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ON THE COMPUTING COMPLEXITY OF ITERATIVE SUBSTRUCTURING ALGORITHMS IN NONLINEAR MAGNETOSTATIC PROBLEMS

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1 Introduction

In this paper we analyse the computational expences for solving the two- and three-dimensional magnetostatics problems in the incomplete-nonlinear formulation [19]. We outline the computational strategy which leads to the almost optimal numerical algorithms which are also highly parallelizable. The rigorous justifications of the computing characteristics for the proposed solvers can be found in [16, 17, 18]. The recent developments in the numerical investigation of the coupled elliptic problems have been done in [7, 11, 14, 16, 17, 23, 28]. Some aspects of the approximation and iterative solution of the nonlinear boundary value problems (BVP) have been considered in [7, 17, 24].

Here we consider the algorithms based on the natural boundary reduction of the original nonlinear BVP in the combined formulation to the nonlinear interface problem defined on the interface boundaries (skeleton). This skeleton defines the multidomain decomposition of the unbounded domain. The basic tool of the iterative substructuring algorithms have been developed in [4, 5, 8, 13, 16, 22] (see also references therein).

Let $\bar{\Omega} = \cup_{i=1}^{M} \bar{\Omega}_i \in R^d$, $d = 2, 3$ be a Lipschitz domain with the boundary Γ_0, which is partitioned into $M \geq 1$ subdomains Ω_i with Lipschitz boundaries $\Gamma_i = \partial\Omega_i$. The exterior domain we denote by $\Omega_0 = R^d \backslash \bar{\Omega}$. When using two scalar potentials for solving the stationary Maxwell equation [17] the following quasi-linear elliptic BVPs arise:

Problem D. *Given* $\Psi_i \in \mathbf{H}^{-1/2}(\Gamma_i)$, *find* $u_i \in \mathbf{C}^2(\Omega_i)$, *such that the equations*

$$\Lambda_i u_i := div\,(\mu_i(x, |\nabla u_i|) \cdot \nabla u_i) = 0 \quad \text{on } \Omega_i \tag{1}$$

$$[u]_{\Gamma_i} = 0, \quad [\partial_\nu u]_{\Gamma_i} = \Psi_i \quad \text{on } \Gamma_i$$

$$|u(x)| = O(|x|^{-\nu}) \quad \text{as } |x| \to \infty, \quad \nu \geq 1$$

hold for $i \in I_0 = \{i : i = 0, 1, \cdots, M\}$ □

Here ∂_ν is the operator of conormal derivative, $[\cdot]_{\Gamma_i}$ is the jump of the corresponding function accross the boundary Γ_i and $\mu_i(x, t)$ for $i \in I_0$ are the given material functions with properties : $\mu_0 = 1$ and for $i \in I_1 = \{i : i = 1, \ldots, M\}$ the following inequalities

$$\mu_i(x, t) \cdot t - \mu_i(x, \tau) \cdot \tau \;\geq\; m_i(t - \tau), \quad t \geq \tau,\, m_i > 0 \tag{2}$$

$$|\mu_i(x,t)\cdot t - \mu_i(x,\tau)\cdot\tau| \le M_i|t-\tau| \qquad (3)$$

hold for almost all $x \in \Omega_i$ and for all $t,\tau \in [0,\infty)$ with some given constants $M_i, m_i > 0$.
Note that some specific choice of the function $\mu(x,|\nabla u|)$ and modification of jump conditions lead to the potential equations for subsonic flow with weak shocks [3] or to a shape design problems [21].

We further restrict ourselves by the case $\mu_i(x,|\nabla u|) = \mu_i(|\nabla u|)$ and define one global function $\mu(x,|\nabla u|)$ by the equation

$$\mu(x,|\nabla u|) = \mu_i(|\nabla u|) \quad \text{for} \quad x \in \Omega_i,\ i \in I_0 \qquad (4)$$

Besides we consider the typical data for magnetostatics with $\Psi_i = 0$ for $i \in I_1$.
The remainder of the paper is organized as follows: In Section 2 we define the weak formulation for the full nonlinear BVP (1) and describe the corresponding weak incomplete-nonlinear formulation [19]. Then we introduce the natural boundary reduction for the above formulation associated with trace space on the skeleton $\Gamma = \cup_{i=1}^{M}\Gamma_i$ equipped with the energy norm. In Section 3 we briefly discuss the mapping properties of the nonlinear interface operator and derive the appropriate Galerkin equations for the interface problem. In Section 4 we formulate the iterative schemes for solving this Galerkin equation and give the asymptotic estimates for computational work and memory needs, underlining the parallel structure of algorithms. We present some graphics illustrating the typical behavior of the magnitostatic scalar potential when moving off the border of nonlinear medium and in Section 5 we draw some concluding remarks.

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2 Nonlinear interface problem

Let us define $\mathbf{W}(\Omega) = \mathbf{H}^1(\Omega)$ for $d = 3$ and $\mathbf{W}(\Omega) = \{u \in \mathbf{H}^1(\Omega) : (u, g_0) = 0\}$ for $d = 2$, where g_0 is the Robin potential on $\Gamma_0 = \partial\Omega$ [26]. If

we introduce the Poincaré-Steklov operator $\mathbf{S}_0^{-1} : \mathbf{H}^{1/2}(\Gamma_0) \to \mathbf{H}^{-1/2}(\Gamma_0)$ associated with the Laplacian in the exterior domain Ω_0 [1], then the weak coupled formulation of the Problem D reads as follows:
Problem C *Find $u \in \mathbf{W}(\Omega)$ such that*

$$\sum_{i=1}^{M}\int_{\Omega_i}\sum_{k=1}^{d}\mu(x,|\nabla u|)\frac{\partial u}{\partial x_k}\cdot\frac{\partial v}{\partial x_k}dx + \left(\mathbf{S}_0^{-1}u, v\right)_{\Gamma_0} = \qquad (5)$$

$$= (\Psi, v)_{\Gamma_0}, \qquad \forall v \in \mathbf{W}(\Omega).$$

Note that the equation (5) is uniquely solvable in $\mathbf{W}(\Omega)$ for any $\Psi \in \mathbf{Y}^*(\Gamma_0)$ [17], where

$$\mathbf{Y}^*(\Gamma_0) = \begin{cases} \mathbf{H}^{-1/2}(\Gamma_0) & \text{for } d = 3\,, \\ u \in \mathbf{H}^{-1/2}(\Gamma_0)\ :(u,1) = 0 & \text{for } d = 2\,. \end{cases}$$

Let us consider the problem (5) in the incomplete-nonlinear formulation which have been proposed and investigated in [19]. The idea is some averaging procedure in any subdomain Ω_i, $i \in I_1$ for the approximation of nonlinearity in (5). The corresponding weak formulation reads as follows:
Problem IN *Given $\Psi \in \mathbf{Y}^*(\Gamma_0)$, find $\bar{u} \in \mathbf{W}(\Omega)$, such that*

$$\sum_{i=1}^{M}\bar{\mu}_i(\bar{u})\int_{\Omega_i}\nabla\bar{u}\cdot\nabla v dx + (\mathbf{S}_0^{-1}u, v)_{\Gamma_0} = (\Psi, v)_{\Gamma_0} \quad \forall v \in \mathbf{W}(\Omega) \qquad (6)$$

where the unknown constants $\bar{\mu}_i(u)$ depending on the desired solution $\bar{u}$ are defined by

$$\bar{\mu}_i(\bar{u}) = \mu_i(\tau_i(\bar{u})), \quad \tau_i(\bar{u}) = \left[\frac{1}{g_i}\int_{\Omega_i}|\nabla\bar{u}|dx\right]^{1/2} \qquad (7)$$

with $g_i = mes\Omega_i$ for $i \in I_1$.
Note that one can assume $\mu_i(t) = \mu_i = const > 0$ for some indices i with corresponding simplifications.

The argument $\tau_i(\bar{u})$ in (7) is the average value of the gradient $|\nabla\bar{u}|$ in Ω_i, $i \in I_1$, where $\bar{u} \in \mathbf{H}^1(\Omega_i)$ is the harmonic extension (in a weak sense) of $u_i = \bar{u}_{|\Gamma_i}$ into Ω_i. So one can easily obtain the boundary reduction of

(6), (7). In fact, if we introduce the "interior" Poincaré-Steklov operators $\mathbf{S}_i^{-1} : \mathbf{H}^{1/2}(\Gamma_i) \to \mathbf{H}^{-1/2}(\Gamma_i)$ associated with the Laplacian in Ω_i for $i \in I_1$ and define the trace space $\mathbf{Y}_\Gamma := \{u = \bar{u}_{|\Gamma} : \bar{u} \in \mathbf{W}(\Omega)\}$ on Γ equipped with the norm

$$\|u\|_{Y_\Gamma} = \inf_{\bar{u}_{|\Gamma}=u} \|\bar{u}\|_{W(\Omega)}, \tag{8}$$

then the desired nonlinear interface problem reads as follows:
Problem INI. *Given $\Psi \in \mathbf{Y}^*(\Gamma_0)$, find $u \in \mathbf{Y}_\Gamma$, such that*

$$\langle \mathbf{A}_{IN} u, v\rangle := \sum_{i=1}^{M} \bar{\mu}_i(u)(\mathbf{S}_i^{-1}u_i, v_i) + (\mathbf{S}_0^{-1}u, v)_{\Gamma_0} = \tag{9}$$

$$= (\Psi, v)_{\Gamma_0} \qquad \forall v \in \mathbf{Y}_\Gamma ,$$

where $\bar{\mu}_i(u) = \mu_i(\tau_i(u))$ with

$$\tau_i(u) = \left[\frac{1}{g_i}(\mathbf{S}_i^{-1}u_i, u_i)\right]^{1/2}, \quad i \in I_1. \quad \square \tag{10}$$

Let $\mu_i > 0$ be some given positive constants for $i \in I_0$ and $\mu_0 = 1$. For the efficient treatment of the problem (9), (10) we introduce following [14, 16, 19], the auxiliary linear interface operator $\mathbf{A}_\Gamma : \mathbf{Y}_\Gamma \to \mathbf{Y}_\Gamma^*$ defined by the associated bilinear form

$$\langle \mathbf{A}_\Gamma u, v\rangle = \sum_{i=0}^{M} \mu_i(\mathbf{S}_i^{-1}u_i, v_i) \qquad \forall u, v \in \mathbf{Y}_\Gamma \tag{11}$$

This operator is continuous, symmetric, positive definite and defines the equivalent "energy" norm in $\mathbf{Y}_\Gamma$ [16]

$$\|u\|_A = (\langle \mathbf{A}_\Gamma u, u\rangle)^{1/2} \qquad \forall u \in \mathbf{Y}_\Gamma .$$

3 The approximate interface problem

The mapping properties of the nonlinear operator $\mathbf{A}_{IN} : \mathbf{Y}_\Gamma \to \mathbf{Y}_\Gamma^*$ defined by (9) have been developed in [19] and further in [14]. In what follows we shall use only the following statement.

Lemma 1 *[19] Under the conditions (2), (3) the operator $\mathbf{A}_{IN} : \mathbf{Y}_\Gamma \to \mathbf{Y}_\Gamma^*$ is Lipschitz continuous with the constant $3M_0$ and strongly monotone with the constant m_0, where*

$$M_0 = \max_{i\in I_1} M_i \,, \quad m_0 = \min_{i\in I_1} m_i > 0 \qquad \square$$

Remark 1 *Under some additional assumption about the solution smoothness for the Problem C the following estimate*

$$\|u_\Delta\|_{Y_\Gamma} \le c(u) \cdot \max_{i\in I_1}(diam\ \Omega_i)$$

holds [19], where u_Δ is the difference between the solutions of the equations (9) and (5). $\square$

We now consider the boundary element Galerkin method for the approximation of (9), (10).

Remark 2 *We further restrict ourselves for brievity only by the two-dimensional case. Our construction can be easily extended on the case $d = 3$ which will be considered separately (see also Chapter 5).* $\square$

Let $\mathbf{W}_h \subset \mathbf{W}(\Omega)$ be some regular finite element space on Ω of piece-wise linear (or bilinear) elements associated with corresponding triangulation (or quadrangulation) of the domain Ω. We assume this triangulation being correlated with the decomposition of $\bar{\Omega} = \cup_{i=1}^{M}\bar{\Omega}_i$ such that $\mathbf{Y}_h = \mathbf{W}_{h_{|\Gamma}} \subset \mathbf{Y}_\Gamma$ is the desired Galerkin subspace of $\mathbf{Y}_\Gamma$. The subspace $\mathbf{Y}_h$ generates the corresponding Galerkin scheme for (9), (10)

$$u_h \in \mathbf{Y}_h : \langle \mathbf{A}_{IN} u_h, v\rangle = (\Psi, v) \qquad \forall v \in \mathbf{Y}_h \tag{12}$$

BEM-Galerkin approximation of the operator (11) can be constructed by the similar way. We refer to the corresponding finite-dimensional operators as $\mathbf{A}_\Gamma^h$ and $\mathbf{A}_{IN}^h$. Both Galerkin schemes are uniquely solvable and admit the standard error analysis [14, 16, 19]. On this way the Lax-Milgram theorem and Lemma 1 can be applied.

Of course, above mentioned naive Galerkin schemes have well known practical rectrictions, but they conserve the properties which are crutial for the analysis of the cost-effectiveness for further developed iterative methods. In fact, it can be shown that the h-harmonic extensions (associated with $\mathbf{W}_h$) of the elements from $\mathbf{Y}_h$ into the interiorities of the Ω_i, $i \in I_0$ lead to the boundary operator spectrally equivalent to $\mathbf{A}_\Gamma^h$ on $\mathbf{Y}_h$ (i.e., by using the Schur complements in subdomains) and so to the same convergence properties of the iterative schemes.

As a result, our further conclusions can be easily extended on the case of direct approximations of the Poincaré-Steklov operators [25]. But here we omit some purely technical details.

4 Fast iterative solver for (12) and its efficient implementation

For fast solving of the equation (12) one can use some iterative methods with preconditioned operator $\mathbf{A}^h_{IN}$. The linear operator $\mathbf{A}^h_\Gamma$ is a good candidate for such preconditioning, as well as any spectrally equivalent to $\mathbf{A}^h_\Gamma$ operator $\mathbf{B}$. Let us formulate the convergence result for the first Richardson method [6, 19].

Theorem 1 *Let under the conditions (2), (3) the constants $M_N, m_N > 0$ are defined from the iequalities*

$$\langle \mathbf{A}^h_{IN}u - \mathbf{A}^h_{IN}v, u - v\rangle \geq m_N \langle \mathbf{A}^h_\Gamma(u-v), u-v\rangle, \tag{13}$$

$$\|\mathbf{A}^h_{IN}u - \mathbf{A}^h_{IN}v\|^2_{Y^*_h} \leq M^2_N \langle \mathbf{A}^h_\Gamma(u-v), u-v\rangle$$

which hold for all $u, v \in \mathbf{Y}_h$, with some given constants $\mu_i > 0$, $i \in I_0$. Then the iterations

$$\langle \mathbf{A}^h_\Gamma \frac{u_{n+1}-u_n}{\tau}, v\rangle = \tag{14}$$

$$= -\langle \mathbf{A}^h_{IN}u_h, v\rangle + (\Psi, v)_{\Gamma_0}, \qquad \forall v \in \mathbf{Y}_h$$

converge for all $\tau \in (0, 2M_N^{-1})$ to the unique solution u^h of (12) with the rate

$$\|u_n - u^h\|_{Y_h} \leq \frac{\tau q^n}{1-q}\|\mathbf{A}^h_{IN}u_0\|_{Y^*_h} \tag{15}$$

for any $u_0 \in \mathbf{Y}_h$, where $q = max(1-\tau m_N, 1-\tau M_N)$. □

Clearly the constants m_N and M_N can be chosen as independent upon the mesh size h and one can substitute in (14) instead of $\mathbf{A}^h_\Gamma$ any easily invertible operator $\mathbf{B}$, spectrally equivalent to $\mathbf{A}^h_\Gamma$. Such operators have been constructed in [16]. In this way for the reduction of the residual in (12) by the factor $\varepsilon > 0$

$$Q_R = \log \varepsilon^{-1} \cdot O(Q(\mathbf{B}^{-1}) + Q(\mathbf{A}^h_{IN})) \tag{16}$$

arithmetic operations are required with storage needs of the order $O(dim\ \mathbf{Y}_h)$. Here $Q(\cdot)$ denotes the expence of matrix time vector multiplication for the corresponding finite dimensional operator.

When using the Newton's type methods for solving the equation (12) (this method have been analysed for this formulation in [19]) the corresponding computational expences Q_N can be presented in the form

$$Q_N = \log(\log \varepsilon^{-1}) \cdot O\Big(Q([\mathbf{A}'^h_{IN}]^{-1}) + Q(\mathbf{A}^h_{IN})\Big) \tag{17}$$

where $\mathbf{A}'_{IN}$ is the Fréchet derivative of the operator $\mathbf{A}_{IN}$. In both cases one can put $\varepsilon = O(h^{-\sigma})$, $\sigma > 0$ and so the dependence of Q_N from $\varepsilon > 0$ can be neglected. The same holds for Q_R if the process (14) is implemented on a sequence of grids with $\sigma \leq 3/2$. Besides, the first term in the right hand side of (16) can be really neglected in compare with the second one (see the details in [16]) for special constructions of the preconditioner $\mathbf{B}$. Both terms in (17) are compatible in the order of magnitude. Finally it turns out that the computational expence for solving of the equation (12) in framework of above mentioned iterative schemes is really proportional to the magnitude $Q(\mathbf{A}^h_{IN})$ when omitting the double logarithmic factor $\log\log h^{-1}$.

For fast computations of the residual in (12) let us consider the special algorithms for rapid computations with Poincaré-Steklov operators on the triangular domains which have been recently proposed in [18]. To that end we restrict ourselves by the regular partitioning of $\bar\Omega = \cup_{i=1}^M \bar\Omega_i$, where Ω_i, $i \in I_1$ are either rectangles or rectangular triangles and the artificial boundary Γ_0 is also rectangular boundary. We further assume for clarity that $N = (dim\mathbf{Y}_{h/\Gamma_i})$ for any $i \in I_1$ and the spaces $\mathbf{Y}_{h|\Gamma_i}$ are generated by the uniform meshs on any Γ_i, $i \in I_1$. Let $M = M_R + M_T$, where M_R and M_T are the numbers of rectangular and triangular subdomains, respectively, with the corresponding splitting $I_1 = I_R \cup I_T$.

Now we are in a position to apply special algorithms for fast treatment of Poincaré-Steklov operators $\mathbf{S}_i^{-1}$ in subdomains to construct some asymptotically almost optimal procedure for computating the residual in (12). In rectangular subdomains we can utilize the method for partial solution of the discrete Laplace equation on the rectangular domain [2] with a cost estimate of the order $O(N\log^2 N)$ for any $i \in I_R$. For the triangular subdomains Ω_i, $i \in I_T$ one can use the special algorithm proposed in [18] with expence of the order $O(N\log^3 N)$. So the matrix-vector multiplication for the total operator $\mathbf{A}^h_{IN}$ with above proposed approach for treatment of the corresponding subproblems can be done by

$$Q(\mathbf{A}_{IN}^{h}) = N \cdot \log^2 N \cdot O(M_R + M_T \cdot \log N) \qquad (18)$$

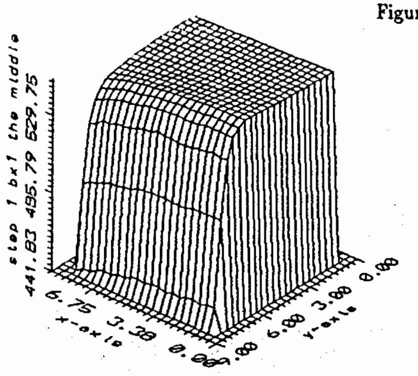
arithmetic operations. Due to above arguments the same estimate holds for the magnitude $Q((\mathbf{A}_{IN}^{h})^{-1})$.

Note that computations of the residual in (12) as well as inverting of the preconditioner $\mathbf{B}$ in (14) can be done in parallel with the number of parallel tasks equal to M. The only exclusion is the transfer of information between substructures on the stage of solving the coarse mesh problem when performing the action $\mathbf{B}^{-1}$.

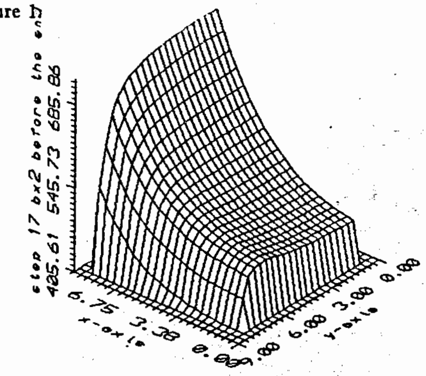
Remark 3 *The treatment of the operator $\mathbf{S}_0^{-1}$ can be performed with asymptotic expences of the order $O(N \log^3 N \cdot (\log\log N)^d)$ for $d = 2, 3$ by using some special decomposition of the unbounded domain $\Omega_0 = R^d \backslash \Omega$ [20] with $M_\infty = O\left[(\log\log N)^d\right]$ subdomains.* □

Note that though the number M_∞ slowly grows with respect to N (see above Remark), in practice it is "almost uniformly bounded" due to real behavior of the magnetostatic scalar potentials as $|x| \to \infty$. We mean the fast damping of the field components when moving off the border of the magnet.
There is another opportunity for enlargement of the coarse mesh which defines the basic decomposition of the domain under consideration. Let the nonlinear magnet medium be space extensive along z-axis (for instance). Then the most meaningfull analysis of the field components has to be done in the vicinity of the magnet border. So the field under computation becomes "almost two-dimensional" when moving off magnet border. Here we present in Figures 1-3 the graphs of the three components of vector $\vec{\mathbf{B}}$ being the induction of the magnet field for dipol superconductor magnet which have been numerically investigated in [29]. In this way $\vec{\mathbf{B}}\,(x) = -\nabla u(x) + \vec{\mathbf{B}}_0, \quad x \in \Omega_0$. The detailed description of the magnet design and the corresponding calculations can be found in [29]. Here we demostrate only that the field is almost two-dimensional in the neighbourhood of the point $z = 0$ (the center of the magnet) being really three-dimensional in the vicinity of the magnet border ($z = 18h$).

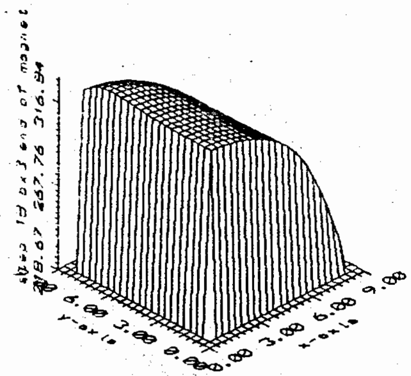
Figure 1



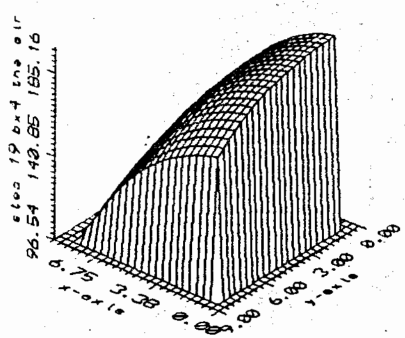
B_X at the step $z = 0$
(the center of the magnet).



B_X at the step $z = 17h$
(before the edge of the magnet).

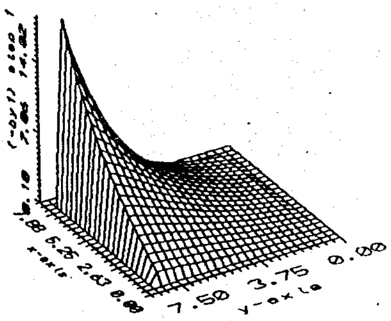


B_X at the step $z = 18h$
(at the edge of the magnet).

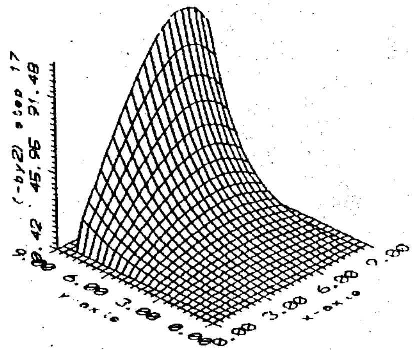


B_X at the step $z = 19h$
(after the edge of the magnet).

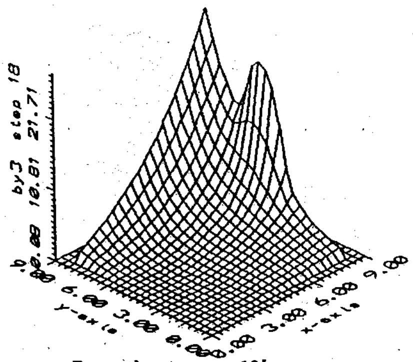
Figure 2.



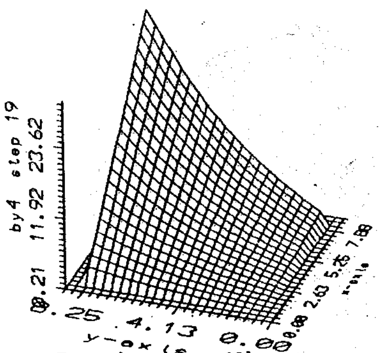
$-B_Y$ at the step $z = 0$
(the center of the magnet).



$-B_Y$ at the step $z = 17h$
(before the edge of the magnet).

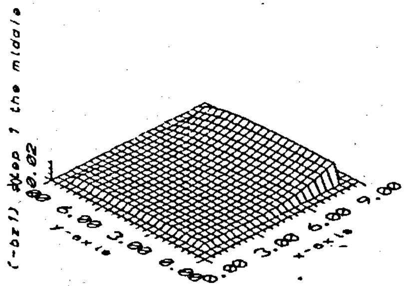


B_Y at the step $z = 18h$
(at the edge of the magnet).

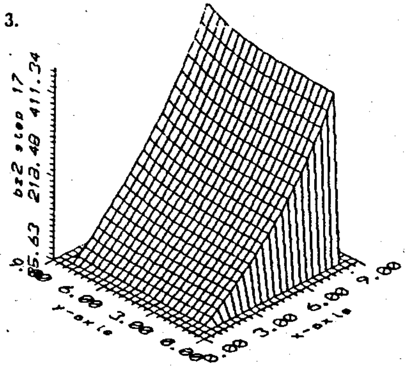


B_Y at the step $z = 19h$
(after the edge of the magnet).

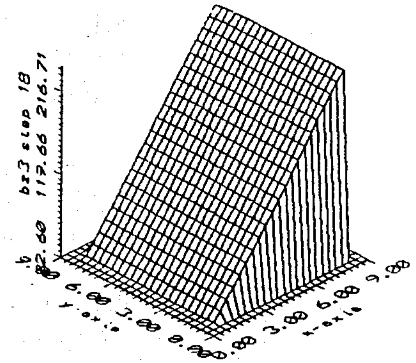
Figure 3.



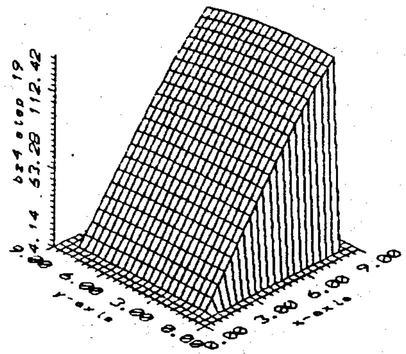
$-B_Z$ at the step $z = 0$
(the center of the magnet).



B_Z at the step $z = 17h$
(before the edge of the magnet).



B_Z at the step $z = 18h$
(at the edge of the magnet).



B_Z at the step $z = 19h$
(after the edge of the magnet).

5 Concluding remarks

The above proposed computing strategy for calculations with iterative substracturing methods in quasi-linear elliptic problems of magnetostatics leads to the asymptotically near optimal algorithms with respect to the operation count and memory needs. These algorithms are highly parallelizable.
The cost estimate (18) can be extended on the three-dimentional case if we set in (18) M_R as a number of parallelepipeds and M_T as a number of rectangular prisms (with the rectangular triangle in the base cutting) which define the decomposition of Ω_0 having also the parallelepiped type boundary. Omitting the thorough justifications, we only note that for such kinds of subdomains the three-dimensional counterparts of the fast algorithms for the treatment of the Poincaré-Steklov operators can be applied [18]. The alternative approach for the fast matrix multiplication in BEM by panel clustering have been developed in [9].

The crutial point for our approach is the well developed efficient preconditioning techniques for linear elliptic operators with piece-wise constant coefficients for $d = 2,3$, including the case with degenerated substructures [10]. This allows to extend our approach on the class of nonquasiuniform decompositions. Some additional flexibility can be achieved by incorporating the nonconformal substructuring techniques. Note that up to day approaches for the construction of the wide range programming systems for the elliptic problems have been considered in [15].

To complete this item we note that some numerical results for the case $d = 3$, $M = 150$, $M_T = 0$ and finite-difference approximations of the Poincaré-Steklov operators have been performed in [20]. These results confirm theoretical conclusions about the asymptotically quadratic grows (only!) of the computing times while twice decreasing of the average mesh size h for the three-dimensional problems.

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