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ITERATIVE PROCEDURE AS EQUATION OF MOTION

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1. Introduction

The majority of realistic problems do not allow exact solutions. The usual way of handling these problems is to resort to an iterative procedure or perturbation theory. The resulting sequence of iterates $\{f_k : k = 0, 1, 2, \dots\}$ is very often poorly convergent or even divergent. If one could calculate many $(k \geq 5)$ first terms of an iterative procedure, then one would be able to improve the convergence or find an effective limit of a divergent sequence by invoking resummation techniques such as Padé approximation, Borel transformation, conformal mapping, continued - fraction representation and so on[1].

For example, when solving the Schrödinger equation with anharmonic potentials by iterating the Brillouin - Winger perturbation formula, one meets with the fact that the convergence rate decreases markedly as whether the coupling constant, anharmonicity power or the energy - level number increases[2]. In such a case, the accuracy can be improved by using the hypervirial theorem and Padé approximations[3, 4].

However, to calculate many iterates is often technically impossible. The standard situation is when one is able to find only a few first iterative terms. In this case, the usual resummation techniques fail.

The technical difficulties arising during an iteration process can be explained as follows. Generally, an iterative operator $\hat{I}_k$ transforming the term f_k into f_{k+1} depends on the iteration number k,

$$f_{k+1} = \hat{I}_k f_k.$$

If the iterative operator is not simple enough, then the calculation of a term

$$f_k = \hat{I}_{k-1} f_{k-1} = \hat{I}_{k-1} \hat{I}_{k-2} f_{k-2} = \ldots = \hat{I}_{k-1} \hat{I}_{k-2} \ldots \hat{I}_0 f_0$$

becomes a rather complicated, if in principle possible, task.

This difficulty could be overcome if we would be able to reconstruct the sequence $\{f_k\}$ to $\{f'_k\}$ for which the iterative operator would be independent of the iteration number, i.e.

$$f'_{k+1} = \hat{I} f'_k.$$

If, in addition, the iterative operator is simple and contracting, then one could easily find any iterate

$$f'_k = \hat{I}^k f'_0.$$

Such iterated maps can be treated as dynamical systems[5]. The motion in discrete time is called a cascade. The iterative transformation being a contracting one implies that the sequence $\{f'_k\}$ converges to an attractor representing the sought solution.

In this report we show that an arbitrary sequence of iterates can be approximated by a cascade whose attractor is the sought limit of this sequence. The advantages of this results are obvious: we need to know only a few initial terms in order to find an effective limit of a sequence, and the convergence of the latter can be checked by considering the stability of motion for the cascade. The approach described

below has been developed[6-8] at first by basing on the renormalization - group ideas, although the analogy with dynamical systems has been also emphasized[8, 9]. Here we demonstrate that the dynamical - theory language makes the interpretation of the method much simpler and permits some important generalizations.

2. Iterative Cascade

Suppose we are interested in a function $f(g)$ with the variable $g \in \mathcal{R}$, which satisfies a very complicated equation to be solved using an iterative procedure. The convergence of the latter is known to depend on the choice of an initial approximation $f_0(g)$. The procedure can be contracting for some values of $f_0(g)$ and divergent for others. All possible initial approximations providing the uniform convergence form an attraction region $\mathcal{L}$. In this way, a sequence of iterates interpreted as a cascade moves to an attractor provided that $f_0(g) \in \mathcal{L}$. The convergence is faster for those sequences whose zero approximations are closer to the attractor.

In order that a sequence of iterates would be in the vicinity of an attractor, this sequence should be governed in some way. To this end, by choosing an initial approximation, introduce into it a set of trial parameters z, so that $f_0(g) = f_0(g, z)$. Then, all further iterates also become dependent on these parameters: $f_k(g) \to f_k(g, z)$. Define a set of functions $z_k(g)$ whose role is to govern the behaviour of the sequence

$\{f_k(g)\}$ formed of the terms

$$f_k(g) \equiv f_k(g, z_k(g)); \quad k = 0, 1, 2, \ldots \tag{1}$$

so as to keep these terms close to an attractor. Because of their role, the functions $z_k(g)$ are called[6-9] the governing functions, and the set

$$\mathcal{G} \equiv \{z_k(g) : k = 0, 1, 2, \ldots; \; g \in \mathcal{R}\} \tag{2}$$

can be named the government. By definition, the governing functions guarantee that all terms

$$f_0(g, z_0(g)) \cong f_1(g, z_1(g)) \cong \ldots = f_*(g, z_*(g))$$

are close to an attractor and move to it as k increases; the attractor representing the sought function $f(g)$,

$$f_*(g) \equiv f_*(g, z_*(g)) \cong f(g). \tag{3}$$

Therefore, the general definition of the governing functions can be given as the relation

$$f_{k+p}(g, z_{k+p}(g)) \cong f_k(g, z_k(g)); \quad k, p \geq 0, \tag{4}$$

which is to be understood in the sense of the Cauchy criterion of the uniform convergence

$$|f_{k+p}(g, z_{k+p}(g)) - f_k(g, z_k(g))| < \epsilon,$$

where $k \geq s(\epsilon)$, $p \geq 0$, $g \in \mathcal{R}$. The relation (4) follows from a particular form of the Cauchy criterion in which $s(\epsilon) = 0$, and which can be called the fastest - convergence criterion[8].

The idea of reconstructing a sequence of approximations in order that this sequence would be convergent has been advanced in Refs.10,11 by introducing additional functions satisfying the fastest - convergence criterion. This is now widely known as renormalized, or modified, perturbation theory. However, an option of the governing functions $z_k(g)$ up to now has been heuristic, thus there has been no firm grounds to prefer one of several known variants[10-15].

Note that the introduction of the governing functions can be combined with the use of integral transformations, like the Borel transformation or the Hubbard transformation

$$\exp(-\varphi^2) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{4} \pm iu\varphi\right) du \; .$$

In the dynamical approach we follow here, different adoptions of the governing functions can be classified and analysed, so that it becomes possible to conclude which of the variants is generally preferable.

The interpretation of the sequence $\{f_k(g)\}$ as a dynamical system assumes the necessity of defining a mapping transmuting a term $f_k(g)$ into $f_{k+1}(g)$. To find such a map, let us introduce some new notation. Define the coupling function $g(f)$ by the equation

$$f_0(g, z_0(g)) = f; \quad g = g(f). \tag{5}$$

Introduce the function

$$x_k(f) \equiv f_k(g(f), z_k(g(f))), \tag{6}$$

whose limiting properties in accordance with (5) and (3) are

$$x_0(f) = f \tag{7}$$

and, respectively,

$$x_*(f) = f_*(g(f)). \tag{8}$$

Defining the inverse function $f^{-1}(g)$ by the equation

$$g(f^{-1}) = g; \quad f^{-1} = f^{-1}(g),$$

we can return from (6) to (1) following the relation

$$x_k(f^{-1}(g)) = f_k(g, z_k(g)) = f_k(g).$$

The fastest - convergence condition (4) in terms of (6) reads

$$x_{k+p}(f) \cong x_k(f). \tag{9}$$

Putting here $k = 0$, we get $x_p(f) = f$ substituting which into the right - hand side of (9) we obtain

$$x_{k+p}(f) = x_k(x_p(f)). \tag{10}$$

This relation is often called the property of functional self - similarity. This is why the method based on (10) has been called the method of self - similar approximations[6–9].

On the other hand, the functional property (10) characterizes, as is known[5], a dynamical system in discrete time, that is a cascade. The

ordered sequence of terms (6) starting at the point (7) is named an orbit, or trajectory,

$$\mathcal{T}_f \equiv \{f, x_1(f), x_2(f), \ldots\}.$$

In this way, to any sequence $\{f_k(g)\}$ satisfying the fastest - convergence criterion (4) it is admissible to put into correspondence a sequence $\{x_k(f)\}$ with the relation (10) characteristic of a cascade.

3. Iterative Flow

An additional information can be extracted if we pass from the cascade to a flow. Introduce a continuous variable

$$t \in \mathcal{R}_+ \equiv [0, +\infty), \tag{11}$$

and make an analytical continuation of (6) to a function $x(t, f)$ such that when t crosses a positive integer, say $t = k$, then $x(t, f)$ coincides with the corresponding value of $x_k(f)$,

$$x(k, f) = x_k(f); \quad k = 0, 1, 2, \ldots \tag{12}$$

The cascade property (10) for the function $x(t, f)$ becomes that for the flow

$$x(t + t', f) = x(t, x(t', f)). \tag{13}$$

The initial condition (7) now is

$$x(0, f) = f. \tag{14}$$

The existence of an attractor expressed in (3) and (8) reads

$$x(s_*, f) = f_*(g(f)), \tag{15}$$

where s_* is a saturation number[8].

The self - similar relation (13) can be presented in the differential form. Differentiating (13) with respect to t and then putting $t \to 0$, $t' \to t$, we get

$$\frac{d}{dt}x(t,f) = v(x(t,f)), \tag{16}$$

where

$$v(t) \equiv \lim_{t\to 0}\frac{d}{dt}x(t,f). \tag{17}$$

The latter function is called the vector field, or velocity. Equation (16) is typical of an autonomous dynamical system, that is of a flow. The trajectory is

$$\mathcal{T}_f \equiv \{x(t,f): \ t \in [0,\infty)\}. \tag{18}$$

Thus, we have shown that an iterative sequence $\{f_k(g)\}$ can be represented as a flow with the equation of motion (16) describing the trajectory (18). Therefore, the sequence

$$\mathcal{T}_{rep}(g) \equiv \{f_k(g): \ k = 0, 1, 2\ldots\}$$

can be called the representation of trajectory (18). We have managed to obtain the equation of motion (16) by introducing the governing functions providing the property of self - similarity (13). It is possible to say that the self - similar symmetry (13) is imposed by the governing functions.

This fact principally distinguishes our approach from continuous analogs of concrete iterative methods[16, 17] as well as from the renormalization - group method[18] of quantum - field theory, based on symmetry properties of particular equations of motion.

Integrating Eq.(16) from a k -th approximation to the saturation point s_*, we have

$$\int_{x(k,f)}^{x(s_*,f)} \frac{dx}{v(x)} = s_* - k. \tag{19}$$

The substitution $f \to f^{-1}(g)$ for the lower and upper limits gives

$$x(k, f^{-1}(g)) = f_k(g), \quad x(s_*, f^{-1}(g)) = f_*(g).$$

As a result, it follows from Eq.(19) that

$$\int_{f_k(g)}^{f_*(g)} \frac{df}{v(f)} = s_* - k. \tag{20}$$

This equation defines the sought self - similar approximation $f_*(g)$.

4. Vector Field

To integrate (20) we need to know the explicit form of the vector field $v(f)$. In reality, all the information we have is related to the discrete representation. Therefore, we are forced to return to it wishing to write an expression for the vector field. The discrete representation for the latter can be written[6–9] as

$$v_{sk}(f) = \frac{\Delta_{sk}(f)}{s-k} \tag{21}$$

by using the finite difference

$$\Delta_{sk}(f) = f_s(g, z_k) - f_k(g, z_k) + (z_s - z_k)\frac{\partial}{\partial z_k} f_k(g, z_k), \tag{22}$$

in which

$$g = g(f), \quad z_k = z_k^*(g(f)), \quad k < s.$$

Then, integral (20) is to be replaced by

$$\int_{f_k(g)}^{f_{sk}^*(g)} \frac{df}{v_{sk}(f)} = s_* - k, \tag{23}$$

the self - similar approximation $f_{sk}^*(g)$ being dependent on the chosen velocity (21).

Define the relative fixed - point distance

$$\delta_{sk} \equiv \frac{s_* - k}{s - k}. \tag{24}$$

With notation (24) the integral (23) takes the form

$$\int_{f_k(g)}^{f_{sk}^*(g)} \frac{df}{\Delta_{sk}(f)} = \delta_{sk}. \tag{25}$$

A more elegant expression can be given for (25) by introducing the function

$$y_{sk}(f) \equiv \{\Delta_{sk}(f)\delta_{sk}\}^{-1}. \tag{26}$$

Then, (25) can be written as the normalization law

$$\int_{f_k(g)}^{f_{sk}^*(g)} y_{sk}(f)df = 1. \tag{27}$$

If we assume that the attractors of the considered cascades and flows are fixed points but not limiting cycles or chaotic and strange attractors, then we can readily derive the corresponding stability conditions[19]. To converge to a fixed point, the self - similar mapping (10) has to be contracting, which implies that the corresponding mapping multipliers must be smaller than unity, these multipliers being defined by

$$M_{sk}^p(g) \equiv \lim_{f \to f_{sk}^*(g)} \left| \frac{d}{df} f_p(g(f), z_p(g(f))) \right|. \tag{28}$$

In addition, the equation of motion (16) can be analyzed with respect to the asymptotic Lyapunov stability[19], which requires that the Lyapunov exponent

$$\Lambda_{sk}(g) \equiv \lim_{f \to f_{sk}^*(g)} \frac{d}{df} v_{sk}(f) \tag{29}$$

has to be negative. Thus, the sufficient conditions for the fixed point $f_{sk}^*(g)$ to be stable are

$$M_{sk}^p(g) < 1, \quad \Lambda_{sk}(g) < 0. \tag{30}$$

These two conditions control the choice of governing functions and of the vector field.

5. Ergodic Sequence

It may happen that the dynamical system described by Eq.(16) may have an attractor which is not a fixed point (stable node or stable focus) but which, e.g., is a stable limit cycle, stable torus, quasiattractor, chaotic attractor or strange attractor. Then, the stability conditions (30) are not valid. How is it possible then to define a correct self - similar approximation of the sought function?

When an attractor is not a fixed point, then the analysis of the Lyapunov stability should be replaced by that of the Poisson stability[20], although for a limit cycle and torus the Lyapunov analysis can be applied. A function $x(t,f)$, real and continuous, given for $t \in \mathcal{R}_+$ is called stable á la Poisson, or Poisson stable, if for each $\epsilon > 0$ and any $t \in \mathcal{R}_+$ one can define an infinite sequence $\{t_p = t_p(\epsilon, t) : \ p = 0, 1, 2, \ldots\}$ for which $t_p \rightarrow \infty$ as $p \rightarrow \infty$, such that

$$|x(t + t_p, f) - x(t, f)| < \epsilon; \quad p = 0, 1, 2 \ldots. \tag{31}$$

A periodic motion corresponding to a limit cycle is, as is obvious, Poisson stable. Then, one may put $t_p = pT$, where T is a period. A quasiperiodic motion corresponds to the motion on a torus. Almost a periodic motion is related to quasiattractor. Both the latter motions are Poisson stable[20]. The motion on a chaotic attractor is mixing. A strange attractor is particular kind of the chaotic attractor with a dimensionality lower than a manifold into which it is embedded. The mixing motion is also Poisson stable.

All kinds of attractors are metrically transitive. Therefore, we can define the ergodic average

$$x_{erg}(f) \equiv \lim_{T\rightarrow\infty} \frac{1}{T} \int_0^T x(t, f) dt, \tag{32}$$

which must be independent of an initial point,

$$x_{erg}(f) = x_{erg}. \tag{33}$$

The discrete analog of (32) is

$$f_{erg}(g) \equiv \lim_{p\rightarrow\infty} \frac{1}{p} \sum_{k=1}^{p} f^*_{sk}(g), \tag{34}$$

where $s = s(k) > k$ and the initial term corresponding to $k = 0$ is omitted in accordance with (33). The ergodic average (34) is the limit

$$\lim_{p\rightarrow\infty} f^*_p(g) = f_{erg}(g)$$

of the sequence $\{f^*_p(g) : \ p = 1, 2 \ldots\}$ composed of the quasiergodic terms

$$f^*_p(g) \equiv \frac{1}{p} \sum_{k=1}^{p} f^*_{sk}(g); \quad s = s(k). \tag{35}$$

Therefore, we may call the sequence $\{f^*_p(g)\}$ the ergodic sequence.

The definition of the ergodic sequence gives us a practical tool for constructing higher orders of self - similar approximations.

References

1. P.M.Stevenson, *Nucl. Phys.* **B231** (1984) 65.

2. F.M.Fernandez, A.M.Meson and E.A.Castro, *J. Phys.* **A18** (1985) 1389.

3. I.K.Dmitrieva and G.I.Plindov, *Phys. Lett.* **A79** (1980) 47.

4. F.M.Fernandez and E.A.Castro, *Phys. Lett.* **A88** (1982) 4.

5. P.Collet and J.Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, (Birkhaüser, Basel, 1980).

6. V.I.Yukalov, *Int. J. Mod. Phys.* **B3** (1989) 1691.

7. V.I.Yukalov, *Phys. Rev.* **A42** (1990) 3324.

8. V.I.Yukalov, *J. Math. Phys.* **32** (1991) 1235.

9. V.I.Yukalov, *Physica* **A167** (1990) 833.

10. V.I.Yukalov, *Mosc. Univ. Phys. Bull.* **31** (1976) 10.

11. V.I.Yukalov, *Theor. Math. Phys.* **28** (1976) 652.

12. P.M.Stevenson, *Phys. Rev.* **D23** (1981) 2916.

13. P.M.Stevenson, *Phys. Rev.* **D24** (1981) 1622.

14. C.Z.An, I.D.Feranchuk, L.I.Komarov and L.S.Nakhamchik, *J. Phys.* **A19** (1986) 1583.

15. A.N.Sissakian and I.L.Solovtsov, *Z.Phys.* **C54** (1992) 263.

16. M.K.Gavurin, *Izv. Vuz. Mat.* **5** (1958) 18.

17. T.Zhanlav and V.I.Puzynin, *Zh. Vych. Mat. Mat. Fiz* **31** (1992) 3.

18. N.N.Bogolubov and D.V.Shirkov, *Introduction to Theory of Quantized Fields*, (Nauka, Moscow, 1973).

19. V.I.Yukalov, *Preprint JINR*, **E5-92-153** (Dubna, 1992)

20. V.I.Zubov, *Dokl. Akad. Nauk* **322** (1992) 28.

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