a_{m,(i)} x^{(i)} = 0, \quad m = 1, 2, \dots, M \tag{10}$$

be a finite set of multivariate polynomial equations, where $(i) = (i_1, \dots, i_n)$, $a_{m,(i)} = a^{(m)}_{i_1,\dots,i_n} \in \mathbf{C}$, $x^{(i)} = x_1^{i_1} \cdots x_n^{i_n}$.

Definition 5.1. The system of polynomial equations (10) is **homogeneous** if there is the scale transformation

$$x_i \longrightarrow \alpha_i x_i, \quad x^{(i)} \longrightarrow \alpha^{(i)} x^{(i)}, \quad 0 < \alpha_i \in \mathbf{R} \tag{11}$$

with at least one factor $\alpha_i \neq 1$ such that each monomial of any given polynomial $f_m(x_1, \dots, x_n)$ acquires the same scale factor.

In other words, for two monomials $x^{(i)}$ and $x^{(j)}$ of polynomial f_m the equality $\alpha^{(i)} = \alpha^{(j)}$ or

$$\sum_{k=1}^{n} (i_k - j_k)\tilde{\alpha}_k = 0, \quad \tilde{\alpha}_k = \log \alpha_k \tag{12}$$

takes place. Generally the common scale factors $\alpha^{(i)}$, might be different for different polynomials f_m.

Collecting equations (12) coming from different polynomials of system (10) we obtain a system of linear algebraic equations with integer coefficients

$$\sum_{k=1}^{n} z_{ik}\tilde{\alpha}_k = 0, \quad z_{ij} \in \mathbf{Z}. \tag{13}$$

From Definition 5.1 and (13) it follows that polynomial system (10) is homogeneous if and only if (13) admits non-trivial solution with at least one $\tilde{\alpha}_i \neq 0$. Obviously, in the latter case, (13) has infinitely many solutions. In addition, some subset

$$\{\tilde{\alpha}_{i_1}, \dots, \tilde{\alpha}_{i_l}\} \subset \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}.$$

can be considered as a set of free parameters, and all the others $\tilde{\alpha}_i$ are uniquely defined in terms of these parameters.

Definition 5.2. Variables x_i corresponding to an arbitrary scale factors $\tilde{\alpha}_i$ are **homogeneous variables** of system (10), and their maximal number is its **homogeneity degree**.

The following theorem establishes homogeneity properties of integrability conditions for multiparametric families of NLEE of the form (1).

Theorem 5.3. Let (1) be NLEE with the uniform rank, and coefficients $\beta_{(i,d)}$ are considered as free parameters. Then a system of nonlinear algebraic equations on these parameters, being the integrability conditions, possesses the following properties:

(i) It is consistent.

(ii) If it admits a solution with at least one non-zero coefficient $\beta_{(i,d)}$, its zero-manifold has a positive dimension.

(iii) It is homogeneous.

Proof. Part (i) of the theorem follows immediately from the fact that any linear evolution equation satisfies all the integrability conditions. Hence, the system has always the trivial solution $\beta_{(i,d)} = 0$. Let there is a nontrivial solution with $\{\beta_1, \dots, \beta_k\}$ as a set of non-zero coefficients of (1) in the given nontrivial solution. Then under the scale transformation

$$u^i \to \mu u^i, \quad x \to \nu x, \quad t \to \nu^N t, \quad \mu, \nu \in \mathbf{R},$$

obviously, preserving the integrability conditions, (1) is transformed to the form

$$u_t = \Lambda u_N + \sum_{(i,d)} \beta_{(i,d)} \mu^{\bar{D}-1} \nu^{N-\bar{I}} \prod_{j=k\geq 0}^{K\leq N-1} (u_j^{i_j})^{d_j}, \tag{14}$$

where $\bar{D}$ and $\bar{I}$ are defined in (4). The rank uniformity condition means that rank (3) of any monomial is the same as rank of the linear term of F in (1). Hence,

$$w_0(\bar{D} - 1) = w_1(N - \bar{I}).$$

Then from (14) it follows

$$u_t = \Lambda u_N + \sum_{(i,d)} \beta_{(i,d)} \left(\mu \nu^{\frac{w_0}{w_1}}\right)^{\bar{D}-1} \prod_{j=k\geq 0}^{K\leq N-1} (u_j^{i_j})^{d_j}. \tag{15}$$

Comparing (1) and (15), and taking into consideration arbitrariness of μ and ν, we come to the following conclusion. If non-zero set $\{\beta_1, \dots, \beta_k\}$ is a solution of the integrability conditions, then its scale transformation of the form (11)

$$\{\beta_1 \rho^{D_1-1}, \dots, \beta_k \rho^{D_k-1}\},$$

where $\bar{D}_i > 1$ is the total degree of a monomial with coefficient β_i, produces other solution for any $\rho \in \mathbf{R}$. □

Theorem 5.1 shows that any substantial, i.e. containing nonlinear integrable equations, classification problem for NLEE of the form (1) is reduced to solving homogeneous polynomial system with infinitely many solutions. The Gröbner bases method [6] is the most universal and constructive method to treat such polynomial systems. Main obstacle in practical use of the Buchberger algorithm for a Gröbner basis construction is its very complexity. For positive dimensional polynomial ideals it is estimated [15] as d^{n*2^n}, where d is the degree of the initial polynomial set and n is a number of variables.

Fortunately, homogeneity allows one to simplify the problem by splitting the initial polynomial system into a finite set of subsystems with the reduced number of variables. We call this splitting the **homogeneity reduction** and give below a short description of an algorithmic procedure for doing such a reduction. This procedure have been implemented in the Reduce package ASYS [13] with use of the Gröbner bases technique. More detailed description and analysis of the algorithm is given in [16].

Homogeneity Reduction

1. Generation and solving linear system (10).
One can use different methods to solve (10). In the ASYS package [13] the Gröbner bases technique is used. The set of homogeneous variables is found as the corresponding one to a maximal independent set [14] modulo ideal generated by l.h.s. of (13). If this set is empty, then the polynomial system (10) is non-homogeneous. Otherwise, a set of arbitrary $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_l$ and the corresponding homogeneous variables $x_1, \ldots, x_l$ are determined. Solution of (13) is given by

$$\tilde{\alpha}_j = \sum_{k=1}^{l} q_{jk}\tilde{\alpha}_k, \quad q_{ij} \in \mathbf{Q}, \quad j = l+1, \ldots, n. \tag{16}$$

2. Transformation of variables.
Substitution

$$x_j = (\prod_{k=1}^{l} x_k^{q_{jk}})\tilde{x}_j \tag{17}$$

for non-homogeneous variables x_j, $j > l$, where q_{jk} are determined in (16), transforms (10) to the form

$$f_m(x_1, \ldots, x_n) = K_m(x_1, \ldots, x_l)\tilde{f}_m(\tilde{x}_{l+1}, \ldots, \tilde{x}_n) \tag{18}$$

with factors

$$K_m(x_1, \ldots, x_l) = \prod_{k=1}^{l} x_k^{i_k + \sum_{j=l+1}^{n} q_{jk} i_j}$$

depending only on homogeneous variables.

3. Reductions by non-zero homogeneous variables. Let $x_i \neq 0$, $i = 1, \ldots, l$. Then (18) is reduced to subsystem

$$\tilde{f}_m(\tilde{x}_{l+1}, \ldots, \tilde{x}_n) = 0, \quad (m = 1, \ldots, M) \tag{19}$$

in $n-l$ variables. Compatibility of this subsystem can be verified by construction of its Gröbner basis G. If $1 \in G$, then one should come to the next step. Otherwise, system (19) may be further investigated by the Gröbner basis method or any other appropriate technique. Finally, back transformation to initial non-homogeneous variables is made by the substitution, inverse to (17).

4. Reduction by zero homogeneous variables. All different sets of homogeneous variables when at least one variable vanishes, are considered with substitution of each such a set in (10). Then the process of homogeneity reduction is performed again as long as new systems with reduced number of variables are generated.

6 Examples

Following to Strategy I of previous section, we illustrate the homogeneity aspects of integrability analysis by two examples of NLEE of the form (1) which were first investigated in [17, 18] (see also [2, 4, 13]). As a result of homogeneity reduction, the output subsystems in the form of the lexicographical Gröbner bases can be solved algebraically.

Example I. Seven-parametric family of the seventh-order KdV-like NLEE [17]

$$u_t = u_7 + \lambda_1 u\, u_5 + \lambda_2 u_1 u_4 + \lambda_3 u_2 u_3 + \lambda_4 u^2 u_3 + \lambda_5 u\, u_1 u_2 + \lambda_6 u_1^3 + \lambda_7 u^3 u_1.$$

The necessary integrability conditions (8) for $i \leq 6$ $(j = 1)$ generate the following system of nonlinear algebraic equations in λ_i:

$$\begin{aligned}
&\lambda_1(\lambda_4 - \lambda_5/2 + \lambda_6) = (2/7\lambda_1^2 - \lambda_4)(-10\lambda_1 + 5\lambda_2 - \lambda_3) = 0, \\
&(2/7\lambda_1^2 - \lambda_4)(3\lambda_4 - \lambda_5 + \lambda_6) = 0, \\
&a_1(-3\lambda_1 + 2\lambda_2) + 21a_2 = a_1(2\lambda_4 - 2\lambda_5) + a_2(-45\lambda_1 + 15\lambda_2 - 3\lambda_3) = 0, \\
&2a_1\lambda_7 + a_2(12\lambda_4 - 3\lambda_5 + 2\lambda_6) = b_1(2\lambda_2 - \lambda_1) + 7b_2 = b_1\lambda_3 + 7b_2 = 0, \\
&b_1(-2\lambda_4 - 2\lambda_5) + b_2(2\lambda_2 - 8\lambda_1) + 84b_3 = 0, \\
&b_1(8/3\lambda_5 + 6\lambda_6) + b_2(11\lambda_1 - 17/3\lambda_2 + 5/3\lambda_3) - 168b_3 = 0, \\
&15b_1\lambda_7 + b_2(5\lambda_4 - 2\lambda_5) + b_3(-120\lambda_1 + 30\lambda_2 - 6\lambda_3) = 0, \\
&-3b_1\lambda_7 + b_2(-\lambda_4/2 + \lambda_5/4 - \lambda_6/2) + b_3(24\lambda_1 - 6\lambda_2) = 0, \\
&3b_2\lambda_7 + b_3(40\lambda_4 - 8\lambda_5 + 4\lambda_6) = 0,
\end{aligned}$$

where

$$\begin{aligned}
&a_1 = -2\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_1\lambda_3 - \lambda_2^2 - 7\lambda_5 + 21\lambda_6, \quad a_2 = 7\lambda_7 - 2\lambda_1\lambda_4 + 3/7\lambda_1^3, \\
&b_1 = \lambda_1(5\lambda_1 - 3\lambda_2 + \lambda_3), \quad b_2 = \lambda_1(2\lambda_6 - 4\lambda_4), \quad b_3 = \lambda_1\lambda_7/2.
\end{aligned}$$

The homogeneity reduction shows that this system is of homogeneous of degree 1 with homogeneous variable λ_1, and there are 6 nonvanishing subsystems and in the final output. One more subsystem gives zero values of all the polynomials at Step 3.

The computing time on an Sun 4 machine is 6 sec. The complete set of 6 solutions giving essentially different parametrization of the tree-dimensional solution space is presented in [2].

The next integrability conditions (8) with $i > 6$ select 3 actually integrable solutions representing the seventh-order symmetries of the well-known low order integrable NLEE: KdV, Sawada-Kotera and Kaup-Kupershmidt equations [17].

Example II. The third-order coupled KdV-like equations [18].

$$\begin{cases} u_t = a_0 u_3 + a_1 u\, u_1 + a_2 v\, v_1 + a_3 u\, v_1 + a_4 v\, u_1, \\ v_t = b_0 v_3 + b_1 v\, v_1 + b_2 u\, u_1 + b_3 v\, u_1 + b_4 u\, v_1, \end{cases} \qquad a_0 \neq b_0, \quad a_0 b_0 \neq 0.$$

Conditions (8) with $0 \le i \le 5$ $j = 1,2$ lead to the following polynomial system

$$e_k = \hat{e}_k = 0, \quad (k = 1 \div 6),$$

where $\hat{e}_k = e_k \,|_{a_i \Leftrightarrow b_i}$ and

$$\begin{aligned}
e_1 &= a_1\ (a_3 - a_4) - a_4\ (b_3 - b_4)\ , \\
e_2 &= (2a_3 - a_4)\ y_1 - b_2 y_2\ , \qquad y_1 = 6a_0 a_3 b_2 + (a_0 - b_0)\ (a_1^2 + a_4 b_2)\ , \\
e_3 &= a_2 y_1 - (2b_3 - b_4)\ y_2\ , \qquad y_2 = 6a_0 a_2 b_3 + (a_0 - b_0)\ (a_1 a_2 + a_4 b_1)\ , \\
e_4 &= 3a_0\ (a_2 b_2 + a_3 b_3) + (a_0 - b_0)\ (a_1 + b_3)\ a_4\ , \\
e_5 &= 2\ (2a_0^2 + 8a_0 b_0 - b_0^2)\ a_3 b_3 + 2\ (a_0 - b_0)\ (4a_0 - b_0)\ a_3 b_4 - \\
&\quad 6a_0\ (a_0 + 2b_0)\ a_2 b_2 + (a_0 - b_0)^2\ (5a_1 a_3 - 5a_1 a_4 + a_4 b_4) - \\
&\quad (a_0 - b_0)\ (7a_0 - b_0)\ a_4 b_3\ , \\
e_6 &= 3a_0\ [\ (a_0 - b_0)^3 - 3a_0\ (a_0 + 2b_0)^2\ ]\ (a_2 b_2 + a_3 b_3) + \\
&\quad (a_0 - b_0)^3\ [\ 3a_0 a_1 a_3 - 2\ (2a_0 + b_0) a_1 a_4\ ] + 9a_0^2\ (a_0 - b_0) \\
&\quad [\ (a_0 - b_0)\ a_4 - (a_0 + 2b_0)\ a_3\ ]\ b_4 - (a_0 - b_0)\ (2a_0^3 - 30a_0^2 b_0 + b_0^3)\ a_4 b_3
\end{aligned}$$

This system with six-dimensional solution space has homogeneity degree 3, and $\{a_3, b_0, b_3\}$ as a set of homogeneous variables. There are 22 different nonvanishing subsystems in the output after homogeneity reduction. Their lexicographical Gröbner bases allow to find the explicit parametrization all the solution space. The running time of homogeneity reduction with the Gröbner bases construction for the output subsystems is 34 sec. on an Sun 4. One should note that construction of the Gröbner basis for the initial polynomial system leads to very large computations with more than 3 Mb output [13, 19]. The homogeneity reduction simplifies this problem dramatically.

The complete list of integrable equations after taking into account the next integrability conditions is given in [18].

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