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SPECIAL BASES FOR DERIVATIONS OF TENSOR ALGEBRAS. III. CASE ALONG SMOOTH MAPS WITH SEPARABLE POINTS OF SELFINTERSECTION

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Специальные базисы для дифференцирований тензорных алгебр.
III. Случай вдоль гладких отображений с отделимыми точками самопересечения

Исследованы необходимые и/или достаточные условия для существования, единственности и голономности базисов, в которых на достаточно общих подмножествах дифференцируемого многообразия обращаются в нуль компоненты дифференцирований тензорной алгебры над ним. С этой точки зрения рассмотрены линейные связности и принцип эквивалентности.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Special Bases for Derivations of Tensor Algebras.
III. Case Along Smooth Maps with Separable Points of Selfintersection

Necessary and/or sufficient conditions are studied for the existence, uniqueness and holonomicity of bases in which on sufficiently general subsets of a differentiable manifold the components of derivations of the tensor algebra over it vanish. The linear connections and the equivalence principle are considered from that point of view.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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I. INTRODUCTION

In connection with the equivalence principle [1], as well as from purely mathematical reasons [2-5], of importance is the problem for existence of local (holonomic or anholonomic [2]) coordinates (bases) in which the components of a linear connection [3] vanish on some subset, usually submanifolds, of a differentiable manifold [3]. This problem has been solved for torsion free, i.e. symmetric, linear connections [3, 4] in the cases at a point [2-5], along a smooth path without selfintersections [2, 5] and on a neighborhood [2, 5]. These results were generalized in our previous works [6, 7] for arbitrary, with or without torsion, derivations of the tensor algebra over a given differentiable manifold [3] and, in particular, for arbitrary linear connections. General results of this kind can be found in [8], where a criteria for the existence of the above-mentioned special bases (coordinates) on submanifolds of a space with a summetric affine connection is given.

The present work generalizes the results from [6-8] and deals with the problems for existence, uniqueness and holonomicity of special bases in which the components of a derivation of the tensor algebra over a differentiable manifold vanish on some its subset of a sufficiently general type (Sect. II and III). In particular, this derivation may be a linear connection (Sect. IV). In this context we also make conclusions concerning the general validity and the mathematical formulation of the equivalence principle in a class of gravity theories (Sect. V).

For further reference purposes, as well as for the exact statement of the above problems, we reproduce below a few simple facts about derivations of a tensor algebras that can be found in [6, 7] or derived from the ones in [3].

Every derivation D of the tensor algebra over a differentiable manifold M admits a unique representation in the form (see [3], ch.I, proposition 3.3) $D=L_X+S$, in which L_X is the Lie derivative along the vector field X and S is tensor field of type (1,1) considered here as a differentiation [3]. Both X and S are uniquely defined by D.

If S is a map from the set of C^1 vector fields into the tensor fields of type (1,1) and $S:X\longmapsto S_X$, then the equation

$$D^S_X=L_X+S_X \tag{1}$$

defines a derivation of the tensor algebra over M for any C^1 vector field X [3]. As the map S will hereafter be assumed fixed, such a derivation, i.e. D^S_X, will be called an S-derivation along X and will be denoted for brevity by D_X. An S-derivation is a map D such that $D:X\longmapsto D_X$ where D_X is an S-derivation along X.

Let $\{E_i,\ i=1,\dots,n:=\dim(M)\}$ be a (coordinate or noncoordinate [2]) local basis of vector fields in the tangent to M bundle. It is (an)holonomic if the vectors $E_1,\ \dots,\ E_n$ (do not) commute [2].

The local components $(W_X)^i_{.j}$ of D with respect to $\{E_i\}$ are defined by the equation

$$D_X(E_j)=(W_X)^i_{.j}E_i, \tag{2}$$

and their explicit form is

$$(W_X)^i_{.j}=(S_X)^i_{.j}-E_j(X^i)+C^i_{.jk}X^k, \tag{3}$$

where $X(f)$ denotes the action of $X=X^kE_k$ on C^1 functions f, i.e.

$X(f):=X^kE_k(f)$, and $C^i_{.jk}$ define the commutators of the basic vectors, i.e.

$$[E_j,E_k]=C^i_{.jk}E_i \tag{4}$$

If we make the change $\{E_i\}\longrightarrow\{E_{i'}=A^i_{i'}E_i\}$, where $A:=\|A^i_{i'}\|=:$ $=:\|A^{i'}_i\|^{-1}$ is a nondegenerate matrix function, then from (2) we can see that $(W_X)^i_{.j}$ transform into

$$(W_X)^{i'}_{..j'}=A^{i'}_iA^j_{j'}(W_X)^i_{\ j}+A^{i'}_iX(A^i_{j'}), \tag{5}$$

which, if we introduce the matrices $W:=\|(W_X)^i_{.j}\|$ and $W':=$ $:=\|(W_X)^{i'}_{..j'}\|$, will read

$$W'_X=A^{-1}[W_XA+X(A)], \tag{5'}$$

where as a first matrix index is understood the superscript and $X(A):=X^kE_k(A)=\|X^kE_k(A^i_{i'})\|$.

If ∇ is a linear connection with local components $\Gamma^i_{.jk}$ (see e.g. [2-4]), then is fulfilled [2, 3]

$$\nabla_X(E_j)=(\Gamma^i_{.jk}X^k)E_i. \tag{6}$$

Hence, comparing (2) and (6), we see that D_X is a covariant differentiation along X iff

$$(W_X)^i_{.j}=\Gamma^i_{.jk}X^k \tag{7}$$

for some functions $\Gamma^i_{.jk}$.

Let D be an S-derivation and X and Y be vector fields. The torsion operator T of D is defined as

$$T^D(X,Y):=D_XY-D_YX-[X,Y]. \tag{8}$$

The S-derivation D is called torsion free if $T^D=0$.

For a linear connection ∇, due to (7), we have

$$(T^{\nabla}(X,Y))^i = T^i_{.jk} X^j Y^k, \qquad (9)$$

where [2, 3] $T^i_{.kl} := -(\Gamma^i_{.kl} - \Gamma^i_{.lk}) - C^i_{.kl}$ are the components of the torsion tensor of ∇.

Further we shall investigate the problem for existence of special bases $\{E_{i'}\}$ in which $W'_X = 0$ for an S-derivation D along any or fixed vector field X. Hence, due to (5′), we shall have to solve the equation $W_X(A) + X(A) = 0$ with respect to A under conditions that will be presented below.

II. DERIVATIONS ALONG ARBITRARY VECTOR FIELDS

This section is devoted to the existence and some properties of special bases $\{E_{i'}\}$, defined in a neighborhood of a subset U of the manifold M, in which the components of an S-derivation D along an *arbitrary* vector field X vanish on U.

Proposition 1. If for some S-derivation D there exists a basis $\{E_{i'}\}$ in which $W'_X|_U = 0$ for every vector field X, then D is a linear connection on the set $U \subset M$.

Remark. On the set $U \subset M$ the derivation D is a linear connection if (cf. (7)) in some, and hence in any, basis $\{E_i\}$ is fulfilled

$$W_X(x) = \Gamma_k(x) X^k(x), \qquad (10)$$

where $x \in U$, $X = X^k E_k$ and Γ_k are some matrix functions on U. (Evidently, a linear connection on M is also on U a linear connection for every U; see (7)).

Proof. If we fix a basis $\{E_i\}$ and $E_{i'} = A^i_{i'} E_i$, then by the

definition of $\{E_{i'}\}$, we have $W'_X|_U = 0$, i.e. $W'_U(x) = 0$ for $x \in U$, which in conformity with (5′) is equivalent to (10) with $\Gamma_k = -(E_k(A))A^{-1}$, $A = \|A^i_{i'}\|$. ■

The opposite statement to proposition 1 is generally not true and for its exact formulation we shall need some preliminary results and explanations.

Let p be an integer, $p \geq 1$, and the Greek indices α and β run from 1 to p. Let J^p be a neighborhood in $\mathbb{R}^p$ and $\{s^\alpha\} = \{s^1, \ldots, s^p\}$ be (Cartesian) coordinates in $\mathbb{R}^p$.

Lemma 2. Let $Z_\alpha : J^p \longrightarrow GL(m,\mathbb{R})$, $GL(m,\mathbb{R})$ being the group of $m \times m$ matrices on $\mathbb{R}$, be a C^1 matrix-valued function on J^p. Then the initial-value problem

$$\frac{\partial Y}{\partial s^\alpha}\Big|_s = Z_\alpha(s) Y, \quad Y|_{s=s_0} = \mathbb{1} := \|\delta^i_j\|^m_{i,j=1}, \qquad (11)$$

in which $\mathbb{1}$ is the unit matrix of the corresponding size, $s \in J^p$, $s_0 \in J^p$ is fixed and Y is $m \times m$ matrix function on J^p, has a solution, denoted by $Y = Y(s, s_0; Z_1, \ldots, Z_p)$, which is unique and smoothly depends on all its arguments, if and only if

$$R_{\alpha\beta}(Z_1, \ldots, Z_p) := \partial Z_\alpha / \partial s^\beta - \partial Z_\beta / \partial s^\alpha + Z_\alpha Z_\beta - Z_\beta Z_\alpha = 0. \qquad (12)$$

Proof. According to the results in [9], ch. VI the integrability conditions for (11) are

$$0 = \partial^2 Y / \partial s^\alpha \partial s^\beta - \partial^2 Y / \partial s^\beta \partial s^\alpha = \partial(Z_\beta Y)\backslash \partial s^\alpha - \partial(Z_\alpha Y)\backslash \partial s^\beta =$$

$$= (\partial Z_\beta \backslash \partial s^\alpha) Y - (\partial Z_\alpha \backslash \partial s^\beta) Y + Z_\beta Z_\alpha Y - Z_\alpha Z_\beta Y = -R_{\alpha\beta}(Z_1, \ldots, Z_p) Y.$$

Hence (11) has a unique solution iff (12) is satisfied. ■

Let $p \leq n := \dim(M)$, $\alpha, \beta = 1, \ldots, p$ and $\mu, \nu = p+1, \ldots, n$. Let $\gamma : J \longrightarrow M$ be a C^1 map. We shall suppose that for any $s \in J^p$ there exists its (p-dimensional) neighborhood $J_s \subset J^p$, $s \in J$ such that the restricted map $\gamma|_{J_s} : J_s \longrightarrow M$ is without selfintersections, i.e. in J_s does not exist points s_1 and $s_2 \neq s_1$ with the property $\gamma(s_1)=\gamma(s_2)$. This assumption is equivalent to the one that the points of selfintersections of γ, if any, can be separated by neighborhoods. With J^p_s we shall denote the union on all neighborhoods J_s with the above property; evidently J^p_s is the maximal neighborhood of s in which γ is without selfintersections.

Let at first suppose $J^p_s=J^p$, i.e. γ to be without silfintersection, and that $\gamma(J^p)$ be contained in only one coordinate neighborhood V of M.

Let us fix some one-to-one C^1 map $\eta: J^p \times J^{n-p} \longrightarrow M$ such that $\eta(\cdot,\underline{t}_0)=\gamma$ for a fixed $\underline{t}_0 \in J^{n-p}$, i.e. $\eta(s,\underline{t}_0)=\gamma(s)$, $s\in J^p$. In $V\cup\eta(J^p,J^{n-p})$ we define coordinates $\{x^i\}$ by putting $(x^1(\eta(s,\underline{t})),\dots,x^n(\eta(s,\underline{t}))):=(s,\underline{t})\in\mathbb{R}^n$, $s\in J^p$, $\underline{t}\in J^{n-p}$.

Proposition 3. Let $\gamma: J^p \longrightarrow M$ be C^1, without selfintersections and $\gamma(J^p)$ lies in only one coordinate neighborhood. Let on $\gamma(J^p)$ the derivation D be a linear connection. Then there exists a defined in a neighborhood of $\gamma(J^p)$ basis $\{E_{i'}\}$ in which the components of D along every vector field vanish on $\gamma(J^p)$ if and only if in the above-defined coordinates $\{x^i\}$ is fulfilled

$$[R_{\alpha\beta}(-\Gamma_1\circ\gamma,\dots,-\Gamma_p\circ\gamma)]\Big|_{J^p}=0, \quad \alpha,\beta=1,\dots,p, \tag{13}$$

where $R_{\alpha\beta}(\dots)$ is defined by (12) for m=n and $(s^1,\dots,s^p)=s\in J^p$, i.e.

$$[R_{\alpha\beta}(-\Gamma_1\circ\gamma,\dots,-\Gamma_p\circ\gamma)](s)=\partial\Gamma_\alpha(\gamma(s))/\partial s^\beta - \partial\Gamma_\beta(\gamma(s))/\partial s^\alpha +$$

$$+ (\Gamma_\alpha\Gamma_\beta - \Gamma_\beta\Gamma_\alpha)|_{\gamma(s)}. \tag{14}$$

Remark. In the case when D is a symmetric affine connection this result was obtained by means of another method in [8].

Proof. The following considerations will be done in the above-

defined neighborhood $V\cap\eta(J^p,J^{n-p})$ and coordinates $\{x^i\}$. Let $E_i=d/dx^i$. We shall look for a basis $\{E_{i'}=A^i_{i'}E_i\}$ in which $W'_X(\gamma(s))=0$, $s\in J^p$. By eq. (5′) the existence of $\{E_{i'}\}$ is equivalent to the existence of the $A=\|A^i_{i'}\|$, transforming $\{E_i\}$ into $\{E_{i'}\}$, and such that $[A^{-1}(W_X A+X(A))]|_{\gamma(s)}=0$ for every X. But as on $\gamma(J^p)$ D is a linear connection, the eq. (10) is valid for some matrix-valued functions Γ_k and $x\in\gamma(J^p)$, consequently A must be a solution of $\Gamma'_{k'}(x)=0$, i.e. of

$$(\Gamma_k(\gamma(s))A(\gamma(s))+(\partial A/\partial x^k)|_{\gamma(s)}=0, \quad s\in J^p. \tag{15}$$

By expanding $A(\eta(s,\underline{t}))$, $s\in J^p$, $\underline{t}\in J^{n-p}$ into a power series with respect to $(\underline{t}-\underline{t}_0)$ it can be shown that (15) has a solution if and only if the integrability conditions (13) are valid. Besides, if (13) take place, than the general solution of (15) is

$$A(\eta(s,\underline{t}))=\{\mathbb{1}-\sum_{\lambda=p+1}^{n}\Gamma_\lambda(\gamma(s))[x^\lambda(\eta(s,\underline{t}))-x^\lambda(\gamma(s))]\}\times$$

$$\times Y(s,s_0;-\Gamma_1\circ\gamma,\dots,-\Gamma_p\circ\gamma)B_0 + \sum_{\mu,\nu=p+1}^{n} B_{\mu\nu}(s,\underline{t};\eta)[x^\mu(\eta(s,\underline{t}))-$$

$$-x^\mu(\gamma(s))][x^\nu(\eta(s,\underline{t}))-x^\nu(\gamma(s))], \tag{16}$$

where $s_0\in J^p$ and the nondegenerate matrix B_0 are fixed and the matrices $B_{\mu,\nu}$, $\mu,\nu=p+1,\dots,n$, together with their derivatives, are bounded when $\underline{t}\longrightarrow\underline{t}_0$. (The fact that into (16) enter only sums from p+1 to n is a consequence from $x^\alpha(\eta(s,\underline{t}))=x^\alpha(\gamma(s))=s^\alpha$, i.e. $x^\alpha(\eta(s,\underline{t}))-x^\alpha(\eta(s,\underline{t}_0))=x^\alpha(\eta(s,\underline{t}))-x^\alpha(\gamma(s))\equiv 0$, $\alpha=1,\dots,p$.)

Thus, bases $\{E_{i'}\}$ in which $W'_X=0$ exist iff (13) is satisfied. If (13) is valid, then the bases $\{E_{i'}\}$ are obtained from $\{E_i=d/dx^i\}$ by means of linear transformations the matrices of which must have the form (16).∎

Now we are ready to consider a general smooth (C^1) map

$\gamma:J^p \longrightarrow M$ whose points of selfintersection, if any, can be separated by neighborhoods. Let for any $r\in J^p$ be chosen a coordinate neighborhood $V_{\gamma(r)}$ of $\gamma(r)$ in M. Let there be fixed a C^1 one-to-one map $\eta_r:J^p_r\times J^{n-p}\longrightarrow M$ such that $\gamma|_{J^p_r}=\eta_r(\cdot,\underline{t}^r_0)$ for some $\underline{t}^r_0\in J^{n-p}$. In the neighborhood $V_{\gamma(r)}\cap\eta_r(J^p_r,J^{n-p})$ of $\gamma(J^p_r)\cap V_{\gamma(r)}$ we introduce local coordinates $\{x^i_r\}$ defined by $(x^1_r(\eta_r(s,\underline{t})),\dots,x^n_r(\eta_r(s,\underline{t}))):=:=(s,\underline{t})\in\mathbb{R}^n$, where $s\in J^p_r$ and $\underline{t}\in J^{n-p}$ are such that $\eta_r(s,\underline{t})\in V_{\gamma(r)}$.

Theorem 4. Let the points of selfintersection of the C^1 map $\gamma:J^p\longrightarrow M$, if any, be separable by neighborhoods and let on $\gamma(J^p)$ the S-derivation D be a linear connection, i.e. eq. (10) to be valid. Then in some neighborhood of $\gamma(J^p)$ exists a basis $\{E_{i'}\}$ in which the components of D along every vector field vanish on $\gamma(J^p)$ if and only if for every $r\in J$ in the above defined local coordinates $\{x^i_r\}$ is fulfilled

$$[R_{\alpha\beta}(-\Gamma_1\circ\gamma,\dots,-\Gamma_p\circ\gamma)](s)=0,\ \alpha,\beta=1,\dots,p, \tag{17}$$

where Γ_α are calculated by means of (10) in $\{x^i_r\}$, $R_{\alpha\beta}$ are given by (14) and $s\in J^p_r$ is such that $\gamma(s)\in V_{\gamma(r)}$.

Proof. For any $r\in J^p$ the restricted map $\gamma|_{'J^p_r}:'J^p_r\longrightarrow M$, where $'J^p_r:=\{s\in J^p_r,\ \ \gamma(s)\in V_{\gamma(s)}\}$, is without selfintersections (see the above definition of J^p_r) and $\gamma|_{'J^p_r}('J^p_r)=\gamma('J^p_r)$ lies in the coordinate neighborhood $V_{\gamma(r)}$.

So, if a basis $\{E_{i'}\}$ with the described property exists, then, by proposition 3, eqs. (17) are identically satisfied.

On the opposite, if (17) are valid, then, again by proposition 3, for every $r\in J^p$ in a neighborhood $'V_r$ of $\gamma('J^p_r)$ in $V_{\gamma(r)}$ exists a basis $\{E^r_{i'}\}$ in which the components of D_X along every vector field X vanish on $\gamma('J^p_r)$. From the neighborhoods $'V_r$ we can construct a neighborhood V of $\gamma(J^p)$, e.g. we can put $V=\bigcup_{r\in J^p}{}'V_r$, but, generally, V is sufficient to be taken as a union of $'V_r$ for some, but not all $r\in J^p$. On V we can obtain a basis $\{E_{i'}\}$ with the needed property by putting $E_{i'}|_x=E^r_{i'}|_x$ if x belongs to only one neighborhood $'V_r$ and if x belongs to more than one neighborhood $'V_r$ we can choose $\{E_{i'}|_x\}$ to be the basis $\{E^r_{i'}|x\}$ for some arbitrary fixed r with this property. (Note that generally the so-obtained basis is not continuous in the regions containing intersections of several neighborhoods $'V_r$.)■

Proposition 5. If on the set U⊂M there exist bases in which the components of some S-derivation along every vector field vanish on U, then all of them are obtained from one another by linear transformations whose coefficients are such that the action on them of the corresponding basic vectors vanishes on U.

Proof. The proposition is a simple corollary from (5').■

Proposition 6. If for some S-derivation D there exists a locally holonomic basis in which the components of D along every vector field vanish on the set U⊂M, then D is torsion free on U. On the opposite, if D is torsion free on U and bases in which the components of D along every vector field vanish on U exist, then all of them are holonomic on U, i.e. their basic vectors commute on U.

Proof. If $\{E_{i'}\}$ is a basis with the mentioned property, i.e. $W'_X(x)=0$ for every X and $x\in U$, then using (2) and (8) (see also eq. (15) from [6]), we fined $T^D(E_{i'},E_{j'})|_U=-[E_{i'},E_{j'}]|_U$ and consequently $\{E_{i'}\}$ is holonomic on U, i.e. $[E_{i'},E_{j'}]|_U=0$, iff $0=T^D(X,Y)|_U= =\{X^{i'}Y^{j'}T^D(E_{i'},E_{j'})\}|_U$ for every vector fields X and Y, which is equivalent to $T^D|_U=0$.

On the opposite, let $T^D|_U=0$. We want to prove that any basis

$\{E_{i'}\}$ in which $W'_X=0$ is holonomic on U. The holonomicity on U means $0=[E_{i'},E_{j'}]|_U=\{-A^{k'}_k(E_{j'}(A^k_{i'})-E_{i'}(A^k_{j'}))E_{k'}\}|_U$. But (see proposition 1 and (10)) the existence of $\{E_{i'}\}$ is equivalent to $W_X|_U=(\Gamma_k X^k)|_U$ for some functions Γ_k and every X. These two facts, combined with (2) and (8), show that $(\Gamma_k)^i_{.j}=(\Gamma_j)^i_{.k}$. Using this and $\{\Gamma_k A+\partial A/\partial x^k\}|_U=0$ (see the proof of proposition 1), we find $E_{j'}(A^k_{i'})|_U= =-A^j_{j'}A^i_{i'}(\Gamma_j)^k_{.i}|_U=E_{i'}(A^k_{j'})|_U$ and therefore $[E_{i'},E_{j'}]|_U=0$ (see above), i.e. $\{E_{i'}\}$ is holonomic on U.■

III. DERIVATIONS ALONG FIXED VECTOR FIELD

As it was said in our previous works [6, 7] the problem for existence and the properties of special bases for derivations along *fixed* vector field is not very interesting from the viewpoint of its applications. By that reason we shall briefly outline only some results concerning it.

The following two propositions are almost evident (cf. resp. propositions 1 and 5).

Proposition 7. If for the S-derivation D_X along a fixed vector field X exists a basis $\{E_{i'}\}$ in which the components of D_X vanish on the set U⊂M, then on U D_X is a covariant differentiation along X, i.e. for the given X the eq. (10) is valid on U.

Proposition 8. If on the set U⊂M there exist bases in which the components of an S-derivation along a fixed vector field vanish, then all of them are obtained from one another by linear transformations, the matrices of which are such that the action of X on them vanishes on U.

The existence of special bases in which the components of D_X, with a *fixed* X, vanish on some set U⊂M significantly differs from the same problem for D_X with an *arbitrary* X (see Sect. II). In fact, if $\{E_{i'}=A^i_{i'}E_i\}$, $\{E_i\}$ being a fixed basis on U, is such a basis on U, i.e. $W'_X|_U=0$, then, due to (5′), its existence is equivalent to the one of $A:=\|A^i_{i'}\|$ for which $(W_X A+X(A))|_U=0$ for the *given* X. As X is fixed, the values of A at two different points, say x,y∈U, are connected through the last equation if and only if x and y lie on one and the same integral path (curve) of X, the part of which between x and y belongs entirely to U. Hence, if $\gamma:J\longrightarrow M$, J being an ℝ-interval, is (a part of) an integral path of X, i.e. at $\gamma(s)$, s∈J the tangent to γ vector field $\dot\gamma$ is $\dot\gamma(s):=X|_{\gamma(s)}$, then along γ the equation $(W_X A+X(A))|_U=0$ reduces to $dA/ds|_{\gamma(s)}=\dot\gamma(A)|_s= =(X(A))|_{\gamma(s)}=-W_X(\gamma(s))A(\gamma(s))$. The general solution of this equation is

$$A(s;\gamma)=Y(s,s_0;-W_X\circ\gamma)B(\gamma), \tag{18}$$

where $s_0\in J$ is fixed, $Y=Y(s,s_0;Z)$, with Z being a C^1 matrix function of s, is the unique solution of the initial-value problem (see [9], ch. IV, §1)

$$dY/ds=ZY,\quad Y|_{s=s_0}=\mathbb{1}, \tag{19}$$

and the nondegenerate matrix $B(\gamma)$ may depend only on γ, but not on s. (Note that (19) is a special case of (11) for p=1 and by lemma 2 it has always a unique solution because of $R_{11}(Z_1)\equiv0$ for p=1.)

From the above considerations follows

Proposition 9. For any S-derivation along a fixed vector field on every set U⊂M there exist bases, in which the components of that derivation vanish on U.

IV. LINEAR CONNECTIONS

The results of Sect. II can directly be applied to the case of linear connections. As this is more or less trivial, we shall present without proofs only three such consequences.

Corollary 10. Let the points of selfintersection of the C^1 map $\gamma: J^p \longrightarrow M$, if any, be separable by neighborhoods, ∇ be a linear connection on M with local components $\Gamma^i_{.jk}$ (in a basis $\{E_i\}$) and $\Gamma_k := \|\Gamma^i_{.jk}\|^n_{i,j=1}$. Then in a neighborhood of $\gamma(J^p)$ exists a basis $\{E_{i'}\}$ in which the components of ∇ vanish on $\gamma(J^p)$, i.e. $\Gamma_{k'}|_{\gamma(J^p)} = 0$, iff for every $r \in J^p$ in the coordinates $\{x^i_r\}$ (defined before theorem 4) (17) is satisfied in which Γ_α, $\alpha=1,\dots,p$ are part of the components of ∇ in $\{x^i_r\}$ and $s \in J^p$ is such that $\gamma(s) \in V_{\gamma(r)}$.

Corollary 11. If on the set $U \subset M$ there exist bases in which the components of a linear connection vanish on U, then all of these bases are obtained from one another by linear transformations, the matrices of which are such that the action of the corresponding basic vectors on them vanishes on U.

Corollary 12. Let for some linear connection on a neighborhood of a set $U \subset M$ exist local continues bases in which the connection's components vanish on U. Then one, and hence any, such basis is holonomic on U iff the connection is torsion free on U.

V. CONCLUDING REMARKS

As the main result of this work is expressed by theorem 4, we shall make some comments on it. First of all, it expresses a sufficiently general necessary and sufficient condition for existence

of the considered here special bases for derivations, in particular linear connections. For instance, it covers that problem on arbitrary submanifolds. In this sense, its special cases are the results in our previous papers [6, 7].

If $p=0$ or $p=1$, then the condition (17) is identically satisfied, i.e. $R_{\alpha\beta}=0$ (see (14)). Hence in these two cases special bases, we are searching for, always exist (respectively at a point or along a path), which was already established in [6] and [7] respectively.

In the other limiting case, $p=n:=\dim(M)$, it is easily seen that the quantities (14) are simply the matrices formed from the compoents of the corresponding curvature tensor (cf. [6, 3, 4]) and that the set $\gamma(J^p)$ consists of one or more neighborhoods in M. Consequently, now theorem 4 states that the investigated here special bases exist iff the corresponding derivation is flat, i.e. if its curvature tensor is zero, a result already found in [6].

In the general case, when $2 \le p < n$ (if $n \ge 3$), special bases, even anholonomic, of the considered here type do not exist if (and only if) the conditions (17) are not satisfied. Besides, in this case the quantities (14) cannot be considered as a "curvature" of $\gamma(J^p)$ as they are something like "commutators" of covariant derivatives of a type ∇_F, where F is a tangent to $\gamma(J^p)$ vector field (i.e. $F \in T(\gamma(J^p))$ if $\gamma(J^p)$ is a submanifold of M), and which act on tangent to M vector fields.

Let us also note that the bases in which the components of some derivation vanish on a set U are generally anholonomic, if any, and only in the torsion free case (the derivation's torsion vanishes on U) they may be holonomic.

The above results outline the general bounds of validity and are the exact mathematical expression of the equivalence principle, which states [1] that the gravitational field strength, theoretically identified with the components of a linear connection, can locally be transformed to zero by a suitable choice of the local reference frame (basis), i.e. it requires the existence of local bases in which the corresponding connection's components vanish.

The above discussion, as well as the results from [6, 7], show the identical validity of the equivalence principle in zero and one dimensional cases, i.e. for $p=0$ and $p=1$. Besides these are the only cases when it is fulfilled for arbitrary gravitational fields. In fact, for $p\geq2$ (for $n\geq2$), as we saw in Sect. IV, bases with the above property do not exist unless the conditions (17) are satisfied. In particular, for $p=n\geq2$ it is valid only for flat linear connections (cf. [6]).

Mathematically the equivalence principle is expressed through corollary 10 (or, in some more general situations, through theorem 4). Thus we see that in gravity theories based on linear connections this principle is identically satisfied at any fixed point or along any fixed path, but on submanifolds of dimension greater or equal two it is generally not valid. Therefore in this class of gravity theories the equivalence principle is a theorem derived from their mathematical background. It may play a role as a principle if one tries to construct a gravity theory based on more general derivations, but, generally, it will reduce this theory to one based on linear connections.

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