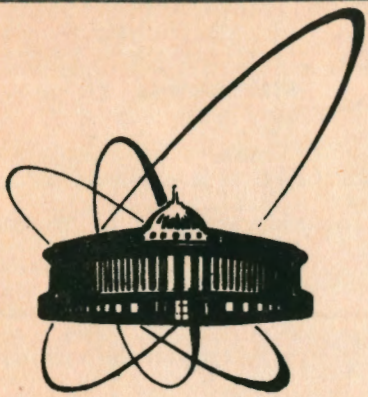

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исследований
дубна

E5-92-508

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SPECIAL BASES FOR DERIVATIONS OF TENSOR ALGEBRAS. II. CASE ALONG PATHS

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1992

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Специальные базисы для дифференцирований
тензорных алгебр.
II. Случай вдоль путей

Доказано существование локальных базисов, в которых равны нулю компоненты дифференцирований тензорных алгебр над дифференцируемым многообразием. Исследована проблема о голономности этих базисов. Полученные результаты применены к случаю линейных связностей. Показаны некоторые связи с принципом эквивалентности.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1992

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E5-92-508

Special Bases for Derivations of Tensor Algebras.
II. Case Along Paths

The existence of local bases in which the components of derivations of tensor algebra over a differentiable manifold vanish along paths is proved. The holonomicity of these bases is investigated. The obtained results are applied to the case of linear connections. Some relations with the equivalence principle are shown.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1992

I. INTRODUCTION

The existence of local coordinates in which the components of symmetric linear connections [1] vanish along a smooth path without selfintersections is a known classical result [2, 3]. In connection with the intensive use of nonsymmetric linear connections [1, 2] in different physical theories it is natural these results to be generalized to the case of nonvanishing torsion.

This paper investigates the mentioned problem from the more general viewpoint of arbitrary derivations of the tensor algebra over a differentiable manifold [1, 2]. In it the existence of special bases (or coordinates) is proved in which the defined below components of derivations, in particular linear connections, vanish along some path. The question when these bases are holonomic or anholonomic [2] is investigated. Also the connection of these topics with the equivalence principle in general relativity [4] and other gravity theories (see, e.g., [5]) is briefly discussed.

For the explicit mathematical formulation of our problem we shall remind from [6] some facts concerning derivations of the tensor algebra over a manifold.

By proposition 3.3 from ch. I in [1] any derivation D of the tensor algebra over a differentiable manifold M admits a unique representation in the form $D=L_X+S$, in which L_X is the Lie derivative along the vector field X and S is tensor field of type (1,1) considered here as a derivation [1]. Both X and S are uniquely determined by D.

If S is a map from the set of C^1 vector fields into the tensor fields of type (1,1) and $S:X\longmapsto S_X$, then the equation

$$D^S_X = L_X + S_X \tag{1}$$

defines a derivation of the tensor algebra over M for any C^1 vector field X [1]. As the map S will hereafter be assumed fixed, such a derivation, i.e. D^S_X will be called an S-derivation along X and will be denoted for brevity by D_X. An S-derivation is a map D such that $D: X \longmapsto D_X$, where D_X is an S-derivation along X.

Let $\{E_i,\ i=1,\ldots n:=\dim(M)\}$ be a (coordinate or noncoordinate [2, 3]) local basis of vector fields in the tangent to M bundle. It is holonomic (resp. anholonomic) if the vectors $E_1, \ldots, E_n$ commute (resp. don't commute) [2].

The local components $(W_X)^i_{.j}$ of D_X with respect to $\{E_i\}$ are defined by the equality

$$D_X(E_j) = (W_X)^i_{.j} E_i. \tag{2}$$

With respect to D_X they play the same role as the components of a linear connection with respect to the connection (see below or [1, 2]). It can easily be computed (see (1) and [1]) that

$$(W_X)^i_{.j} := (S_X)^i_{.j} - E_j(X^i) + C^i_{.kj} X^k \tag{3}$$

in which with X(f) we denote the action of $X = X^k E_k$ on C^1 scalar functions f, i.e. $X(f) := X^k E_k(f)$, and $C^i_{.kj}$ defines the commutators of the basic vectors, i.e.

$$[E_j, E_k] = C^i_{.jk} E_i. \tag{4}$$

If we make the change $\{E_i\} \longrightarrow \{E_{i'} := A^i_{i'} E_i\}$, where $A := \|A^i_{i'}\| =: \|A^{i'}_i\|^{-1}$ is a nondegenerate matrix function, then from (2) we can see that $(W_X)^i_{.j}$ transform into

$$(W_X)^{i'}_{.j'} = A^{i'}_i A^j_{j'} (W_X)^i_{.j} + A^{i'}_i X(A^i_{j'}), \tag{5}$$

which, if we introduce the matrices $W_X := \|(W_X)^i_{.j}\|$ and $W'_X := \|(W_X)^{i'}_{.j'}\|$, will read

$$W'_X = A^{-1}\left(W_X A + X(A)\right), \tag{5'}$$

where as a first matrix index is understood the superscript and $X(A) := X^k E_k(A) = \|X^k E_k(A^i_{i'})\|$.

If ∇ is a linear connection with local components $\Gamma^i_{.jk}$ (see, e.g., [1-3]), it is fulfilled

$$\nabla_X(E_j) = (\Gamma^i_{.jk} X^k) E_i. \tag{6}$$

Hence, comparing (2) and (6), we see that D_X is a covariant differentiation along X iff

$$(W_X)^i_{.j} = \Gamma^i_{.jk} X^k \tag{7}$$

for some functions $\Gamma^i_{.jk}$.

Let D be an S-derivation and X and Y be vector fields. The torsion operator T^D of D is defined as

$$T^D(X,Y) := D_X Y - D_Y X - [X,Y], \tag{8}$$

which in a case of linear connection reduces to the corresponding classical quantity (see [1-3] and below (11)). The S-derivation D will be called torsion free if $T^D = 0$.

By means of (1) and (2), we find the local expressions:

$$(T^D(X,Y))^i = (W_X)^i_{.l} Y^l - (W_Y)^i_{.l} X^l - C^i_{.kl} X^k Y^l. \tag{9}$$

If D is a linear connection ∇, then, due to (7), we have:

$$(T^\nabla(X,Y))^i = T^i_{.kl} X^k Y^l, \tag{10}$$

where [1-3]

$$T^i_{.kl} := -(\Gamma^i_{.kl} - \Gamma^i_{.lk}) - C^i_{.kl}, \tag{11}$$

are the components of the torsion tensor of ∇.

The task of this work is the problem to be investigated when along a given path $\gamma: J \longrightarrow M$, J being a real interval, exist special bases $\{E_i,\}$ in which the components W'_X of an S-derivation D along some or all vector fields X vanish. In other words, we are going to solve eq. (5′) with respect to A under certain conditions, which will be specified below.

II. Derivations along arbitrary vector fields

In this section we shall investigate the problems for existence and some properties of special bases $\{E_i,\}$ in which the components of a given S-derivation D_X along *arbitrary vector field* vanish along a path $\gamma: J \longrightarrow M$, with J being an $\mathbb{R}$-interval. Let's note that $\{E_i,\}$ are supposed to be defined *in a neighborhood* of $\gamma(J)$, while the components of D vanish *on* $\gamma(J)$.

Proposition 1. Along a path $\gamma: J \longrightarrow M$ the S-derivation is a linear connection if and only if along γ exists a basis in which the components of D along every vector field vanish along γ, i.e. on $\gamma(J)$.

Remark. We say that along γ D is a linear connection if in some, and hence in any, basis $\{E_i\}$, we have (cf. (7))

$$W_X(\gamma(s)) = \Gamma_k(\gamma(s)) X^k(\gamma(s)) \tag{12}$$

for some defined on $\gamma(J)$ matrix functions Γ_k and every vector field X, i.e. if (7) is valid for $x \in \gamma(J)$, but for $x \notin \gamma(J)$ eq. (7) may not be true.

Proof. Let along γ the derivation D be a linear connection, i.e. (12) to be valid. Let at first assume that γ is without self-

intersections and that $\gamma(J)$ is contained in only one coordinate neighborhood U in which some local coordinate basis $\{E_i = \partial/\partial x^i\}$ is fixed.

Due to (5′) we have to prove the existence of a matrix $A = \|A^i_i,\|, \|: = \|A^{i'}_i\|^{-1}$ for which in the basis $\{E_i, = A^i_i, E_i\}$ $W'_X(\gamma(s)) = 0$ to be fulfilled for every $X = X^k E_k$. Substituting (12) into (5′), we see that the last equation is equivalent to

$$\Gamma_k(\gamma(s)) A(\gamma(s)) + E_k(A)\big|_{\gamma(s)} = 0, \quad E_k = \partial/\partial x^k. \tag{13}$$

The general solution of this equation can be constructed as follows.

Let $V := J \times \cdots \times J$, where J is taken $n-1 = \dim(M) - 1$ times. Let us fix a one-to-one C^1 map $\eta: J \times V \longrightarrow M$ such that $\eta(\cdot, \underline{t}_0) = \gamma$ for some fixed $\underline{t}_0 \in V$, i.e. $\eta(s, \underline{t}_0) = \gamma(s)$, $s \in J$, which is possible iff γ is without selfintersections. In $U \cap \eta(J, V)$ we introduce coordinates $\{x^1\}$ by putting $(x^1(\eta(s, \underline{t})), \ldots, x^n(\eta(s, \underline{t}))) = (s, \underline{t})$, $s \in J$, $\underline{t} \in V$, which again is possible iff γ is without selfintersections.

If we expand $A(\eta(s, \underline{t}))$ into power series with respect to $(\underline{t} - \underline{t}_0)$, we find the general solution of (13) in the form

$$A(\eta(s, \underline{t})) = \{\mathbb{1} - \sum_{k=2}^{n} \Gamma_k(\gamma(s)) [x^k(\eta(s, \underline{t})) - x^k(\eta(s, \underline{t}_0))]\} Y(s, s_0; -\Gamma_1 \circ \gamma) \times$$

$$\times B(s_0, \underline{t}_0; \eta) + B_{k1}(s, \underline{t}; \eta) [x^k(\eta(s, \underline{t})) - x^k(\eta(s, \underline{t}_0))] \times$$

$$\times [x^1(\eta(s, \underline{t})) - x^1(\eta(s, \underline{t}_0))]. \tag{14}$$

Here: $\mathbb{1}$ is the unit matrix, $s_0 \in J$ is fixed, B is any nondegenerate matrix function of its arguments, B_{k1} and their derivatives are bounded matrix functions when $\underline{t} \longrightarrow \underline{t}_0$ and $Y = Y(s, s_0; Z)$, with Z being a continuous matrix function of s, is the unique solution of the matrix initial-value problem [7]

$$\frac{dY}{ds}=ZY,\quad Y|'_{s=s_0}=\mathbb{1}. \tag{15}$$

So, a matrix A, and consequently a basis $\{E_i\}$, with the needed property exist.

If $\gamma(J)$ doesn't lie in only one coordinate neighborhood, then by means of the above described method we can obtain local bases with the looked for property in different coordinate neighborhoods which form a neighborhood of $\gamma(J)$. From these local bases we can construct a global basis along γ with the needed property, but generally this basis will not be continuous in the regions of intersection of two coordinate neighborhoods. For example, if for some γ there doesn't exist coordinate neighborhood containing $\gamma(J)$ but there are coordinate neighborhoods U′ and U″ such that $\gamma(J)\subset(U')\cup(U'')$, then in U′ and U″ there are (see above) bases $\{E_i'\}$ and $\{E_i''\}$ in which the components of D_X vanish for every X along γ and a global basis $\{E_i^0\}$ in $(U')\cup(U'')$ with the same property may be obtained by putting $E_i^0|_x=\delta_i^{i'}E_{i'}|_x$ for $x\in U'$ and $E_i^0|_x=\delta_i^{i''}E_{i''}|_x$ for $x\in U''\setminus U'$ (note that $(U'')\cap(U')$ is not empty as γ is C^1 path).

Analogously, if γ has selfintersections, then on any "part" of γ without selfintersections exist bases with the needed property. From these bases can be constructed a global basis with the same property along γ. (At the points of selfintersections of γ we can arbitrary fix these bases to be the bases obtained above for some fix part of γ which is without selfintersections.)

Consequently, if along γ D is a linear connection, then in a neighborhood of $\gamma(J)$ a basis $\{E_i'\}$ exists in which the components of D_X for every X vanish along γ.

On the opposite, let us assume the existence of a basis $\{E_i'\}$ in which $W'_X=0$ for every X. Fixing some basis $\{E_i\}$ such that

$E_{i'}=A^i_{i'}E_i$, from (5′) we find $(W_XA+X(A))|_{\gamma(s)}=0$ or $W_X(\gamma(s))=-[(X(A))A^{-1}]|_{\gamma(s)}$ which means that (12) is valid for $\Gamma_k(\gamma(s))=-[(E_k(A))A^{-1}]|_{\gamma(s)}$.■

Proposition 2. All bases in which along a path γ the components of an S-derivation along every vector field vanish are obtained along γ from one another by linear transformations whose coefficients are such that the action of the vectors from these bases on them vanish along γ (i.e. on $\gamma(J)$).

Proof. If $\{E_i\}$ and $\{E_{i'}\}$ are such bases, then $W'_X(\gamma(s))=W_X(\gamma(s))=0$. So, from (5′) follows $X(A)|_{\gamma(s)}=0$ for every $X=X^kE_k$, i.e. $E_k(A)|_{\gamma(s)}=0$.■

Proposition 3. If along a path $\gamma:J\longrightarrow M$ for some S-derivation D there is a local *holonomic* (on $\gamma(J)$) basis in which the components of D along every vector field vanish on $\gamma(J)$, then D is torsion free on $\gamma(J)$. On the opposite, if D is torsion free on $\gamma(J)$ and a continuous basis with the mentioned property along γ exists, then all bases with the same property are holonomic along γ.

Remark. In the second part of the proposition we say that the basis must be continuous. This is necessary as any holonomic basis is such.

Proof: If $\{E_{i'}\}$ is a basis with the mentioned property, i.e. $W'_X(\gamma(s))=0$ for every X and $s\in J$, then, using (8), we find $T^D(E_{i'},E_{j'})|_{\gamma(s)}=-[E_{i'},E_{j'}]|_{\gamma(s)}$ and consequently $\{E_{i'}\}$ is holonomic at $\gamma(s)$, i.e., $[E_{i'},E_{j'}]|_{\gamma(s)}=0$, iff $0=T^D(X,Y)|_{\gamma(s)}=X^{i'}(\gamma(s))Y^{j'}(\gamma(s))(T^D(E_{i'},E_{j'})|_{\gamma(s)})$ (see proposition 1 and (12)) for every vector fields X and Y, which is equivalent to $T^D|_{\gamma(J)}=0$.

On the opposite, let $T^D|_{\gamma(J)}=0$. We want to prove that any basis $\{E_{i'}\}$ along γ in which $W'_X(\gamma(s))=0$ is holonomic at $\gamma(s)$, $s\in J$. The holonomicity at $\gamma(s)$ means $0=[E_{i'},E_{j'}]|_{\gamma(s)}=[-A^{k'}_k(E_{j'}(A^k_{i'})-$

$-E_{i'}(A^k_{j'},))E_{k'},]\big|_{\gamma(s)}$. But (see proposition 1) the existence of $\{E_{i'},\}$ is equivalent to $W_X(\gamma(s))=(\Gamma_k X^k)\big|_{\gamma(s)}$ for every X. These two facts, combined with (8), show that $(\Gamma_k)^i_{.j}=(\Gamma_j)^i_{.k}$. Using this and $(\Gamma_k A+\partial A/\partial x^k)\big|_{\gamma(s)}=0$ (see the proof of proposition 1), we find $E_{j'}(A^k_{i'})\big|_{\gamma(s)}=-A^j_{j'}A^i_{i'}(\Gamma_j)^k_{.i}\big|_{\gamma(s)}=E_{i'}(A^k_{j'})\big|_{\gamma(s)}$ and therefore $[E_{i'},E_{j'}]\big|_{\gamma(s)}=0$ (see above), i.e $\{E_{i'},\}$ is holonomic on $\gamma(J)$.■

It can be proved (see below lemma 7) that for any path $\gamma:J\longrightarrow M$ every basis $\{E^\gamma_i\}$ defined only on $\gamma(J)$ can be extended to a *holonomic* basis $\{E^h_i\}$ defined in a neighborhood of $\gamma(J)$ and such that $E^h_i|_{\gamma(J)}=E^\gamma_i$. In particular, this is true for the restriction $E^\gamma_{i'}=E_{i'}|_{\gamma(J)}$ of the considered above special bases $\{E_{i'},\}$. But in the general case, the extended holonomic bases $\{E^h_{i'},\}$ will not have the special property that $\{E_{i'},\}$ has.

III. DERIVATIONS ALONG PATHS

Let $\gamma:J\longrightarrow M$, J being an $\mathbb{R}$-interval, be a C^1 path and X be a C^1 vector field defined on a neighborhood of $\gamma(J)$ in such a way that on $\gamma(J)$ it reduces to the tangent to γ vector field $\dot\gamma$, i.e. $X|_{\gamma(s)}=\dot\gamma(s)$, $s\in J$. The restriction of an S-derivation D_X on $\gamma(J)$ we shall call (S-)derivation along γ and denote it by $\mathcal{D}^\gamma$. Of course, $\mathcal{D}^\gamma$ generally depends on the values of X outside $\gamma(J)$, but, as this dependence is insignificant for the following, it will not be written explicitly. So, if T is a C^1 tensor field in a neighborhood of $\gamma(J)$, then

$$(\mathcal{D}^\gamma T)\big|_{\gamma(s)}:=\mathcal{D}^\gamma_s T:=(D_X T)\big|_{\gamma(s)},\quad X|_{\gamma(s)}=\dot\gamma(s). \tag{16}$$

It is easily seen that $\mathcal{D}^\gamma_s T$ depends only on the values of $T|_x$ for $x\in\gamma(J)$, but not on the ones for $x\notin\gamma(J)$, and also that $\mathcal{D}^\gamma$ is a

generalization of the usual covariant differentiation along curves (see [2-4] or Sect. V).

When restricted on $\gamma(J)$, the components of D_X will be called components of $\mathcal{D}^\gamma$.

Proposition 4. Along any C^1 path $\gamma:J\longrightarrow M$ there exists a basis $\{E_{i'},\}$ in which the components of a given S-derivation $\mathcal{D}^\gamma$ along γ are zeros.

Proof. Let $\{E_i\}$ be some fixed basis in a neighborhood of $\gamma(J)$. We have to prove the existence of transformation $E_i\longrightarrow E_{i'}=A^i_{i'}E_i$ such that $W'_X|_{\gamma(J)}=0$, which, by (5'), is equivalent to the existence of a matrix function $A=\|A^i_{i'}\|$ satisfying along γ the equation $(A^{-1}(W_X A+X(A)))\big|_{\gamma(J)}=0$, $s\in J$, or

$$\dot\gamma(A)\big|_{\gamma(s)}\equiv\frac{dA(\gamma(s))}{ds}=-W_X(\gamma(s))A(\gamma(s)) \tag{17}$$

as $X|_{\gamma(s)}=\dot\gamma(s)$. Along γ the general solution of this equation with respect to A is

$$\underline{A}(s;\gamma)=Y(s,s_0;-W_X\circ\gamma)B(\gamma), \tag{18}$$

where Y is the unique solution of the initial-value problem (15), $s_0\in J$ is fixed and $B(\gamma)$ is a nondegenerate matrix function of γ.

So, if we take any matrix function A with the property $A(\gamma(s))=\underline{A}(s;\gamma)$ for some s_0 and B (e.g. using the notation of the proof of proposition 1, in any coordinate neighborhood in which γ is without selfintersections, we can put $A(\eta(s,\underline{t}))=Y(s,s_0;-W_X\circ\gamma)\times$ $\times B(s_0,\underline{t}_0,\underline{t};\gamma)$ for a fixed nondegenerate matrix function B), we see that A carries out the needed transformation. Hence, the basis $E_{i'}=A^i_{i'}E_i$ is the looked for one.■

Proposition 5. The bases along $\gamma:J\longrightarrow M$ in which the components of $\mathcal{D}^\gamma$ vanish are obtained from one another by linear transformations whose coefficients on $\gamma(J)$ are constant or may depend only on γ.

Proof. If $\{E_i\}$ and $\{E_{i'}\}$ are such bases, then $W_X(\gamma(s))=W'_X(\gamma(s))=0$, $X|_{\gamma(s)}=\dot\gamma(s)$ and from (5′) follows $\dot\gamma(A)|_{\gamma(s)}=dA(\gamma(s))/ds=0$, i.e. $A(\gamma(s))$ is a constant or depends only on the map γ.■

From propositions 4 and 5 we infer that the requirement for the components of $\mathcal{D}^\gamma$ to vanish along a path γ determines the corresponding special bases with some arbitrariness only on $\gamma(J)$ and leaves them absolutely arbitrary outside the set $\gamma(J)$. For this reason we can talk for bases defined only on $\gamma(J)$ in which the components of $\mathcal{D}^\gamma$ vanish.

Proposition 6. Let for a C^1 path $\gamma:J\longrightarrow M$ the defined on $\gamma(J)$ basis $\{E_{i'}\}$ be such that the components of some S-derivation $\mathcal{D}^\gamma$ along γ vanish. Let U be a coordinate neighborhood such that on $U\cap(\gamma(J))\neq\emptyset$ the path γ is without selfintersections. Then there is a neighborhood of $U\cap(\gamma(J))$ in U in which $\{E_{i'}\}$ can be extended to a coordinate basis, i.e. in this neighborhood exist local coordinates $\{x^{i'}\}$ such that $E_{i'}|_{\gamma(s)}=\partial/\partial x^{i'}|_{\gamma(s)}$.

Remark. Said in other words, this proposition means that locally any spacial on $\gamma(J)$ basis for $\mathcal{D}^\gamma$ can be thought (extended) as a coordinate, and hence holonomic, one (see proposition 5).

Proof. The proposition is a trivial corollary from the proof of proposition 4 and the following lemma.

Lemma 7. Let the path $\gamma:J\longrightarrow M$ be without selfintersections and such that $\gamma(J)$ is contained in some coordinate neighborhood U, i.e. $\gamma(J)\subset U$, and let $\{E_{i'}\}$ be defined on $\gamma(J)$ continuous basis, i.e. $E_{i'}|_{\gamma(s)}$ depends continuously on s. Then there is a neighborhood of $\gamma(J)$ in U in which coordinates $\{x^{i'}\}$ exists such that $E_{i'}|_{\gamma(s)}=\partial/\partial x^{i'}|_{\gamma(s)}$, i.e. $\{E_{i'}\}$ can be extended in it to a coordinate basis.■

Proof of lemma 7. Let us take a one-to-one C^1 map $\eta:J\times V\longrightarrow U$,

$V:=J\times\cdots\times J$ (n-1 times), such that $\eta(\cdot,\underline{t}_0)=\gamma$ for some fixed $\underline{t}_0\in V$, i.e. $\eta(s,\underline{t}_0)=\gamma(s)$, $s\in J$ (cf. the proof of proposition 1). In the neighborhood $\eta(J,V)\subset U$ we introduce coordinates $\{x^i\}$ by putting $(x^1(\eta(s,\underline{t})),\dots,x^n(\eta(s,\underline{t})))=(s,\underline{t})$, $s\in J$, $\underline{t}\in V$. Let the nondegenerate matrix $\|A^i_{j'}(s;\gamma)\|=:\|A^{i'}_j(s;\gamma)\|^{-1}$ defines the decomposition of $\{E_{i'}\}$ over $\{\partial/\partial x^i\}$, i.e.

$$E_{j'}|_{\gamma(s)}=A^j_{j'}(s;\gamma)\left(\frac{\partial}{\partial x^j}\Big|_{\gamma(s)}\right). \tag{19}$$

If we define the functions $x^{i'}:\eta(J,V)\longrightarrow\mathbb{R}$ by

$$x^{i'}(\eta(s,\underline{t})):=\delta^{i'}_i x^i_0+\int_{s_0}^{s}A^{i'}_1(u;\gamma)du+A^{i'}_j(s;\gamma)[x^j(\eta(s,\underline{t}))-x^j(\gamma(s))]+$$

$$f^{i'}_{jk}(s,\underline{t};\gamma)[x^j(\eta(s,\underline{t}))-x^j(\gamma(s))][x^k(\eta(s,\underline{t}))-x^k(\gamma(s))] \tag{20}$$

in which $s_0\in J$ and $x_0\in V$ are fixed and the functions $f^{i'}_{jk}$ together with their derivatives are bounded when $\underline{t}\longrightarrow\underline{t}_0$, then, because of $\eta(\cdot,\underline{t}_0)=\gamma$, we find

$$\frac{\partial x^{i'}}{\partial x^j}\Big|_{\gamma(s)}=\frac{\partial x^{i'}}{\partial x^j}\Big|_{\eta(s,\underline{t}_0)}=A^j_{j'}(s;\gamma). \tag{21}$$

As $\det\|A^i_{j'}(s;\gamma)\|\neq0,\infty$, from (21) follows that the transformation $\{x^i\}\longrightarrow\{x^{i'}\}$ is a diffeomorphism in some lying in U neighborhood of $\gamma(J)$. So, in this neighborhood $\{x^{i'}\}$ are local coordinates, the basic vectors of which on $\gamma(J)$ are (see (21) and (19))

$$\frac{\partial}{\partial x^{j'}}\Big|_{\gamma(s)}=\left(\frac{\partial x^i}{\partial x^{j'}}\Big|_{\gamma(s)}\right)\frac{\partial}{\partial x^i}\Big|_{\gamma(s)}=A^i_{j'}(s;\gamma)\frac{\partial}{\partial x^i}\Big|_{\gamma(s)}=E_{j'}|_{\gamma(s)}.$$

Hence $\{x^{i'}\}$ are the coordinates we are looking for.■

Lemma 7 has also a separate meaning: according to it any locally continuous basis defined on $\gamma(J)$ can locally be extended to a *holonomic* basis in a neighborhood of $\gamma(J)$. Evidently, such an extension can be done and in an anholonomic way. Consequently, the holonomicity problem for a basis defined only on $\gamma(J)$ depends on the way this basis is extended in a neighborhood of $\gamma(J)$.

IV. DERIVATIONS ALONG FIXED VECTOR FIELD

Results, analogous to the ones of Sect. II, are true and with respect to S-derivations D_X along a *fixed vector field* X, i.e. (see Sect. I) for a fixed derivation. As this case is not very interesting from the view-point of its application, we shall present below without proofs only three results about it.

Proposition 8. Along a path $\gamma: J \longrightarrow M$ the S-derivation D_X along a fixed vector field X is a linear connection, i.e. (12) is valid for that fixed X, iff along γ exists basis $\{E_{i'}\}$ in which the components of D_X vanish on $\gamma(J)$.

Proposition 9. The bases along γ in which the components of D_X for a fixed X vanish on $\gamma(J)$ are connected by linear transformations whose matrices are such that the action of X on them vanish on $\gamma(J)$.

In a case of D_X for a fixed X the analog of proposition 3 is, generally, not true. But if for D_X, X being fixed, is valid (16) on $\gamma(J)$, then we can construct a class of S-derivations $\{'D\}$ whose components for every X are given by (12). Evidently, for these derivations the proposition 3 holds, i.e. we have

Proposition 10. If along γ for D_X, X being fixed, is valid (12) and there is a local holonomic (on $\gamma(J)$) basis in which the components of D_X vanish on $\gamma(J)$, then the above described derivations $'D$ are torsion free on $\gamma(J)$. Vice versa, if $'D$ are torsion free on $\gamma(J)$ and there exists a continuous bases in which the components of D_X vanish, then between them exist holonomic ones, but generally not all of them are such.

V. THE CASE OF LINEAR CONNECTIONS

Here we shall apply the results from the previous sections to the special case of a linear connection ∇. As the corresponding proofs are almost evident, they will be omitted.

Corollary 11. Along every path $\gamma: J \longrightarrow M$ for any linear connection ∇ there exist bases in which the components of ∇ vanish on $\gamma(J)$. These bases are connected with one another in the way described in proposition 2.

Corollary 12. One, and hence any, continuous basis along a path $\gamma: J \longrightarrow M$ in which the components of a linear connection ∇ vanish on $\gamma(J)$ is holonomic if and only if ∇ is torsion free on $\gamma(J)$.

Remark. When γ is without selfintersections and $\gamma(J)$ lies in only one coordinate neighborhood, then holonomic bases (coordinates) in which the components of ∇ vanish on $\gamma(J)$ exist if ∇ is *torsion free* and vice versa, which is a well known fact; cf. [1-3, 7].

Corollary 13. For a torsion free linear connection ∇ along any path $\gamma: J \longrightarrow M$ without selfintersections and lying in only one coordinate neighborhood there exist coordinates, or equivalently holonomic bases, in which the components of ∇ vanish on $\gamma(J)$.

Remark. This corollary reproduces a classical theorem that can be found, for instance, in [3] and in [2], ch. III, §8, in the latter of which references to original works are given.

Corollary 14. If $\frac{D}{ds}|_\gamma := \nabla_{\dot\gamma}$ is the associated to ∇ covariant derivative along the C^1 path $\gamma: J \longrightarrow M$, then on $\gamma(J)$ exist bases, obtained from one another by linear transformations whose coefficients are constant or depend only on γ, in which the

components of $'\nabla_{\dot{\gamma}}$ vanish (on $\gamma(J)$). If γ is without selfintersections and $\gamma(J)$ lies in only one coordinate neighborhood, then in some neighborhood of $\gamma(J)$ all of these bases could be extended in a holonomic way. These results generalize a series of analogous ones, which began with [4] and concern linear connections.

VI. CONCLUSION

The above investigation shows that under sufficiently general conditions for derivations of the tensor algebra over a differentiable manifold M along a path $\gamma: J \longrightarrow M$ generally anholonomic bases exist in which the derivation's components vanish on $\gamma(J)$. It is important to be noted that when the derivations are along paths, then the corresponding special bases always can be taken as holonomic (or coordinate) ones.

A feature of the considered here case along paths is its independence of the derivations's curvature, which even wasn't introduced here. The cause for this is the one dimensionality of the paths as submanifolds of M. In connection with this it is interesting the same problems to be investigated as here but on arbitrary submanifolds of M, which will be done elsewhere.

At the end we want to say a few words for the relation of the obtained in this paper results with the equivalence principle [4, 5], according to which the gravitational field strength, usually identified with the components of some linear connection, is locally transformable to zero by an appropriate choice of the local (called inertial, normal, Riemannian or Lorentz) reference frame (basis or coordinates). So, from mathematical point of view,

the equivalence principle states the existence of local bases in which the corresponding connection's components vanish. The results of this investigation show the strict validity of this statement along any path (on any curve). Hence, we can make the following two conclusions: Firstly, any gravity theory based on space-time with a linear connection is compatible with the equivalence principle along every path, i.e. in it exist (local) inertial frames along paths; these frames are generally anholonomic, but under some, not very restrictive from physical point of view, conditions on the paths (see lemma 7) there exist such holonomic frames of reference. Secondly, in the gravity theories based on linear connections the equivalence principle must not be considered as a principle as it is identically fulfilled because of their mathematical background.

ACKNOWLEDGEMENTS

The author is grateful to Professor Stancho Dimiev (Institute of Mathematics of Bulgarian Academy of Sciences) and to Dr. Sava Manoff (Institute for Nuclear Research and Nuclear Energy of Bulgarian Academy of Sciences) for the valuable comments and stimulating discussions. He thenks Professor N. A. Chernikov (Joint Institute for Nuclear Research, Dubna, Russia) for the interest of the problems put in this work.

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Received by Publishing Department

on December 7, 1992.