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SPECIAL BASES FOR DERIVATIONS OF TENSOR ALGEBRAS. I. CASES IN A NEIGHBORHOOD AND AT A POINT

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Специальные базисы для дифференцирований тензорных алгебр.
I. Случаи в окрестности и в точке

Исследованы необходимые и достаточные условия для существования локальных базисов, в которых равны нулю компоненты дифференцирования тензорных алгебр над дифференцируемым многообразием в окрестности и в отдельной точке. Рассмотрена проблема о том, когда эти базисы голономны или неголономны. Обращено внимание на случай линейных связностей. Показаны связи этих вопросов с принципом эквивалентности.

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Special Bases for Derivations of Tensor Algebras.
I. Cases in a Neighbourhood and at a Point

Necessary and sufficient conditions are investigated for the existence of local bases in which the components of derivations of tensor algebras over a differentiable manifold vanish in a neighbourhood or only at a single point. The problem when these bases are holonomic or anholonomic is considered. Attention is paid to the case of linear connections. Relations of these problems with the equivalence principle are pointed out.

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I. INTRODUCTION

In the theory of linear connections [1, 2] problems connected with existence of (local) bases in which the connection's components vanish at a point [2-7], along a curve [3, 6] or in a neighborhood [3, 5, 6] have been considered. But with a very rare exceptions (see e.g. [7]) in the literature only the torsion free case has been investigated. The aim of the present work is generalization of these problems and their results to the case of arbitrary derivations of the tensor algebra over a differentiable manifold (see [2] or Sec. II), the curvature and torsion of which (as defined below in Sec. II) are not a priori restricted somehow.

Mathematically the main purpose of this work is necessary and sufficient conditions to be found for the existence of local (holonomic or anholonomic [3]) bases (coordinates) in which the components of some derivation (of the tensor algebra over a manifold) must vanish. If such bases exist, we investigate the problem when they (or part of them) are holonomic.

Physically the goal of the paper is to be shown that in gravity theories, based, first of all, on linear connections, the equivalence principle is identically satisfied because of their underlying mathematical structure.

The work is organized as follows. In Sec. II some notations and definitions are introduced. Sec. III deals with the above pointed problems in a neighborhood and Sec. IV investigates them at a single point. In Sec. V the connection of the considered mathematical problems with the equivalence principle is shown. Sec. VI contains some concluding remarks.

II. DERIVATIONS, THEIR COMPONENTS, CURVATURE AND TORSION

Let D be a derivation of the tensor algebra over a given manifold M [1, 2]. By proposition 3.3 from ch. I from [2] there exists unique vector field X and unique tensor field S of type (1,1) such that $D=L_X+S$, where L_X is the Lie derivative along X [1, 2] and S is considered as a derivation of the tensor algebra over M [2].

If S is a map from the set of C^1 vector fields into the tensor fields of type (1,1) and $S:X\longmapsto S_X$, then the equation

$$D^S_X=L_X+S_X \tag{1}$$

defines a derivation of the tensor algebra over M for any C^1 vector field X [2]. As the map S will hereafter be assumed fixed, such a derivation will be called an S-derivation along X and will be denoted by D_X. An S-derivation is a map D such that $D:X\longmapsto D_X$, where D_X is an S-derivation along X.

Let $\{E_i,\ i=1,\dots n:=\dim(M)\}$ be a (coordinate or not [3, 4]) local basis of vector fields in the tangent to M bundle. It is holonomic (anholonomic) if the vectors $E_1, \dots, E_n$ commute (don't commute) [3, 4]. Let T be a C^1 tensor field of type (p,q), p and q being integers or zero(s), with local components $T^{i_1\dots i_p}_{j_1\dots j_q}$ with respect to the tensor basis associated to $\{E_i\}$. Here and below all latin indices, may be with some subscripts, run from 1 to $n:=\dim(M)$. Using the explicit action of L_X and S_X on tensor fields [2] and the usual summation rule on repeated on different levels indices, we find the components of D_XT to be

$$(D_XT)^{i_1\dots i_p}_{j_1\dots j_q}=X\left(T^{i_1\dots i_p}_{j_1\dots j_q}\right)+\sum_{a=1}^{p}(W_X)^{i_a}_{..k}T^{i_1\dots i_{a-1}ki_{a+1}\dots i_p}_{j_1\dots j_q}+$$

$$+\sum_{b=1}^{q}(W_X)^{k}_{.j_b}T^{i_1\dots i_p}_{j_1\dots j_{b-1}kj_{b+1}\dots j_q}, \tag{2}$$

where with $X(f)$ we denote the action of $X=X^kE_k$ on the C^1 scalar function f, i.e. $X(f):=X^kE_k(f)$, and

$$(W_X)^i_{.j}:=(S_X)^i_{.j}-E_j(X^i)+C^i_{.kj}X^k \tag{3}$$

in which $C^i_{.kj}$ defines the commutators of the basic vectors, i.e.

$$[E_j,E_k]=C^i_{.jk}E_i. \tag{4}$$

By definition we shall call $(W_X)^i_{.j}$ components of D_X as they play with respect to D_X the same role as the components of a linear connection with respect to itself. In fact, from (2) we see that (cf. (7) below)

$$D_X(E_j)=(W_X)^i_{.j}E_i. \tag{5}$$

If we make the change $\{E_i\}\longrightarrow\{E_{i'}:=A^i_{i'}E_i\}$, where $A:=\|A^i_{i'}\|=:\|A^{i'}_i\|^{-1}$ is a nondegenerate matrix function, then from (5) we can see that $(W_X)^i_{.j}$ transform into

$$(W_X)^{i'}_{..j'}=A^{i'}_iA^j_{j'}(W_X)^i_{.j}+A^{i'}_iX(A^i_{j'}), \tag{6}$$

which, if we introduce the matrices $W_X:=\|(W_X)^i_{.j}\|$ and $W'_X:=\|(W_X)^{i'}_{..j'}\|$, will read

$$W'_X=A^{-1}\left(W_XA+X(A)\right), \tag{6'}$$

where as a first matrix index is understand the superscript and $X(A):=X^kE_k(A)=\|X^kE_k(A^i_{i'})\|$.

If ∇ is a linear connection with local components $\Gamma^i_{.jk}$ (see, e.g., [1-3]), then it is fulfilled

$$\nabla_X(E_j)=(\Gamma^i_{.jk}X^k)E_i. \tag{7}$$

Hence, comparing (5) and (7), we see that D_X is a covariant differentiation along X iff

$$(W_X)^i_{.j}=\Gamma^i_{.jk}X^k, \tag{8}$$

for some functions $\Gamma^i_{.jk}$. Due to (3) or (1), a linear connection ∇ is characterized by the map $S:X\longmapsto S_X$ such that

$$S_X=\Sigma_X,\quad \Sigma_X(Y):=\nabla_X(Y)-[X,Y], \tag{9}$$

where $[X,Y]=L_XY$ is the commutator of the vector fields X and Y [2].

Let D be an S-derivation and X, Y and Z be vector fields. The curvature operator R^D and the torsion operator T^D of D are

$$R^D(X,Y):=D_XD_Y-D_YD_X-D_{[X,Y]}, \tag{10}$$

$$T^D(X,Y):=D_XY-D_YX-[X,Y], \tag{11}$$

which in a case of linear connection reduce to the corresponding classical tensorial quantities (see [2, 3] and below (16) and (17)). The S-derivation D will be called flat (=curvature free) or torsion free if $R^D=0$ or $T^D=0$ respectively (cf. [2]).

If we use the representation (1) and $L_XY=[X,Y]$, we get

$$(R^D(X,Y))Z=\{S_XS_Y-S_YS_X+L_X(S_Y)-L_Y(S_X)\}Z+S_Y[X,Z]-S_X[Y,Z]-S_{[X,Y]}Z \tag{12}$$

$$T^D(X,Y)=S_XY-S_YX+[X,Y]. \tag{13}$$

Analogously, by means of (2), we find the local expressions:

$$(R^D(X,Y))^i_{.k}=X((W_Y)^i_{.k})-Y((W_X)^i_{.k})+(W_X)^i_{.l}(W_Y)^l_{.k}-(W_Y)^i_{.l}(W_X)^l_{.k}-$$

$$-(W_{[X,Y]})^i_{.k}, \tag{14}$$

$$(T^D(X,Y))^i=(W_X)^i_{.l}Y^l-(W_Y)^i_{.l}X^l-C^i_{.kl}X^kY^l. \tag{15}$$

If we introduce the matrix $\underline{R}^D(X,Y):=\|(R^D(X,Y))^i_{.k}\|$, then (14) takes the form

$$\underline{R}^D(X,Y)=X(W_Y)-Y(W_X)+W_XW_Y-W_YW_X-W_{[X,Y]}. \tag{14'}$$

If D is a linear connection ∇, then, due to (8), we have:

$$(R^\nabla(X,Y))^i_{.j}=R^i_{.jkl}X^kY^l, \tag{16}$$

$$(T^\nabla(X,Y))^i=T^i_{.kl}X^kY^l, \tag{17}$$

where [1-6]

$$R^i_{.jkl}:=-E_l(\Gamma^i_{.jk})+E_k(\Gamma^i_{.jl})-\Gamma^m_{.jk}\Gamma^i_{.ml}+\Gamma^m_{.jl}\Gamma^i_{.mk}-\Gamma^i_{.jm}C^m_{.kl} \tag{18}$$

$$T^i_{.kl}:=-(\Gamma^i_{.kl}-\Gamma^i_{.lk})-C^i_{.kl}, \tag{19}$$

are the components of the curvature and torsion tensors of ∇.

In the next sections we shall look for special bases $\{E_{i'}\}$ in which the components W'_X of an S-derivation D vanish along some or along all vector fields X. Evidently, for this purpose we shall have to solve (6') with respect to A under certain conditions.

III. SPECIAL BASES FOR DERIVATIONS IN A NEIGHBORHOOD

In this section we shall solve the problems for existence, uniqueness and holonomicity of basis or bases $\{E_{i'}\}$ in which the components of a given (S-)derivation vanish in some neighborhood U.

Proposition 1: The following three statements are equivalent:

(a) In U the S-derivation D is a flat linear connection.

(b) D is in U curvature free, i.e. $R^D=0$, and $D_X|_{X=0}=0$.

(c) For D in U exists basis $\{E_{i'}\}$ such that $W'_X=0$ for every X.

Proof: We shall proof this proposition according to the implications (a)⇒(b)⇒(c)⇒(a).

If (a) is true, then (8) and (16) take place, from where, evidently, follows (b) as the flatness of ∇ means $R^i_{.jkl}=0$ in U.

Let (b) be valid. The existence of $\{E_{i'}\}$, in which $W'_X=0$, is equivalent to the existence of a matrix $A:=\|A^i_{i'}\|$ transforming $\{E_i\}$ into $\{E_{i'}\}$ and such that (see (6'))

$$0=W'_X=A^{-1}(W_X A+X(A)), \qquad (20)$$

i.e. A must be a solution of $X(A)=-W_X A$ for every vector field X. The integrability conditions for this equation are $0=[X,Y]A=X(Y(A))-$ $-Y(X(A))$ for all commuting vector fields X and Y, i.e. $[X,Y]=0$. Using $X(A)=-W_X A$ it can easily be calculated that

$$[X,Y]A=-(R^D(X,Y)+W_{[X,Y]})A. \qquad (21)$$

So, if (b) is valid, then, due to (1) and (3), we get $W_X|_{x=x_0}=0$, therefore the integrability conditions for (20) are satisfied and, consequently, the above pointed transformation exists, i.e. in $\{E_{i'}\}$ we have $W'_X=0$. Hence, from (b) follows (c).

Let (c) be fulfilled. If we arbitrary fix some basis $\{E_i\}$, then the existence of $\{E_{i'}\}$, in which $W'_X=0$, is equivalent to the existence of matrix A transforming $\{E_i\}$ into $\{E_{i'}\}$ and which, due to (6'), is such that (20) is valid. From this equation, we get $W_X=-(X(A))A^{-1}$, i.e. $(W_X)^i_{.j}=-[X^k(E_k(A^i_{i'}))]A^{i'}_j$ which, due to (8), means that the S-derivation D is a linear connection with local components $\Gamma^i_{.jk}=-(E_k(A^i_{i'}))A^{i'}_j$. If in (14') we substitute $W_X=$ $=-(X(A))A^{-1}$ and use $X(A^{-1})=-A^{-1}(X(A))A^{-1}$, we get $R^D=R^\nabla=0$, i.e. D is a flat linear connection. So, from (c) follows (a).■

The main consequence from proposition 1 is that the question for the existence and the properties of special bases in which the components of an S-derivation along every vector field to be zeros in a neighborhood is equivalent to that question for (flat) linear connections as it is expressed by

Corollary 2: In a neighborhood there exists a basis in which the components of an S-derivation along every vector field vanish

iff $R^D=0$ and $D_X|_{x=x_0}=0$, or iff D is a flat linear connection (whose components vanish in the same basis).

The following two propositions are almost evident (see (6') and (15) respectively):

Proposition 3: All local bases in which the components of a flat S-derivation along every vector field vanish in a neighborhood are obtained from one another by linear transformations with constant coefficients.

Proposition 4: The local bases in which the components of a flat S-derivation along every vector field vanish in a neighborhood are all holonomic or all anholonomic iff the S-derivation has zero or nonzero torsion respectively in that neighborhood.

Remark: A stronger result is that the mentioned bases are all holonomic or anholonomic at a given point iff the torsion vanishes or is not a zero respectively at that point. We consider only the above statement as it is the most widely used one of that kind.

Now we shall make some conclusions concerning *linear connections*.

Corollary 5: In a neighborhood there exists a basis in which the components of a linear connection vanish if and only if this connection is flat in that neighborhood.

Remark 1: If the connection is torsion free, this is an old classical result that can be found, e.g., in §106 from [6], in [3], p.142, or in [5].

Remark 2: In [7] an analogous statement is pointed out in the U_4 gravity theory, which states that the U_4 connection components can "always be transformed to zero with respect to a suitable anholonomic system in U_4". This statement suffers from two defects: firstly, generally it is not valid in a neighborhood unless the U_4 connection is not flat, a condition which is not even mentioned in [7], and, secondly, in [7] one finds a "proof" of the cited statement not in a neighborhood, but at a fixed point, which in fact is not a real proof but only a hint for it as it is a simple counting of the number of conditions which must be satisfied by some independent quantities.

Proof: If there is a basis $\{E_{i'}\}$ in which the components of a linear connection ∇ are $\Gamma^{i'}{}_{..j'k'}=0$, then, in accordance with (18), we have $R^{i'}{}_{..j'k'l'}=0$, i.e ∇ is flat. On the opposite, let ∇ be flat. Then the S-derivation D defined by $D_X:=\nabla_X$ has components $W_X=\Gamma_k X^k$ at every point (see (8)) and, due to (14) and (16), is also flat as $R^D=R^\nabla=0$ and besides $D_X|_{x=0}=\nabla_X|_{x=0}=0$. Consequently, by corollary 2, there exists a basis $\{E_{i'}\}$ such that $W'_X=0$ for every X. But $W'_X=\Gamma_{k'}X^{k'}$, so that $\Gamma_{k'}=0$, i.e. $\Gamma^{i'}{}_{..j'k'}=0$.■

In the proof of proposition 1 we constructed a bases in which the components of a flat S-derivation vanish by proving that the equation $X(A)+W_X A=0$, X being arbitrary, is integrable with respect to A, and hence $W'_X=0$ for $E_{i'}=A^i_{i'}E_i$. But as D is in this case a (flat) linear connection ∇ (see corollary 2), we have $W_X=\Gamma_k X^k$ for every X. So, A is a solution of $\Gamma_k A+E_k(A)=0$, which doesn't depend on X, or (see (6')) $\Gamma'_k=A^{-1}(\Gamma_k A+E_k(A))=0$, i.e. in $\{E_{i'}\}$ the components of ∇ vanish. In this way one can construct a basis, which generally is anholonomic, and in which the components of a flat linear connection vanish.

Corollary 6: All local bases in which the components of a flat linear connection vanish in a neighborhood are obtained from one another by linear transformations with constant coefficients.

Proof: The result follows from (8) and proposition 3.■

Corollary 7: If in a neighborhood there exists a holonomic local basis in which the components of a linear connection vanish, then this connection is torsion free in that neighborhood. On the

opposite, if a flat linear connection is torsion free in some neighborhood, then all bases in this neighborhood in which the connection's components vanish are holonomic in it.

Remark: In different modifications this result can be found, for instance, in [2, 3, 5, 6].

Proof: This result is a consequence from (8), (17), (18) and proposition 4.■

The question for the existence and the properties of basis (bases) in which the components of an S-derivation D_X along a *fixed* vector field X vanish in a neighborhood is not so interesting as the considered until now problem. That is why we shall only sketch briefly the existence of such bases in the case $X|_x\neq 0$ for every x from the neighborhood.

If $\{E_{i'}=A^i_{i'}E_i\}$ is the looked for basis with the needed property, $W'_X=0$, then its existence, due to (6'), is equivalent to the existence of $A:=\|A^i_{i'}\|$ satisfying $W_X A+X(A)=0$ for a given X. As X is fixed, the values of A at different points are connected through the last equation iff these points lie on one and the same integral curve (path) of X. So, if $\gamma_y:J\longrightarrow M$ (J being an $\mathbb{R}$ interval) is the integral curve for X passing through $y\in M$, i.e. $\gamma_y(s_0)=y$ and $\dot\gamma_y(s)=X|_{\gamma_y(s)}$, $\dot\gamma_y$ being the tangent to γ_y vector field, for $s\in J$ and a fixed $s_0\in J$, then the equation $W_X A+X(A)=0$ along γ_y reduces to $(dA/ds)|_{\gamma_y(s)}=-W_X(\gamma_y(s))A(\gamma_y(s))$. The general solution of this equation is

$$A(\gamma_y(s))=Y(s,s_0;-W_X\circ\gamma_y)B(\gamma_y), \tag{22}$$

in which $Y=Y(s,s_0;Z)$, Z being a matrix function of s, is the unique solution of the initial-value problem [8]

$$\frac{dY}{ds}=ZY, \quad Y|_{s=s_0}=\mathbb{1} \tag{23}$$

and the nondegenerate matrix B doesn't depend on s.

Therefore along the integral curves of X, bases exist in which the components of D_X vanish. Hence this is true and at any point at which X is defined. Due to (6′), every two such bases $\{E_i\}$ and $\{E_{i'}\}$ are connected by a linear transformation the matrix A of which is such that $X(A)=0$.

IV. SPECIAL BASES FOR DERIVATIONS AT A POINT

The purpose of this section is problems analogous to the ones in the previous section to be investigated but in the case describing the behavior of derivations at a given point.

At first we shall consider S-derivations with respect to a *fixed vector field*, i.e. we shall deal with a *fixed* derivation.

Proposition 8: At every point $x_0\in M$ and for every fixed vector field X such that $X|_{x_0}\neq 0$ there exists a defined in a neighborhood of x_0 local basis $\{E_{i'}\}$ in which the components of a given S-derivation D_X along X vanish at x_0, i.e. $W'_X(x_0)=0$ in $\{E_{i'}\}$. If $X|_{x_0}=0$, then such a basis exists if in some basis $\{E_i\}$ we have $W_X(x_0)=0$.

Proof: Let $\{x^\alpha\}$, $\alpha,\beta=1,\dots n$ be local coordinates in a neighborhood of x_0 and $E_i:=B^\alpha_i\partial/\partial x^\alpha$, $B:=\|B^\alpha_i\|=:\|B^i_\alpha\|^{-1}$ being a nondegenerate matrix function. The existence of $\{E_{i'}\}$ is equivalent to the existence of transformation $\{E_i\}\longrightarrow\{E_{i'}=A^i_{i'}E_i\}$ such that $W'_X(x_0)=0$ which, as a consequence of (6′), is equivalent to the existence of $A:=\|A^i_{i'}\|$ with the property

$$W_X(x_0)A(x_0)+X(A)|_{x_0}=0. \qquad (24)$$

If the basis $\{E_{i'}\}$ exists, from (24) follows that if $X|_{x_0}=0$,

then $W_X(x_0)=0$. Vice versa, if $X|_{x_0}=0$ and $W_X(x_0)=0$, then (24) is identically satisfied, i.e in this case any basis has the needed property.

So, let us below suppose $X|_{x_0}\neq 0$.

In this case with respect to A the equation (24) has infinite number of solutions, a class of which can be formed by putting

$$A^j_{j'}(x)=a^j_{j'k}X^k(x_0)-a^l_{j'k}(W_X(x_0))^j_{.l}B^k_\alpha(x_0)(x^\alpha-x^\alpha_0)+$$

$$+b^j_{j'\alpha\beta}(x)(x^\alpha-x^\alpha_0)(x^\beta-x^\beta_0), \qquad (25)$$

where $\|a^j_{j'k}X^k(x_0)\|$ is a nondegenerate matrix and the C^1 functions $b^j_{j'\alpha\beta}(x)$ and their derivatives are bounded when $x\longrightarrow x_0$. In fact, from (25), we find

$$A^j_{j'}(x_0)=a^j_{j'k}X^k(x_0),\quad E_k(A^j_{j'})|_{x_0}=-a^l_{j'k}(W_X(x_0))^j_{.l} \qquad (26)$$

which convert (24) into identity.■

Proposition 9: If for some S-derivation D_X along a fixed vector field X there is a local basis in which the components of D_X vanish at a given point, then there exist holonomic bases with this property.

Proof: If $X|_{x_0}=0$, then according to the proof of proposition 8 any basis, if any, including the holonomic ones, has the mentioned property.

If $X|_{x_0}\neq 0$, then a class of holonomic bases with the needed property can be found as follows. For the constructed in the proof of proposition 8 basis $\{E_{i'}\}$, we get

$$E_{k'}(A^j_{j'})|_{x_0}=A^k_{k'}(x_0)E_k(A^j_{j'})|_{x_0}=-a^k_{k'm}X^m(x_0)a^l_{j'k}(W_X(x_0))^j_{.l}. \qquad (27)$$

Hence, if we take $a^k_{k'm}$ to be of the form $a^k_{k'm}=a_{k'}a^k_m$, with $a_{k'}\neq 0$ and $\det\|a^k_m\|\neq 0,\infty$, we see that the quantities (27) are symmetric with respect to k' and j'. Consequently, choosing appropriately B^k_α (e.g. $B^k_\alpha=\delta^k_\alpha$), there exist classes of local coordinates $\{y^i\}$ and $\{y^{i'}\}$ in a neighborhood of x_0 partially fixed by the conditions:

$$\frac{\partial y^k}{\partial x^\alpha}\Big|_{x_0}=B^k_\alpha(x_0),\ \text{i.e.}\ E_k\Big|_{x_0}=\frac{\partial}{\partial y^k}\Big|_{x_0},$$

$$\frac{\partial y^j}{\partial y^{j'}}\Big|_{x_0}=A^j_{j'}(x_0)=a_{j'}a^j_kX^k(x_0),\ \text{i.e.}\ E_{k'}\Big|_{x_0}=\frac{\partial}{\partial y^{k'}}\Big|_{x_0},$$

$$\frac{\partial^2 y^j}{\partial y^{k'}\partial y^{j'}}\Big|_{x_0}=E_{k'}(A^j_{j'})\Big|_{x_0}=-a_{k'}a_{j'}[a^k_mX^m(x_0)a^l_k(W_X(x_0))^j_{.l}].$$

Evidently, the coordinate, and hence holonomic, basis $\{\partial/\partial y^{j'}\}$ defined by some of the coordinates $\{y^{j'}\}$, satisfying the above conditions, has the needed property.■

Let us now turn our attention to S-derivations with respect to *arbitrary vector fields*.

Proposition 10: An S-derivation D is at some $x_0\in M$ a linear connection iff there is a local basis $\{E_{i'}\}$ in which the components of D along every vector field vanish at x_0.

Remark: The S-derivation D at x_0 is *a* linear connection if for all X and some, and hence any, basis $\{E_i\}$ we have (cf. (8))

$$W_X(x_0)=\Gamma_kX^k(x_0), \tag{28}$$

where Γ_k are constant matrices. This means (8) to be valid at x_0, but it may not be true at $x\neq x_0$.

Proof: Let $\{x^i\}$ be local coordinates in a neighborhood of x_0 and let D be at x_0 a linear connection, i.e (28) to be valid for some Γ_k. We search for a basis $\{E_{i'}=A^i_{i'}\partial/\partial x^i\}$ in which $W'_X=0$. Due to (6′) this is equivalent to $\Gamma_kA(x_0)+\partial A/\partial x^k|_{x_0}=0$. So, if we define

$$A(y)=B-\Gamma_kB(x^k(y)-x^k(x_0))+B_{kl}(y)(x^k(y)-x^k(x_0))(x^l(y)-x^l(x_0)), \tag{29}$$

where B=const and B_{kl} and their derivatives are bounded functions when $y\longrightarrow x_0$, we find

$$A(x_0)=B,\ \partial A/\partial x^k|_{x_0}=-\Gamma_kB. \tag{30}$$

Hence $\Gamma_kA(x_0)+\partial A/\partial x^k|_{x_0}\equiv0$ for all A defined by (30), i.e. the bases $\{E_{i'}=A^i_{i'}\partial/\partial x^i\}$ with $\|A^i_{i'}\|=A$ have the needed property.

On the opposite, let in some $\{E_{i'}\}$ be valid $W'_X=0$ for every X. Then, fixing a basis $\{E_i=A^{i'}_iE_{i'}\}$, from (6′) we get $W_X(x_0)A(x_0)+$ $+X(A)|_{x_0}=0$, i.e. $W_X(x_0)=-X(A)|_{x_0}A^{-1}(x_0)$, which means that (28) is satisfied for $\Gamma_k=-E_k(A)|_{x_0}A^{-1}(x_0)$ and consequently D is at x_0 a linear connection.■

Proposition 11: If there exist bases in which the components of an S-derivation along every vector field vanish at a given point, then they are obtained from one another by linear transformations whose coefficients are such that the action of the vectors of these bases on them vanish at the given point.

Proof: If $\{E_i\}$ and $\{E_{i'}\}$ are such bases at the point x_0, then $W'_X(x_0)=W_X(x_0)=0$. Therefore, from (6′), we get $X(A)|_{x_0}=0$ for every X, i.e $E_i(A)|_{x_0}=0$.■

Proposition 12: If for some S-derivation D there is a local holonomic basis in which the components of D along every vector field vanish at a point x_0, then the torsion of D is zero at x_0. On the opposite, if D is torsion free at x_0 and bases with the mentioned property exist, then all of them are holonomic at x_0.

Proof: If $\{E_{i'}\}$ is a basis with the mentioned property, i.e. $W'_X(x_0)=0$ for every X, then, using (15), we find $T^D(E_{i'},E_{j'})|_{x_0}=$ $=-[E_{i'},E_{j'}]|_{x_0}$ and consequently $\{E_{i'}\}$ is holonomic at x_0, i.e. $[E_{i'},E_{j'}]|_{x_0}=0$, iff $0=T^D(X,Y)|_{x_0}=X^{i'}(x_0)Y^{j'}(x_0)(T^D(E_{i'},E_{j'})|_{x_0})$

(see proposition 10 and (28)) for every vector fields X and Y, which is equivalent to $T^D|_{x_0}=0$.

On the opposite, let $T^D|_{x_0}=0$. We want to prove that any basis $\{E_{i'}\}$ in which $W'_X(x_0)=0$ is holonomic at x_0. The holonomicity at x_0 means $0=[E_{i'},E_{j'}]|_{x_0}=-A_k^{k'}(E_{j'}(A_{i'}^k)-E_{i'}(A_{j'}^k))E_{k'}|_{x_0}$. But (see proposition 1) the existence of $\{E_{i'}\}$ is equivalent to $W_X(x_0)=$ $=\Gamma_k X^k(x_0)$ for every X. These two facts, combined with (15), show that $(\Gamma_k)^i{}_{.j}=(\Gamma_j)^i{}_{.k}$. Using this and $\Gamma_k A(x_0)+\partial A/\partial x^k|_{x_0}=0$, (see the proof of proposition 10), we find $E_{j'}(A_{i'}^k)|_{x_0}=-A_{j'}^j A_{i'}^i(\Gamma_j)^k{}_{.i}|_{x_0}=$ $=E_{i'}(A_{j'}^k)|_{x_0}$ and therefore $[E_{i'},E_{j'}]|_{x_0}=0$ (see above), i.e $\{E_{i'}\}$ is holonomic at x_0.■

Now we shall apply the above-obtained results to the theory of *linear connections*.

Corollary 13: For every point x_0 and every linear connection ∇ there exist in a neighborhood of x_0 local bases in which the components of ∇ vanish at x_0.

Remark: For torsion free linear connections this result is well known and is valid in holonomic bases (normal coordinates); see, e.g.: [2], ch. III, §8; [4], p. 120; or [5], §36.

Proof: From proposition 10, its proof, (28) and (8) follows the existence of bases $\{E_{i'}\}$ such that $0=W'_X(x_0)=\Gamma_{k'}(x_0)X^{k'}(x_0)$ for every X, where $\Gamma_{k'}(x_0)=\|\Gamma^{i'}{}_{..j'k'}(x_0)\|$ is the matrix of the components of ∇ in $\{E_{i'}\}$ at x_0. Consequently, as X is arbitrary, $\Gamma_{k'}(x_0)=0$, i.e. $\Gamma^{i'}{}_{..j'k'}(x_0)=0$.■

One can easily prove the following three corollaries:

Corollary 14: The bases in which the components of a linear connection ∇ vanish at a point x_0 are obtained from one another by linear transformations the coefficients of which are such that the action of the vectors of these bases on them vanishs at x_0.

Corollary 15: In a neighborhood of a given point x_0 there exist holonomic bases in which the components of a linear connection ∇ vanish at x_0 iff the torsion of ∇ vanishes at x_0.

Remark: This is a classical result that can be found, for instance, in [2], ch. III, §8 or in [4], p. 120. The same is valid and for corollary 16 below.

Corollary 16: For a torsion free linear connection in neighborhood of any point, local coordinates exist (or, equivalently, holonomic bases) in which its components vanish at that point.

If ∇ is arbitrary linear connection, then, generally, its torsion is not zero. But if we define a linear connection ${}^s\nabla$ whose components are the symmetric part of the ones of ∇, then ${}^s\nabla$ is torsion free. By corollary 15 for ${}^s\nabla$, local holonomic bases exist in which its components vanish at any given point. Thus we have proved the known result (see, e.g., [2], ch. III, §8 and [4], p. 120) that if ∇ isn't torsion free, then there doesn't exist local holonomic basis in which the components of ∇ vanish at some point, but there exist local holonomic bases (coordinates, called normal [2, 4, 5]) in which the symmetric part of the components of ∇ vanish at that point.

V. THE EQUIVALENCE PRINCIPLE

Physically the above results are important in connection with the equivalence principle (see, e.g. [7, 9] and the references in them).

Usually in a local frame (basis) the gravitational field strength is identified with the components of some linear connection which may be with or without torsion (e.g. the Riemannian one in general relativity [9] or the one in Riemann-Cartan space-times [7]). This linear connection must be compatible with the equivalence principle in a sense that there must exist "local" inertial, called also Lorentz, frames of reference (bases) in which the gravity field strength is "locally" transformed to zero. In our terminology this means the existence of special "local" basis (or bases) in which the connection's components vanish "locally". Above we have put the words "local" and "locally" in inverted commas as they are not well defined here, which is usual for the "physical" literature [9], where they often mean "infinitesimal surrounding of a fixed point of space-time" [7]. The strict meaning of "locally" may be at a point, in a neighborhood, along a path (curve) or on some other submanifold of the space-time. As in the present work we have used the first two of these meanings of "locally" we can make the following conclusions:

(1) All gravity theories based on space-times endowed with a linear connection (e.g. the general relativity [9] and the U_4 theory [7]) are compatible with the equivalence principle at any fixed space-time point, i.e. at any point there exist (local) inertial frames, which generally are anholonomic and may be holonomic ones iff the connection is torsion free (as is,e.g., the case of general relativity [9]).

(2) Any gravity theory based on space-time endowed with a flat linear connection is compatible with the equivalence principle in some neighborhood of any space-time point, i.e. for every point there exist its neighborhoods in which there exist (local) inertial frames (basses) which are holonomic iff the connection is torsion free.

(3) In the above cases the equivalence principle is not at all a principle because it is identically satisfied, namely, it is

a corollary from the underlying mathematics for the corresponding gravity theories.

(4) The equivalence principle becomes important if one tries to formulate gravity theories on the base of some (class of) derivations. Generally, it will select those theories which are based on linear connections, i.e.,only those in which it is identically valid.

VI. REMARKS AND GENERALIZATIONS

As we have seen the linear connections are remarkable among all derivations with their property that in a number of considered here sufficiently general cases they are the only derivations for which special bases in which their components vanish exist.

If one tries to construct a gravity theory based, for example, on linear connections, then he needn't to take into account the equivalence principle for it is identically fulfilled.

This formalism seems to be applicable also to other fields, not only to the gravitational one, namely at least to those of them which are described by linear connections. This suggests the idea for extending the aria of validity of the equivalence principle outside the gravity interaction (cf. [10]).

It should be possible to generalize this formalism along paths or on some other submanifolds of the space-time, which will be done elsewhere.

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