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CONSISTENCY BETWEEN METRICS AND LINEAR TRANSPORTS ALONG CURVES

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I. INTRODUCTION

Let be defined a general linear transport (I-transport) on the differentiable manifold M, i.e., to any curve $\gamma: J \to M$, J is a nondegenerate $\mathbb{R}$-interval, there corresponds a map $I^{\gamma}_{x\to y}: T^{p}_{\cdot q}|_x(M) \longrightarrow T^{p}_{\cdot q}|_y(M)$, where $x, y \in \gamma(J)$ and $T^{p}_{\cdot q}|_x(M)$ is the set of tensors of the type $\binom{p}{q}$ at x on M having the properties described in [5]. Let also M be endowed with a metric, i.e., a nondegenerate, symmetric bilinear mapping $g_x: T_x(M) \times T_x(M) \longrightarrow T^{0}_{\cdot 0}|_x(M)$, where $x \in M$, $T_x(M) := T^{1}_{\cdot 0}|_x(M)$ and $\times$ is a Cartesian product sign so that the scalar product of $A_x, B_x \in T_x(M)$ is $A_x \cdot B_x := g_x(A_x, B_x)$.

Definition 1.1. The I-transport and the metric will be called consistent (resp. along the curve $\gamma: J \longrightarrow M$) if the I-transport preserves the scalar products of the vectors along any curve (resp. along γ), i.e., if the scalar product of $A_x, B_x \in T_x(M)$, $x \in \gamma(J)$ is equal to the scalar product of the vectors obtained from them by means of an I-transport along $\gamma: J \longrightarrow M$ to any point $y \in \gamma(J)$

$$g_x(A_x, B_x) = g_y(I^{\gamma}_{x\to y} A_x, I^{\gamma}_{x\to y} B_x), \quad x, y \in \gamma(J). \tag{1.1}$$

Important examples of transports along curves consistent with some metric are the parallel and Fermi-Walker transports which are consistent with the Riemannian metric (defining them). This result has been proved, e.g., in [2,7] (see also below Sec.3).

It can be shown that in physics the concept for consistency of I-transports with metrics naturally appears when investigating the relative energy of two point particles (cf. [7]).

The aim of this paper, which is an extended version with proofs of [6], is to consider some necessary and/or sufficient conditions for consistency of I-transports and metrics. The results and their proofs are constructive, so they may find practical usage when working on that problem. Moreover, important results concerning the general structure of I- and S-transports of vectors are obtained.

In Sec.2 the general case of I-transports is investigated. Necessary and sufficient conditions for their consistency with metrics are proved. All metrics (resp. I-transport) consistent with a given I-transport (resp. metric) are obtained. Sec.3 is devoted to a special case of S-transports. Here, the same problems as in Sec.2 are considered but for the case of S-transports with specific and effective methods. Sec.4 contains some concluding remarks concerning one possible generalization of the problems studied in the present article.

2. GENERAL CASE

Let us fix some basis $\{E_i(x)\}$ in $T_x(M)$. Because of (see [5]) $I^{\gamma}_{x\to y}E_i(x)\in T_y(M)$, $x,y\in\gamma(J)$ there exist uniquely defined functions $H^{i}_{\cdot j}(y,x;\gamma)$ which are components of a two-point tensor from $T^{0}_{\cdot 1}|_y(M)\otimes T^{1}_{\cdot 0}|_x(M)$ such that

(2.1) $$I^{\gamma}_{x\to y}E_j(x)=H^{i}_{\cdot j}(y,x;\gamma)E_i(y),$$

where henceforth in our text all latin indices run from 1 to $n:=\dim M$ and summation from 1 to n over the indices repeated

on different levels is understood. Let $H(y,x;\gamma):=\|H^{i}_{\cdot j}(y,x;\gamma)\|$. From the basic properties of the I-transport (see [5]) it is not difficult to derive the equalities

(2.2) $$H(x,x;\gamma)=1,\quad x\in\gamma(J),$$

(2.3) $$H(z,x;\gamma)=H(z,y;\gamma)H(y,x;\gamma),\quad x,y,z\in\gamma(J).$$

The components of the metric g_x in $\{E_i(x)\}$ are $(g_x)_{ij}:=g_x(E_i(x),E_j(x))$. Let $G(x):=\|(g_x)_{ij}\|$.

<u>Proposition 2.1.</u> The metric and the I-transport are consistent if and only if

(2.4) $$G(x)=H^{T}(y,x;\gamma)\,G(y)\,H(y,x;\gamma),\quad x,y\in\gamma(J),$$

where the superscript T means transposition of the matrices (e.g. $(H^{T}(\cdots))_{i}^{\cdot j}:=H^{j}_{\cdot i}(\cdots)$, $(G^{T}(x))_{ij}=(G(x))_{ji}=(g_x)_{ji}=(g_x)_{ij}$).

Proof. Let $A_x,B_x\in T_x(M)$ so that $A_x=A^{i}_{x}E_i(x)$ and $B_x=B^{i}_{x}E_i(x)$. From the linearity of the metric and the I-transport there follow

(2.5) $$g_x(A_x,B_x)=A^{i}_{x}B^{j}_{x}\,g_x(E_i(x),E_j(x))=A^{i}_{x}B^{j}_{x}(g_x)_{ij},$$

(2.6) $$I^{\gamma}_{x\to y}A_x=\left(H^{i}_{\cdot j}(y,x;\gamma)A^{j}_{x}\right)E_i(y).$$

Taking into account the arbitrariness of A^{i}_{x} and B^{j}_{x} and using (2.5) and (2.6) we see that (1.1) is equivalent to

(2.4') $$(g_x)_{ij}=H^{k}_{\cdot i}(y,x;\gamma)(g_y)_{kl}H^{l}_{\cdot j}(y,x;\gamma)$$

which is simply the matrix eq. (2.4) but written in a component form.

Due to the dependence on two points x and y eq. (2.4) seems to be complicated but by means of the following lemma it can be put into an equivalent form depending only on one point.

Lemma 2.1. The matrix function $H(y,x;\gamma)$ obeys eqs. (2.2) and (2.3) iff there exists a nondegenerate matrix function $F(x;\gamma)$ such that

$$H(y,x;\gamma)=F^{-1}(y;\gamma)F(x;\gamma),\quad x,y\in\gamma(J), \tag{2.7}$$

where $F^{-1}(y;\gamma)$ means the inverse matrix of $F(y;\gamma)$.

Remark. It can be proved that the functions $H^{i}_{.j}(y,x;\gamma)$ define by (2.6) an I-transport of the vectors iff the matrix $H(y,x;\gamma)$ can be put in the form (2.7) in which $F(x;\gamma)$ satisfies the condition $F(x;\gamma')=D(\gamma,\gamma')F(x;\gamma)$, where $x\in\gamma'(J')=\gamma(J')$, γ' is the restriction of γ on any subinterval $J'\subset J$ and $D(\gamma,\gamma')$ is independent of x nondegenerate matrix (see (2.9)). This condition for F will be assumed below.

Proof. If (2.7) is true, then evidently (2.2) and (2.3) are satisfied. On the opposite, let (2.2) and (2.3) be valid. Hence, for $z=x$ there follows

$$H^{-1}(x,y;\gamma)=H(y,x;\gamma), \tag{2.8}$$

so (2.3) is equivalent to

$$H(z,x;\gamma)=H^{-1}(y,z;\gamma)\,H(y,x;\gamma).$$

Let x_0 be some fixed point from $\gamma(J)$ and $F(x;\gamma):=H(x_0,x;\gamma)$, then for $y=x_0$ we get (2.7) from the last equation.

Remark. As consequence of the proof of lemma 2.1 (see also (2.3)) we see that the appearing in (2.7) function F is defined within a constant nondegenerate left multiplier, i.e., if for some F_0 we have $H(y,x;\gamma)=F_0^{-1}(y;\gamma)F_0(x;\gamma)$, then any other function F for which (2.7) holds is obtained by the formula

$$F(x;\gamma)=DF_0(x;\gamma),\quad x\in\gamma(J),\ D=D(\gamma), \tag{2.9}$$

where $D=D(\gamma)$ is some constant (along γ) nondegenerate matrix. (Here and below by a constant matrix we understand an independent of $x\in\gamma(J)$ matrix which may depend on γ, but the argument γ, will not be written explicitly.)

Proposition 2.2. The metric and the I transport are consistent iff

$$(F^{-1}(x;\gamma))^{T}G(x)F^{-1}(x;\gamma)=C,\quad x\in\gamma(J),\ C^{T}=C\ (=C(\gamma)), \tag{2.10}$$

where C is a constant, nondegenerate and symmetric ($C^T=C$) matrix and F defines the given I-transport by means of (2.7).

Proof. Substituting (2.7) into (2.4) and multiplying the so-obtained equality by $(F^{-1}(x;\gamma))^{T}$ from the left and by $F^{-1}(x;\gamma)$ from the right, we see that by proposition 2.1 the metric and the I-transport are consistent iff

$$(F^{-1}(x;\gamma))^{T}G(x)F^{-1}(x;\gamma)=(F^{-1}(y;\gamma))^{T}G(y)F^{-1}(y;\gamma) \tag{2.11}$$

for every $x,y\in\gamma(J)$. If the metric and the I-transport are consistent, then putting here $y=y_0$ for some fixed $y_0\in\gamma(J)$ and $C=C(\gamma):=(F^{-1}(y_0;\gamma))^{T}G(y_0)F^{-1}(y_0;\gamma)$ we obtain (2.10). The symmetry of C is a corollary from its definition. On the opposite, if (2.10) holds, then (2.11) is obviously valid but it is equivalent to (2.4). In this way we see that eqs.(2.4) and (2.11) are equivalent, i.e., propositions 2.1 and 2.2 are equivalent, the former being already proved.

As a direct consequence of proposition 2.2 we derive

Proposition 2.3. Let some I-transport be fixed on M and F defines it by eq. (2.7). Then, all consistent with it metrics along $\gamma: J\to M$ are obtained by the equality

$$G(x;\gamma) = F^T(x;\gamma)\,C\,F(x;\gamma)\,,\quad x\in\gamma(J)\,,\quad C^T = C\;(=C(\gamma))\,, \tag{2.12}$$

where $C=C(\gamma)$ is nondegenerate, symmetric and independent of x matrix. That is, a given metric is consistent with the fixed I-transport iff there exists a constant symmetric matrix C for which (2.12) holds.

Evidently, the transformation $F\mapsto DF$ of F (see (2.9)) implies the change $C\mapsto D^T C D$ of the matrix C appearing in (2.I0):

$$F\mapsto DF \Longrightarrow C\mapsto D^T C D\,. \tag{2.13}$$

Let on M be given a metric and a consistent with it I-transport described in $\{E_i(x)\}$ by the matrices $G(x)$ and $F_0(x;\gamma)$ (see (2.1),(2.7),(2.9)), respectively. Due to (2.I0) we have $(F_0^{-1}(x;\gamma))^T G(x) F_0^{-1}(x;\gamma) = C_0$ for some constant matrix $C_0=C_0(\gamma)=C_0^T$. Due to the symmetry of C_0 there exists an orthogonal matrix D_0 such that $D_0^T C_0 D_0$ is a constant diagonal matrix [1]. Then, (see (2.9) and (2.13)) the matrix $F(x;\gamma) := D_0 F_0(x;\gamma)$ describes the same I-transport and due to the last equation it satisfies (2.I0) for $C=D_0^T C_0 D_0$, i.e., one can choose F in (2.I0) in such a way that C is a diagonal which means that $G(x)$ can be transformed to one and the some independent of x diagonal form by transformation of the type $D^T(x)\,G(x)\,D(x)$ for some nondegenerate matrix $D(x)$. But because of $G^T(x)=G(x)$ there exists an orthogonal matrix $D_1(x)$ such that $D_1^T(x)G(x)D_1(x) = \mathrm{diag}(g_1(x),\dots,g_n(x))$, $g_i(x)\neq 0$ being the eigenvalues of $G(x)$ [1]. Here of and from the inertia law of Jacobi-Sylvester [1] it follows that the number p of positive and the number $q\;(=n-p)$ of negative eigenvalues of $G(x)$ (i.e., of the metric) are equal to the numbers of positive and negative, respectively, diagonal elements of $D^T(x)G(x)D(x) = \mathrm{const}$, and consequently, they do not depend on $x\in M$.

On the opposite, let on M be defined a metric for which the numbers p and $q\;(=n-p)$ are independent of the points of M. From $G^T(x)=G(x)$ the existence of an orthogonal matrix $D_1(x)$ follows such that $D_1^T(x)G(x)D_1(x) = \mathrm{diag}(g_1(x),\dots,g_n(x))$, $g_i(x)\neq 0$ being the eigenvalues of $G(x)$ [1]. Let $D_2(x) := \mathrm{diag}\left(1/\sqrt{|g_1(x)|},\dots,1/\sqrt{|g_n(x)|}\right)$ and $D(x) := D_1(x)D_2(x)$. Then, $D^T(x)G(x)D(x) = \mathrm{diag}(\varepsilon_1,\dots,\varepsilon_n)$, where p of the numbers $\varepsilon_1,\dots,\varepsilon_n$ are equal to +1 and the others $q=n-p$ are equal to –1. So from proposition 2.2 for $C=\mathrm{diag}(\varepsilon_1,\dots,\varepsilon_n)$ we infer that the defined by (2.7) for $F(x;\gamma) = D^{-1}(x),\ x\in\gamma(J)$ I-transport is consistent with the given metric. Thus, we have proved the following important result.

<u>Proposition 2.4.</u> A necessary and sufficient condition for the existence of an I-transport consistent with a given metric is the independence of the number of positive (or negative: $\mathrm{rank}\,G(x)=n$) eigenvalues of the metric (i.e. of the matrix $G(x)$) of the point of the manifold at which they are evaluated (i.e. of $x\in M$).

<u>Proposition 2.5.</u> Let on M be fixed a metric for which the numbers p and $q=n-p$ of its positive and negative, respectively, eigenvalues do not depend on the point of M at which they are evaluated and the basis $\{E_i(x)\}$ be chosen in such a way that the first p eigenvalues of the matrix $G(x)$ defining the metric in it (see (2.5)) be positive (see below (2.16)). Then a given I-transport is consistent with this metric if and only

if it is defined by (2.7) in which the matrix $F(x;\gamma)$ has the form

$$(2.14) \qquad F(x;\gamma) = Y Z(x;\gamma) D^{-1}(x),$$

where $Y=Y(\gamma)$ is a constant nondegenerate matrix, $Z(x;\gamma)$ is a pseudo-orthogonal matrix of the type (p,q) $(Z(x;\gamma)\in O(p,q))$, i.e.

$$(2.15) \qquad Z^T(x;\gamma) G_{p,q} Z(x;\gamma) = G_{p,q} := diag(\underbrace{1,\dots,1}_{p\text{-times}},\underbrace{-1,\dots,-1}_{q\text{-times}}),$$

and $D(x)$ is any fixed matrix such that

$$(2.16) \qquad D^T(x) G(x) D(x) = G_{p,q}.$$

In other words, all I-transports consistent with the given metric are defined by (2.7) in which the matrix $F(x;\gamma)$ has the form (2.14).

Remark. The case, when in some basis not all the first eigenvalues of $G(x)$ are positive, is obtained from the above case by transformation (renumbering) of the basis $\{E_i(x)\}$.

Proof. To prove the necessity we have in fact to solve eq. (2.I0) with respect to $F(x;\gamma)$. From the choice of $\{E_i(x)\}$, $G^T(x) = G(x)$ and the independence of p and q of x there follows the existence of $D(x)$ satisfying (2.16) (e.g., one can put $D(x) = D_1(x) D_2(x)$ where D_1 and D_2 are defined in the proof of proposition 2.4). Let $F(x;\gamma) =: F_1(x;\gamma) D^{-1}(x)$ and x_0 be a fixed point from $\gamma(J)$. Then, from (2.I0) and from the constantcy of C, we get

$$(F_1^{-1}(x;\gamma))^T G_{p,q} F_1^{-1}(x;\gamma) = C = (F_1^{-1}(x_0;\gamma))^T G_{p,q} F_1^{-1}(x_0;\gamma).$$

Putting here $F_1(x;\gamma) =: Y Z(x;\gamma)$, where $Y = Y(\gamma) := F(x_0;\gamma)$, we see that $Z(x;\gamma)$ obeys (2.15), i.e. $F(x;\gamma)$ has the form (2.14).

The sufficiency is evident: if (2.14) is valid, then (2.I0) is satisfied for $C = (Y^{-1})^T G_{p,q} Y^{-1}$ and by proposition 2.2 the I-transport and the metric are consistent.

At the end of this section we shall investigate the problem when for a given I-transport there are metrics globally consistent with it (cf. proposition 2.3).

If a given I-transport is consistent with a globally defined metric (i.e. if (1.1) holds for any x,y and γ), then due to proposition 2.5 the function F defining it by (2.7) has the form (2.14). So (2.14) is a necessary condition for the existence of a globally defined metrics consistent with the I-transport. But this condition is not sufficient in a sense that not all metrics defined by I-transports for which (2.14) holds (for Z satisfying (2.15) for some p and q, $p+q=n$ and arbitrary nondegenerate matrix $D(x)$) are globally defined, i.e., there are F of the type (2.14) such that $G(x;\gamma)$ (see (2.12)) depends on γ. In fact, let (2.14) hold for arbitrary Y and $D(x)$ (i.e., $\det Y \cdot \det D(x) \neq 0$) and Z satisfy (2.15) for some p and q. Substituting (2.14) into (2.12), we get

$$(2.17) \qquad G(x;\gamma) = (D^{-1}(x))^T Z^T(x;\gamma) Y^T C Y Z(x;\gamma) D^{-1}(x), \; C^T = C = C(\gamma), Y = Y(\gamma).$$

Evidently, a necessary and sufficient condition for the independence of $G(x;\gamma)$ of γ is the existence of a symmetric nondegenerate matrix $Q(x)$ depending only on $x\in M$ such that

(2.18) $$Z^T(x;\gamma)Y^TCYZ(x;\gamma)=Q(x),\ Q^T(x)=Q(x).$$

For example, if $Y^TCY=G_{p,q}$ then due to (2.15) we have $Q(x)=G_{p,q}$. If (2.18) is valid, then

(2.19) $$G(x;\gamma)=G(x)=(D^{-1}(x))^T Q(x) D^{-1}(x)$$

does not depend on γ.

Equality (2.18) means that Z is a solution of

(2.18') $$(Z^{-1}(x;\gamma))^T Q(x) Z^{-1}(x;\gamma)=Y^TCY.$$

Thus by the proof of proposition 2.5 it follows

(2.20) $$Z(x;\gamma)=Y_0 Z_0(x;\gamma) D_0^{-1}(x),$$

where Y_0 is a constant nondegenerate matrix, $D_0(x)$ is any fixed matrix such that $D_0^T(x)Q(x)D_0(x)=G_{r,s}$ for some integers $r,s\geq 0$, $r+s=n$ (r is the number of positive eigenvalues of Y^TCY) and $Z_0(x;\gamma)$ is a pseudoorthogonal matrix of the type (r,s) (i.e., it satisfies (2.15) for $p=r$ and $q=s$).

From the above results we derive

Proposition 2.6. Let for some I-transport (2.14) hold for Y and $D(x)$ being nondegenerate matrices and $Z(x;\gamma)$ being a pseudo-orthogonal matrix of type (p,q), $p+q=n$. Then, the metric (2.12) does not depend on γ if $Z(x;\gamma)$ has the form (2.20) for some nondegenerate matrices Y_0 and $D_0(x)$ and $Z_0(x;\gamma)$ being a pseudoorthogonal matrix of type (r,s), $r+s=n$, where r and s are the numbers of positive and negative, respectively, eigenvalues of Y^TCY. In this case the metric in $\{E_i(x)\}$ is given by (2.19) for $Q(x)=(D_0^{-1}(x))^T G_{r,s} D_0^{-1}(x)$ and has r (s) positive (negative) eigenvalues.

3. THE CASE OF S-TRANSPORTS

As the S-transports [4] are a special case of the general I-transport [5] all the preceding results for them are valid. The aim of this section is to consider with adequate methods some specific for the S-transports conditions for their consistency with metrics.

Let (for the corresponding definitions see [4]) $\gamma : J\to M$ be a C^1-curve, $T^p{}_q(M)$ be the set of tensor fields of the type $\binom{p}{q}$ on M, V be a C^1 vector field defined in some neighbourhood of $\gamma(J)$, $V_{\gamma(s)}:=\dot\gamma(s)$, $s\in J$ be the tangent to γ vector $\dot\gamma$ at $\gamma(s)$ $(\dot\gamma^i(s)=d\gamma^i(s)/ds)$, $S:T^1{}_0(M)\longrightarrow T^1{}_1(M)$, $S:V\longmapsto S_V$, $D^S_V|_x=(L_V+S_V)_x$, $x\in\gamma(J)$, L_V be the Lie derivative along V to be the S-differentiation along γ with respect to V and $S^\gamma_{x\to y}:T^p{}_q|_x(M)\longrightarrow T^p{}_q|_y(M)$, $x,y\in\gamma(J)$ be the S-transport along γ (from x to y) defined by D^S_V.

By definition 1.1 the C^1 metric $g\in T^0{}_2(M)$ and the S-transport $S^\gamma_{x\to y}$ are consistent along γ iff

(3.1) $$g_x(A_x,B_x)=g_y(S^\gamma_{x\to y}A_x, S^\gamma_{x\to y}B_x),\ x,y\in\gamma(J),\ A_x,B_x\in T_x(M).$$

Introducing the contraction operator C^p_q over the p-th superscript and q-th subscript, we see that

(3.2) $$g(A,B)=(C^1_1)^2 A\otimes g\otimes B,\ A,B\in T^1{}_0(M),$$

where $\otimes$ is the tensor product sign. Taking into account that $S^{\gamma}_{x\to y} f(x) \equiv f(x)$ for any $f: M \to \mathbb{R}$ and the commutativity of $S^{\gamma}_{x\to y}$ and C^p_q (see [4], propositions 3.6 and 3.3) from (3.1) and (3.2) we derive

<u>Proposition 3.1.</u> The C^1 metric g and the S-transport $S^{\gamma}_{x\to y}$ are consistent along $\gamma: J \to M$ iff

(3.3) $$g_y = S^{\gamma}_{x\to y} g_x, \quad x, y \in \gamma(J).$$

Due to the definition of an S-transport [4] this results is equivalent to

<u>Proposition 3.2.</u> The S-transport defined by means of an S-derivative $D^S_V|_x$ along $\gamma: J \to M$ is consistent with the C^1 metric g iff

(3.4) $$D^S_V g|_x \equiv D^S_V|_x g_x = 0, \quad x, y \in \gamma(J).$$

<u>Remark 1.</u> This statement also follows directly from the definition of an S-transport and eq. (3.2): if A and B undergo S-transport along γ, i.e. if $D^S_V A|_x = D^S_V B|_x = 0$, $x \in \gamma(J)$, then due to the commutativity of D^S_V and C^p_q (see also [4]) the scalar product $g(A,B)$ undergoes an S-transport along γ (i.e. $D^S_V g(A,B)|_x = 0$, $x \in \gamma(J)$) iff (3.4) is true.

<u>Remark 2.</u> Eq. (3.4) is useful and effective for practical checking the consistency between S-transports and metrics, e.g. for the Riemannian parallel transport (3.4) is identically satisfied [2,7]: in this case $\nabla^{\{\}}_X g \equiv 0$, where $\nabla^{\{\}}_X$ is the covariant derivative defined by the Cristoffel symbols along $X \in T^1_{\cdot 0}(M)$ in the Riemannian space.

Let us write (3.4) in terms of local components in some (local along γ) coordinate basis $\{E_i(x)\}$ in $T_x(M)$.

We have ($x = \gamma(s)$, $s \in J$, $\partial_i := \partial/\partial x^i = E_i(x)$)

$$(L_V g|_x)_{ij} = \partial_k (g_x)_{ij} V^k|_x + (g_x)_{ik} \partial_j V^k|_x + (g_x)_{kj} \partial_i V^k|_x =$$
$$= d(g_{\gamma(s)})_{ij}/ds + (g_x)_{ik} \partial_j V^k|_x + (g_x)_{kj} \partial_i V^k|_x,$$

$$(S_V g|_x)_{ij} = -(S_V|_x)^k_{\cdot i} (g_x)_{kj} - (S_V|_x)^k_{\cdot j} (g_x)_{ik},$$

hereof we get

(3.5) $$(D^S_V g|_x)_{ij}|_{x=\gamma(s)} = (L_V g|_x + S_V|_x)_{ij}|_{x=\gamma(s)} = d(g_{\gamma(s)})_{ij}/ds +$$
$$+ (g_{\gamma(s)})_{ik} (\partial_j V^k|_{\gamma(s)} - (S_V|_{\gamma(s)})^k_{\cdot j}) + (g_{\gamma(s)})_{kj} (\partial_i V^k|_{\gamma(s)} - (S_V|_{\gamma(s)})^k_{\cdot i}).$$

Introducing the matrices $G(x) := \|(g_x)_{ij}\| = G^T(x)$ and $W(x) = \|W_{ij}(x)\| := \|\partial_j V^i|_x - (S_V|_x)^i_{\cdot j}\|$ we see that in $\{E_i(x)\}$ eq. (3.4) is equivalent to

(3.6) $$\frac{dG(\gamma(s))}{ds} + G(\gamma(s)) \cdot W(\gamma(s)) + W^T(\gamma(s)) \cdot G(\gamma(s)) = 0.$$

<u>Proposition 3.3.</u> Let $D^S_V = L_V + S_V$ be an S-differentiation which defines the S-transport $S^{\gamma}_{x\to y}$ along the C^1-curve $\gamma: J \to M$. Then, all metrics consistent with this S-transport have along γ the form

(3.7) $$G(\gamma(s)) = Y(s, s_0; -W^T)\, C\, Y^T(s, s_0; -W^T),$$
$$s, s_0 \in J, \quad C^T = C,$$

where C is a constant nondegenerate symmetric matrix,

$Y(s,s_0;Z)$, Z is a nondegenerate matrix function on J , is the fundamental solution [1,2] of the linear matrix differential equation

(3.8a) $$dY/ds = Z(s)Y$$

satisfying the initial condition

(3.8b) $$Y(s_0,s_0;Z) = \mathbb{1} = \mathrm{diag}(1,\dots,1),$$

i.e., $Y = \|Y_{ij}\|$ is the matrixant of the equation $dx/ds = Z(s)x$, $x = (x^1(s),\dots,x^n(s))$, and in (3.7) we have written W instead of $W \circ \gamma$, i.e. we have put $W(\gamma(s)) \equiv W(s)$, $s \in J$.

Remark. Equality (3.7) is simply the explicit form of eq. (3.3) for $x = \gamma(s_0)$ and $y = \gamma(s)$ (note that

(3.9) $$G(\gamma(s_0)) = C),$$

from which there follows [4] eq. (3.3) for every $x, y \in \gamma(J)$.

Proof (first version). This proposition is in fact a special case of proposition 2.3 (compare (2.12) and (3.7)). Indeed, by considering $S^{\gamma}_{\gamma(s_0)\to\gamma(s)}A$, $A \in T_{\gamma(s_0)}(M)$ as a solution of $D^S_V B|_{\gamma(s)} = 0$, $B_{\gamma(s_0)} = A$ (see [4])one finds (see [2] and the notation of Sec.2)

(3.10) $$\|H^i{}_j(\gamma(s_2),\gamma(s_1);\gamma\| = Y(s_2,s_1;W),\quad s_1,s_2 \in J$$

from which because of $Y(s_3,s_2;W)Y(s_2,s_1;W) = Y(s_3,s_1;W)$ and $Y(s_1,s_2;W) = Y^{-1}(s_2,s_1;W)$ (see Sec.2 and [2]) there follows

(3.11) $$F(\gamma(s);\gamma) = D\,Y(s_0,s;W),\quad s \in J,$$

where s_0 is any fixed point from J and $D = D(\gamma) := F(\gamma(s_0);\gamma)$ is a constant nondegenerate matrix (cf. (2.9)).

From $Y(s_1,s_2;W) = Y^{-1}(s_2,s_1;W)$, $dY^{-1}/ds = -Y^{-1}(dY/ds)Y^{-1}$ (see [1] , ch.IO, § 2) and (3.8) we derive that

(3.12) $$Y(s,s_0;-W^T) = Y^T(s_0,s;W),\quad s_0,s \in J.$$

Substituting (3.11) into (2.12), taking into account the last equality and denoting $D^T C D$ simply by C, we derive (3.7).

Proof (second version). One can easily check that (3.7) is the general solution of eq.(3.6) with respect to G satisfying the initial condition $G(\gamma(s_0)) = C$ (cf: [2] and [1], ch. IO, § 16, exercise 2). So the metric g satisfies eq. (3.4) iff in $\{E_i(x)\}$ it has the form (3.7).

Now we want to solve just the opposite problem of the one stated in proposition 3.3, i.e., we are going to find all S-transports along a given smooth curve which are consistent with a given metric.

It is not difficult to see that the matrix $W(x)$ appearing in (3.6) defines in $\{E_i(x)\}$ the action of the S-differentiation D^S_V along $\gamma : J \to M$ on any C^1 tensor field $T \in T^p{}_q(M)$ and so the S-transport along γ . In fact, if $T^{i_1\dots i_p}_{j_1\dots j_q}(x)$ are the local components of T at $x \in \gamma(J)$, then following the derivation of (3.5) (see also [4]), we find the local components of $D^S_V T$ at $x = \gamma(s), s \in J$:

(3.13) $$\left(D^S_V T|_{\gamma(s)}\right)^{i_1\dots i_p}_{j_1\dots j_q} = dT^{i_1\dots i_p}_{j_1\dots j_q}(\gamma(s))/ds - \\ -\sum_{a=1}^{p}\sum_{k=1}^{n} W_{i_a k}(\gamma(s))\,T^{i_1\dots i_{a-1}k\,i_{a+1}\dots i_p}_{j_1\dots j_q}(\gamma(s)) +$$

$$+\sum_{b=1}^{q}\sum_{k=1}^{n} W_{kj_b}(\gamma(s))\, T^{i_1\cdots i_p}_{j_1\cdots j_{b-1}kj_{b+1}\cdots j_q}(\gamma(s)).$$

From where (see also (3.8) and [4]), we get the following explicit expression for the components of $S^{\gamma}_{\gamma(s_0)\to\gamma(s)}\, T(\gamma(s_0))$ (cf. (3.10) and (3.7)):

$$\left(S^{\gamma}_{\gamma(s_0)\to\gamma(s)}\, T(\gamma(s_0))\right)^{i_1\cdots i_p}_{j_1\cdots j_q} =$$

$$= \sum_{\substack{k_1\cdots k_p\\ l_1\cdots l_q}} \left(\prod_{a=1}^{p} Y_{i_a k_a}(s,s_0;W)\right)\left(\prod_{b=1}^{q} Y_{j_b l_b}(s,s_0;-W^T)\right)\times \tag{3.14}$$

$$\times\, T^{k_1\cdots k_p}_{l_1\cdots l_q}(\gamma(s_0)), \quad s_0, s\in J.$$

From the above we infer that the problem of finding all S-transports consistent along some curve γ with a given metric is equivalent to the solution of eq. (3.6) along γ with respect to W.

Proposition 3.4. Let $\gamma: J\to M$ be a C^1 curve, g be a C^1 metric (which may be defined only on $\gamma(J)$) represented in $\{E_i(x)\}$ by $G(x)$ whose number of positive (or negative) eigenvalues does not depend on x, $D(x)$ be any fixed orthogonal matrix such that [1]

$$D^T(x)G(x)D(x) = \tilde{G}(x) := diag(g_1(x),\dots,g_n(x)), \quad D^T(x) = D^{-1}(x), \tag{3.15}$$

where $g_i(x)\neq 0$ are the eigenvalues of $G(x)$ and

$$K(\gamma(s)) := -D^T(\gamma(s))\frac{dG(\gamma(s))}{ds}D(\gamma(s)) = K^T(\gamma(s)), \quad s\in J. \tag{3.16}$$

Then, all S-transports consistent with the metric g are defined in $\{E_i(x)\}$ by the matrix $W(x), x\in\gamma(J)$ (see (3.14)) which is obtained by the equality

$$W(x) = D(x)\left(P(x) + Q(x)\tilde{G}(x) + R(x)\right)D^T(x) = \tag{3.17}$$

$$= D(x)\left(P(x)+R(x)\right)D^T(x) + D(x)Q(x)D^T(x)G(x),$$

where P is symmetric, Q and R are skewsymmetric matrices $(P^T=P,\ Q^T=-Q,\ R^T=-R)$ whose components are: if the values of i and $j\neq i$ are such that $g_i(x)+g_j(x)=0$, then $Q_{ij}(x):=0$ and $R_{ij}(x):=K_{ij}(x)/2g_i(x)$; for any pair i and j for which $g_i(x)+g_j(x)\neq 0$ we have $P_{ij}(x):=$ $:=K_{ij}(x)/(g_i(x)+g_j(x))$ and $R_{ij}(x):=0$; the other components of $P(x)$ and $Q(x)$ can be chosen arbitrarily only if the conditions $P^T=P$, $Q^T=-Q$ and $R^T=-R$ are fulfilled.

Remark. It is easy to see that if $g_i(x)+g_j(x)\neq 0$ for every i and j (of such a type are all Euclidean metrics), then (3.17) can be put in the form

$$W(x) = W_1(x) + W_2(x)G(x), \quad x\in\gamma(J), \tag{3.18}$$

where $W_2(x) := -W_2^T(x)$ is an arbitrary skewsymmetric matrix and $W_1(x) = W_1^T(x)$ is any fixed symmetric solution of (3.6) with respect to W, e.g., one can take (see [1] , ch. 12, §13)

$$W_1(x) = \int_0^{+\infty}\left(exp(G(x)t)\right)\frac{dG(x)}{ds}\left(exp(G(x)t)\right)dt, \quad x=\gamma(s),\ s\in J. \tag{3.19a}$$

Evidently, a necessary and sufficient condition for the existence of the representation (3.18) is the existence of a symmetric solution $W_1 = W_1^T$ of eq. (3.6) with respect to W. Using the method, applied below in the proof of proposition 3.4, one can prove that such a solution exists iff for any pair i and j for which $g_i(x)+g_j(x)=0$ simultaneously is satisfied and the equality $K_{ij}(x)=0$. If this is the case and such i and j exist, then the integral in the right-hand side of (3.16) does not exist[1] and $W_1(x)$ may be taken, e.g., as

(3.19b) $$W_1(x) = D(x)\, W_0(x)\, D^T(x), \quad x \in \gamma(J),$$

where the components of $W_0(x) := W_0^T(x)$ are: $(W_0(x))_{ij} := K_{ij}(x)/(g_i(x)+g_j(x))$ if i and j are such that $g_i(x)+g_j(x) \neq 0$ and $(W_0(x))_{ij}$ are arbitrary for any i and j for which $g_i(x)+g_j(x) = K_{ij}(x) = 0$.

<u>Proof</u>. In fact, we have to prove that (3.17) is the general solution of eq. (3.6) with respect to W along γ.

Multiplying (3.6) by $D^T(\gamma(s))$ from the left and by $D(\gamma(s))$ from the right, we get

(3.20) $$\tilde{G}\tilde{W} + \tilde{W}^T\tilde{G} = K,$$

where $\tilde{W} := D^T W D$ and here, as well as below, we omit the common argument $\gamma(s)$ of all quantities.

Due to (3.15) eq. (3.20) in component form reads

(3.20') $$g_i \tilde{W}_{ij} + g_j \tilde{W}_{ji} = K_{ij}$$

(do not sum over i and j).

Let us first consider all pairs i and j for which $g_i+g_j=0$. For them (3.20') reduces to $g_i(\tilde{W}_{ij} - \tilde{W}_{ji}) = K_{ij}$, so by using the identity $\tilde{W}_{ij} \equiv \frac{1}{2}(\tilde{W}_{ij} + \tilde{W}_{ji}) + \frac{1}{2}(\tilde{W}_{ij} - \tilde{W}_{ji})$ we see that for these pairs eq. (3.20') defines only the skewsymmetric part of $\tilde{W}_{ij}$, i.e., $\tilde{W}_{ij} = P_{ij} + R_{ij} + Q_{ij}g_j$, $R_{ij} := K_{ij}/2g_i =: -R_{ji}$, the quantities $P_{ij} = P_{ji}$ being arbitrary and $Q_{ij} := -Q_{ji} := 0$.

Let us now consider all pairs i and j for which $g_i+g_j \neq 0$. In this case, we define the quantities $P_{ij} = P_{ji}$ (i.e., the remaining components of $P = P^T$) by $g_i P_{ij} + g_j P_{ji} = K_{ij}$, i.e., as a symmetric in i and j solution of (3.20), so $P_{ij} = K_{ij}/(g_i+g_j) = P_{ji}$. Then, putting in (3.20') $\tilde{W}_{ij} =: P_{ij} + Q_{ij}g_j$, we obtain that the only restriction for Q_{ij} is to be skewsymmetric, i.e., $Q_{ij} = -Q_{ji}$. Thus, we derive $\tilde{W}_{ij} = P_{ij} + R_{ij} + Q_{ij}g_j$ where by definition $R_{ij} := -R_{ji} := 0$.

From the above we derive $\tilde{W} = P + R + Q\tilde{G}$ to be the general solution of (3.20) with respect to $\tilde{W}$. But then $W = (D^T)^{-1}\tilde{W}D^{-1} = D\tilde{W}D^T$ is the same W as it is given by (3.7) and consequently it is the general solution of (3.6) with respect to W.

4. CONCLUSION

In the present paper we have investigated from a general viewpoint some important problems concerning the task formulated in Sec.1 for consistency of metrics and I-transports (and S-transports as their special case). In our opinion, all this is essential in connection with the existing and possible applications of the theory of linear transports of tensors along curves.

From the standpoint of the theoretical developments of the considered in this work ideas we want to pay attention to the following problem whose evident special case we have studied in the present work.

Let

$$(4.1)\qquad \omega_x : T^{p_1}_{\cdot q_1}|_x(M)\otimes\cdots\otimes T^{p_m}_{\cdot q_m}|_x(M)\longrightarrow \mathbb{R},\ x\in M$$

be a multilinear (m-linear) mapping of m tensor arguments. We shall say that ω is consistent with the I-transport $I^{\gamma}_{x\to y}$ along $\gamma : J\to M$, $x,y\in\gamma(J)$ if for any tensors ${}^{a}T_x\in T^{p_a}_{\cdot q_a}|_x(M), a=1,\dots,m$ is satisfied by the equality (cf. (1.1))

$$(4.2)\qquad \omega_x({}^{1}T_x,\dots,{}^{m}T_x)=$$

$$=\omega_y(I^{\gamma}_{x\to y}{}^{1}T_x,\dots,I^{\gamma}_{x\to y}{}^{m}T_x),\quad x,y\in\gamma(J).$$

Here, for instance, one can state the following problems: to be found neccessary and/or sufficient condition for consistency of I-transports and maps ω ; to be found all maps ω (resp. I-transports) which are consistent with a given I-transport (resp. a map ω) and so on. It is not difficult to see that all such problems can be investigated and solved by an evident generalization of the developed in the present work methods, but this is an item of another article.

At the end we want to mention that in some cases the following definition may turn out to be useful (cf. Definition 1.1):

Definition 4.1. The I-transport I^{γ} and the metric g will be called scalar consistent (resp. along $\gamma : J\to M$) if for any curve γ (resp. only for γ) we have

$$(4.3)\qquad I^{\gamma}_{x\to y}(g_x(A_x,B_x))=g_y(I^{\gamma}_{x\to y}A_x,\ I^{\gamma}_{x\to y}B_x)$$

for every $A_x,B_x\in T_x(M)$, $x,y\in\gamma(J)$.

Let us compare definitions 1.1 and 4.1.

If $f_0:\gamma(J)\to\mathbb{R}$ then due to the linearity of an I-transport [5] we have $I^{\gamma}_{x\to y}(f_0(x))=f_0(x)\,I^{\gamma}_{x\to y}(1)$, $x,y\in\gamma(J)$, i.e.,

$$(4.4)\qquad I^{\gamma}_{x\to y}f_0(x)=h(y,x;\gamma)f_0(x),$$

where the two-point scalar function $h(y,x;\gamma):=I^{\gamma}_{x\to y}(1)$ uniquely defines the I-transport of scalars. From the basic properties of the I-transport [5] one can easily derive that (cf. (2.2) and (2.3))

$$(4.5)\qquad h(x,x;\gamma)=1,\quad x\in\gamma(J),$$

$$(4.6)\qquad h(z,x;\gamma)=h(z,y;\gamma)h(y,x;\gamma),\quad x,y,z\in\gamma(J)$$

from where there follows (see Lemma 2.1 and its proof , where one has to write h for H and f for F)

Lemma 4.1. The function $h(y,x;\gamma)$ defines an I-transport of the scalars along γ iff there exist $f_\gamma:\gamma(J)\to\mathbb{R}\setminus\{0\}$ which may depend on γ , such that

$$(4.7)\qquad h(y,x;\gamma)=f(x;\gamma)/f(y;\gamma),\quad f(x;\gamma):=f_\gamma(x),\ x,y\in\gamma(J).$$

This result along with (4.4) shows that (4.3) is equivalent to

$$(4.8)\qquad f(x;\gamma)g_x(A_x,B_x)=f(y;\gamma)g_y(I^{\gamma}_{x\to y}A_x,\ I^{\gamma}_{x\to y}B_x)$$

and hence we derive the searching connection between the definitions 1.1 and 4.1 in the form of

<u>Proposition 4.1</u>. If the metric g and the I-transport I^γ are scalar consistent (may be along γ), then the metric $\tilde{g} := f\cdot g$ and the I-transport I^γ are consistent (resp. along γ) and on the opposite, if the metric g and the I-transport I^γ are consistent (resp. along γ), then the metric $\underset{\sim}{g} := \frac{1}{f}\cdot g$ and the I-transport I^γ are scalar consistent (resp., along γ).

This proposition shows that any result concerning the consistency between metrics and I-transports may be put into an equivalent form concerning the scalar consistency between others but equivalent to the initial metrics and the same I-transports and vice versa.

Evidently, the concepts of consistency and scalar consistency between metrics and I-transports coincide if and only if $h(y,x;\gamma)=1$ or equivalently iff $f_\gamma =$ const $\in \mathbb{R}\setminus\{0\}$ (i.e., $f_\gamma(\gamma(J))=c(\gamma)$ where $c(\gamma)\in\mathbb{R}\setminus\{0\}$ may depend only on γ). It is important to note that due to proposition 3.6 from [4] this condition is satisfied for any S-transport, but the opposite statement is not true (see, e.g., Proposition 3.2 from [5]).

References

1. Bellmann R., Introduction to matrix analysis, McGraw-Hill book comp., New York – Toronto – London, 1960.
2. Dubrovin B.A., Novikov S.P., Fomenko A.T. Modern geometry, Nauka, Moscow, 1979, Part I, 29, point 3 (in Russian).

3. Hartman Ph., Ordinary differential equations, John Willey & Sons, New-York – Sydney, 1964, ch.III, 1.
4. Iliev B.Z., Linear transports of tensors along curves: General S-transport, Comp.rend.Acad.Bulg.Sci., Vol.40, No.7, 1987, pp.47-50.
5. Iliev B.Z., General linear transport (I-transport), Comp.rend.Acad.Bulg.Sci., Vol.40, No.8, 1987, pp.45-48.
6. Iliev B.Z., Consistency between metrics and linear transports along curves, Comp.rend.Acad.Bulg.Sci., vol.42, No.3, 1989, pp.17-20.
7. Synge J.L., Relativity: The general theory, North-Holl. publ.comp., Amsterdam, 1960.

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