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ON INVARIANT MEASURES FOR SOME INFINITE-DIMENSIONAL DYNAMICAL SYSTEMS

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1. Introduction

Recently several papers have been published on invariant measures for dynamical systems (DS) defined by nonlinear partial differential equations [1-5]. In paper [1] that measure is constructed for the periodic problem for the nonlinear Klein-Gordon equation and in paper [2] the similar construction is performed for a certain physical system. Unfortunately, in paper [1] some important steps of the proof are omitted. In the author's paper [3] the invariant measure is constructed for a nonlinear Schrödinger equation (NSE) under some strict assumptions on the nonlinearity. Partially these difficulties are avoided in paper [4] where the power nonlinearities are admissible. The next author's paper [5] contains simpler approach to the same problem. The nonlinear wave equation is considered. However, as it is remarked, one can easily apply this technique for the investigation of NSE.

Invariant measures play an important role in the theory of dynamical systems. It is well known that the whole ergodic theory is based on this concept. On the other hand, they are necessary in various physical cosiderations. In paper [6] they are used for the construction of the statistical mechanics corresponding to the NSE (however the proof of the invariance is not presented). Similar considerations are made in papers [7-10] where the Kubo-Martin-Schwinger states are constructed but without the proof of the invariance, too.

For the author, the first point which directs him to this investigation was the so-called Fermi-Past-Ulam phenomenon consisting in the return of an arbitrary trajectory of the DS defined by any "soliton" equation to the initial data with time with an arbitrary accuracy (see [11-13], for example). Using the finite invariant measure one can apply the Poincare recurrence theorem which explains this phenomenon.

In the present paper we consider the abstract Hamiltonian system introduced in paper [14] for the investigation of the soliton stability. In particular, a wide class of the "soliton" equations to be studied later for application admits this representation.

2. Notation. Basic results

Let $Y \subset X$ be real Hilbert spaces with the scalar products $(\, , \,)_Y$ and $(\, , \,)_X$ and the norms $||g||_Y = (g,g)_Y^{\frac{1}{2}}$ and $||g||_X = (g,g)_X^{\frac{1}{2}}$ respectively, satisfying the condition

$$||g||_X \leq C \; ||g||_Y$$

with $C > 0$ independent of $g \in Y$. Let Y be a dense set in X. Let $X_1 \subset X_2 \subset ... \subset X_n \subset ...$ be a sequence of finite-dimensional subspaces of Y, $dim \; X_n = d_n < \infty$, and let $\bigcup_n X_n$ be a dense set in Y. Let H be a C^1 functional on Y with real values and let $J: \; X^* \to X$ be a (generally unbounded) linear operator defined on the dense set $D \subset X^*$ with

$$g(Jh) = -h(Jg)$$

for any $g, h \in D$ where $g(h)$ is the value of $g \in X^*$ of $h \in X$. It is clear that any $g \in X^*$ belongs to Y^*.

Consider the problem

$$\dot{u}(t) = JH'(u(t)), \qquad t \in R, \tag{1}$$

$$u(t_0) = \phi \in X. \tag{2}$$

Here the dot means the derivative with respect to t, $t_0 \in R$ and $u(t)$ is the unknown function. In addition we consider the sequence of finite-dimensional problems

$$\dot{u}^n(t) = P_n[JH'(P_n u^n(t))], \qquad t \in R, \tag{3}$$

$$u^n(t_0) = P_n\phi\,, \tag{4}$$

where P_n is the orthogonal projector onto X_n in X.

It is obvious that $P_n^* X^*$ is the adjoint space for X_n if P_n^* is the adjoint operator for P_n on X. We assume that J is defined on any X_n^* and

$$P_n J = J P_n^* \qquad (n = 1, 2, 3, \ldots).$$

Remark 1

As it is well known, the norms $\|\ \|_X$ and $\|\ \|_Y$ are equivalent on any X_n.

We denote $I = [t_0 - T, t_0 + T]$ for any $T > 0, t_0 \in R$ and by $C(I; B)$ the space of continuous bounded functions from I into B with the norm $\|g(t)\|_{C(I;B)} = \sup_{t \in I} \|g(t)\|_B$, where B is an arbitrary Banach space with the norm $\|\ \|_B$. By the above assumptions the operator from the right-hand side of (3) is of the class C^1 as the map from X_n into X_n. Hence, for any $\phi \in X$ there exists $T > 0$ such that there exists a unique solution of the problem (3)-(4) of the class $u^n(t) \in C(I; X_n)$.

Remark 2

In particular, the above solution $u^n(t)$ belongs to $C(I; X)$.

Assumption 1

Let $u^n(t)$ be defined for all $t \in R$ and for all $\phi \in X$.

Definition 1

Let for any $\phi \in X$ there exist $u(t) \in C(I; X)$ such that there exists a sequence $u^n(t)$ converging to $u(t)$ in $C(I; X)$ for any I. Then, we call $u(t)$ the solution of the problem (1)-(2).

Assumption 2

Let there exist a unique solution $u(t)$ of the problem (1)-(2) for any $\phi \in X$.

Assumption 3

Let for any $t_0 \in R, \epsilon > 0, T > 0$ there exist $\delta > 0$ such that

$$\|u_1^n(t) - u_2^n(t)\|_X < \epsilon \qquad (n = 1, 2, 3, \ldots)$$

for any two solutions of equation (3) such that

$$\|u_1^n(t_0) - u_2^n(t_0)\|_X < \delta$$

and for any $t \in I$.

Corollary 1

For any $t_0 \in R, \epsilon > 0, T > 0$ there exists $\delta > 0$ such that

$$\|u_1(t) - u_2(t)\|_X < \epsilon$$

if

$$\|u_1(t_0) - u_2(t_0)\|_X < \delta$$

for all $t \in I$ and for any two solutions u_1 and u_2 of the problem (1)-(2).

Now we briefly remind the general construction of a Gaussian measure on a Hilbert space (for details see [15-17]). For a Hilbert space consider X. Let $\{e_k\}$ be the orthonormal basis in X which consists of eigenvectors of some operator $S = S^* > 0$ with corresponding eigenvalues $\{\lambda_k\}$ $(k = 1, 2, 3, \ldots)$. We call a set $M \subset X$ the cylindrical set iff

$$M = \{x \in X | [(x, e_{j_1})_X, \ldots, (x, e_{j_m})_X] \in F\}$$

for some Borel set $F \subset R^m$, some integer m and j_i is not equal to j_l if i is not equal to l. We define the measure w as follows:

$$w(M) = (2\pi)^{-\frac{m}{2}} \prod_{l=1}^{m} \lambda_{j_l}^{\frac{1}{2}} \int_F e^{-\frac{1}{2}\sum_{i=1}^{m} \lambda_{j_i} y_i^2}\, dy\,,$$

where $y = (y_1, \ldots, y_m) \in R^m$ and dy is the Lebesque measure in R^m. One can easily verify that the class $\mathcal{A}$ of all cylindrical sets is an algebra on which the function w is additive. The function w is called the centred Gaussian measure with the correlation operator S^{-1} on X. The basic result is as follows:

Statement

The measure w is σ-additive on the algebra $\mathcal{A}$ iff S^{-1} is a nuclear operator.

If the measure w is σ-additive on $\mathcal{A}$ one can continue this measure to the minimal σ-algebra $\mathcal{M}$ containing $\mathcal{A}$ using standard methods. In fact, $\mathcal{M}$ is the Borel σ-algebra of X (see [15-17]).

Assumption 4

Let $H(u) = \frac{1}{2}(Su, u)_X + g(u)$ where $S^* = S > 0$ is the (unbounded) operator on X mapping X_n into X_n $\quad (n = 1, 2, 3, \ldots)$ and $g(u)$ is a continuous real functional on X. Let S^{-1} be a nuclear operator on X.

Definition 2

We denote by $f(\phi, t)$ the function from X into X mapping ϕ into $u(t+t_0)$, where $u(t)$ is a solution of the problem (1)-(2). By analogy, let $f_n(\phi, t)$ be the function from X into X_n mapping $\phi \in X$ into $u^n(t+t_0)$, where $u^n(t)$ is a solution of the problem (3)-(4). It is clear that $f(\phi, 0) = \phi$, $f(f(\phi, \tau), t) = f(\phi, t + \tau)$ and $f_n(\phi, 0) = P_n\phi$, $f_n(f_n(\phi, \tau), t) = f_n(\phi, t + \tau)$ for any $\phi \in X$, $t, \tau \in R$. So, we call f and f_n the dynamical systems (DS) on the phase spaces X and X_n, respectively. We call a Borel measure μ on X

the invariant measure for DS f iff $\mu(\Omega) = \mu(f(\Omega, t))$ for any Borel set $\Omega \subset X$ and $t \in R$.

The basic result of this paper is the following:

Theorem 1

Let Assumptions 1-4 be valid and let μ be a Borel measure on X defined for any Borel set $\Omega \subset X$ by the rule

$$\mu(\Omega) = \int_{\Omega} e^{-g(u)}\, w(du)\,,$$

where w is the centred Gaussian measure corresponding to the correlation operator S^{-1}. Then, μ is the invariant measure for DS f.

Remark 3

Since we do not claim the boundedness of the functional g, generally the measure μ is not finite. It is not difficult to formulate the conditions for the finiteness of μ. For example, μ is a finite measure if g is bounded from below in addition to the above assumptions.

As we will see further, Assumption 3 makes the class of nonlinearities of the admissible partial differential equations very narrow. So, we present one more result which helps to prove the invariance of the measure μ for a more wide class of nonlinearities.

Let $H_N(u) = \frac{1}{2}(Su, u)_X + g_N(u)$ $(N = 1, 2, 3, \ldots)$. Consider the sequence of the problems

$$\dot{u}_N(t) = JH'_N(u(t)), \qquad t \in R, \tag{5}$$

$$u(t_0) = \phi \in X. \tag{6}$$

Let for any N the Assumptions 1-4 be valid for the problem (5)-(6). We denote solutions of this problem by $u_N(t)$.

Assumption 5

Let $G(u)$ be a real functional on X such that $e^{-g_N(u)}$ converges to $G(u)$ for any $u \in X$ when N tends to ∞. Let for any $\phi \in X$ $u_N(t)$ tends to some $u(t)$ when $N \to \infty$ in X for any $\phi \in X$ and $t \in R$. Then, one can call $u(t)$ the solution of the problem (1)-(2). So, DS f will be defined in this case, too.

Theorem 2

Under Assumption 5 the measure $\mu(\Omega) = \int_{\Omega} G(u)w(du)$ is invariant for DS f.

Remark 4

In each situation, one should verify that the measure μ is non-trivial, i.e. that μ is not equal to zero for any set. In particular, in the case of NSE that proof was made in paper [4].

3. Proof of theorem 1

Since S maps X_n into X_n there exists an orthonormal basis $\{e_k\}$ of its eigenvectors with respect to the product of the space X with corresponding eigenvalues $\{\lambda_k\}$ $(k = 1, 2, 3, \ldots)$ such that $e_1, \ldots, e_{d_n}$ is the basis of X_n for any n. Let $u^n(t) = \sum_{k=1}^{d_n} a_k(t)e_k$, $h(\mathbf{a}) = H(\sum_{k=1}^{d_n} a_k e_k)$, where $\mathbf{a} = (a_1, \ldots, a_{d_n})$ and let J_n be the matrix of the operator J from $X_n^\star$ into X_n in the bases $\{e_k^\star\}$ and $\{e_k\}$, where $\{e_k^\star\}$ is the dual basis to $\{e_k\}$. Then, $J_n^\star = -J_n$ and the problem (3)-(4) takes the form

$$\dot{\mathbf{a}}(t) = J_n \nabla_{\mathbf{a}} h, \tag{7}$$

$$a_k(t_0) = (\phi, e_k) \qquad (k = 1, 2, \ldots, d_n). \tag{8}$$

Then, R^{d_n} is the phase space for this problem. We use the following result. Consider a dynamical system of the form

$$\dot{z} = f(z) \tag{9}$$

where $z = z(t) \in R^r$ with some integer r and a smooth function $f = (f_1, \ldots, f_r)$. Let for any Borel set $C \subset R^r$

$$\rho(C) = \int_C \lambda(z)\, dz\,,$$

where $\lambda(z) > 0$ is a smooth function and dz is the Lebesque measure in R^r. Then, the measure ρ is invariant for the system (9) iff

$$\sum_{i=1}^{r} \frac{\partial}{\partial z_i}(\lambda f_i) = 0$$

for all z. (For the proof, see [18]).

Using this result one can easily verify that the Borel measure

$$\mu'_n(A_n) = (2\pi)^{-\frac{d_n}{2}} \prod_{k=1}^{d_n} \lambda_k^{\frac{1}{2}} \int_A e^{-\frac{1}{2}\sum_{k=1}^{d_n} \lambda_k a_k^2 - g(\sum_{k=1}^{d_n} a_k e_k(x))}\, d\mathbf{a}$$

is the invariant measure for the problem (7)-(8). Also, we introduce the measures

$$w'_n(A) = (2\pi)^{-\frac{d_n}{2}} \prod_{k=1}^{d_n} \lambda_k^{\frac{1}{2}} \int_A e^{-\frac{1}{2}\sum_{k=1}^{d_n} \lambda_k a_k^2}\, d\mathbf{a}.$$

Let $\Omega_n \subset X_n$ and $\Omega_n = \{u \in X \,|u = \sum_{k=1}^{d_n} a_k e_k, \ \mathbf{a} \in A\}$, where $A \subset R^{d^n}$ is a Borel set. We define $\mu_n(\Omega_n) = \mu'_n(A)$; by analogy, $w(\Omega_n) = w'_n(A)$. Since μ'_n is the invariant measure for (7)-(8), μ_n is invariant for the problem (3)-(4).

Although w_n and μ_n are the measures on X_n we can define them on the Borel σ-algebra of X by the rule: $w_n(\Omega) = w_n(\Omega \bigcap X_n)$ and $\mu(\Omega) = \mu(\Omega \bigcap X_n)$. Since the set $\Omega \bigcap X_n$ is open as a set in X_n for any open set $\Omega \subset X$, it is correct.

Lemma1

The sequence $\{w_n\}$ weakly converges to w in X.

Proof

Since S^{-1} is a nuclear operator, the trace $Tr\ S^{-1} = \sum_k \lambda_k^{-1} < \infty$. It is clear that there exists a continuous positive function $p(x)$ defined on $(0,\infty)$ with the property $\lim_{x\to\infty} p(x) = +\infty$ such that $\sum_k \lambda_k^{-1}p(\lambda_k) < +\infty$. We define a (unbounded) operator $T = p(S)$ and let $Q = S^{-1}T$. By the definition $0 < Tr\ Q < \infty$. Consider $R > 0$. Let $B_R = \{u \in X|\ \|T^{\frac{1}{2}}u\|_X \leq R\}$ and let B be the closure of B_R in X.It is clear that B is compact for any $R > 0$. By the well-known inequality (see [15])

$$w_n(X \setminus B) = w_n(u : (Tu,u)_X > R^2) \leq \frac{TrQ}{R^2}.$$

Hence, by the Prokhorov theorem $\{w_n\}$ is weakly compact on X.

By the definition $w_n(M) \to w(M)$ for any cylindrical set $M \subset X$ (because $w_n(M) = w(M)$ for all sufficiently large n). Then, by the uniqueness of the continuation of a measure from an algebra to a minimal σ-algebra

Lemma 1 is proved.

Lemma 2

$\lim_{n\to\infty} inf\ \mu_n(\Omega) \geq \mu(\Omega)$ for any open set $\Omega \subset X$ such that $\mu(\Omega) < \infty$.

Proof is usual. Let $\Omega \subset X$ be open. For any $\epsilon > 0$ there exists a function $\phi(u)$finite in Ω: $0 \leq \phi(u) \leq 1$ such that

$$\int_\Omega \phi(u)e^{-g(u)}\ w(du) \geq \mu(\Omega) - \epsilon.$$

Then,

$$\lim_{n\to\infty} inf\ \mu_n(\Omega) \geq \lim_{n\to\infty} inf \int_\Omega \phi(u)e^{-g(u)}\ w_n(du) = \int_\Omega \phi(u)e^{-g(u)}w(du) \geq \mu(\Omega) - \epsilon$$

and due to the arbitrariness of $\epsilon > 0$ Lemma 2 is proved.

Lemma 3

Let $\Omega \subset X$ be open, $t \in R$. Then $\mu(\Omega) = \mu(\Omega_1)$ where $\Omega_1 = f(\Omega,t)$.

Proof

Using Assumption 2 and Corollary 1 one has that Ω_1 is open, too. First, let us assume that $\mu(\Omega) < \infty$, $\mu(\Omega_1) < \infty$.

Let us fix $\epsilon > 0$.Then, there exists compact $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \epsilon$. Let $K_1 = f(K,t)$. Then $K_1 \subset \Omega_1$ is compact. Let $\alpha = min\{dist(K,\partial\Omega);\ dist(K_1,\partial\Omega_1)\}$ where $dist(A,B) = \inf_{x\in A, y\in B} \|x-y\|_X$ and ∂A is the boundary of a set $A \subset X$. One obviously has $\alpha > 0$. By Assumption 3 for any $u \in K$ there exists a ball $B(u)$ with the center in u, $B(u) \subset \Omega$, such that $dist(f_n(u,t); f_n(g,t)) < \frac{\alpha}{3}$ for all $g \in B(u)$ and for all n.Let $\Omega_\beta = \{g \in \Omega_1 | dist(g,\partial\Omega_1) \geq \beta\}$ for any $\beta > 0$ and let $B(u_1), ..., B(u_l)$ be a finite covering of K by the balls, $D = \bigcup_{i=1}^{l} B(u_i)$. Since $f_n(u_i,t) \to f(u_i,t)$ $(n \to \infty)$ for

any i, using Assumption 2 one gets: $f_n(D,t) \subset \Omega_{\frac{\alpha}{3}}$ for all sufficiently large n. Then, by lemma 2

$$\mu(\Omega) \leq \mu(D)+\epsilon \leq \lim_{n\to\infty} inf\ \mu_n(D)+\epsilon = \lim_{n\to\infty} inf\ \mu_n(f_n(D \bigcap X_n, t))+\epsilon \leq \mu(\Omega_1)+\epsilon.$$

Due to the arbitrariness of $\epsilon > 0$ one gets the inequality

$$\mu(\Omega) \leq \mu(\Omega_1).$$

Since $\Omega = f(\Omega_1,-t)$, the opposite inequality is valid ,too:

$$\mu(\Omega) \geq \mu(\Omega_1).$$

Thus, we proved the equality

$$\mu(\Omega) = \mu(\Omega_1)$$

for any two open sets with finite measures. If Ω has an infinite measure, we take the sequence $\Omega^k = \Omega \bigcap \{u \in X|\ |g(u)| + |g(f(u,t))| < k\}$ $(k = 1,2,3,...)$ and let $\Omega_1^k = f(\Omega^k,t)$. One has $\mu(\Omega^k) = \mu(\Omega_1^k) < \infty$. Taking the limit when k tends to infinity we get the statement of the lemma.

Lemma 3 is proved.

For any Borel set $\Omega \subset X$ we get the equality $\mu(\Omega) = \mu(\Omega_1)$ approximating Ω and Ω_1 by open sets from outside and by closed sets from inside.

Thus, theorem 1 is proved.

4. Proof of theorem 2

We denote by $f_N(u,t)$ the DS defined by the problem (1)-(2) corresponding to $H = H_N$. Let μ_N be the corresponding invariant measure from Theorem 1 and let $\mu(\Omega) = \int_\Omega G(u)w(du)$ for any Borel set $\Omega \subset X$. Since $G(u)$ is a limit of continuous functionals, it is measurable. Then, the measure μ is defined. By the classical result

$$\lim_{N\to\infty} \mu_N(\Omega) = \mu(\Omega)$$

for any measurable $\Omega \subset X$.

Let us fix $t \in R$ and a measurable $\Omega \subset X$. Let $\Omega_N = f_N(\Omega,t)$, $A_k = \bigcap_{N\geq k} \Omega_N$, $A = \bigcup_{k\geq 1} A_k$. It is clear that $A_1 \subset A_2 \subset A_3 \subset ... \subset A_k \subset ....$

Lemma 4

Let $\Omega_1 = f(\Omega,t)$ be open. Then, $\Omega_1 \subset A$.

Proof

Let $u \in \Omega_1$. By Assumption 5 $f_N(u,-t) \in \Omega$ for all sufficiently large numbers N. Hence, $u \in A_k$ for sufficiently large k, and Lemma 4 is proved.

Let Ω and Ω_1 be open. Using Lemma 4 we get

$$\mu_N(\Omega) = \mu_N(\Omega_N) \geq \mu_N(A_k)$$

for $N \geq k$. Taking the limit over $N \to \infty$ we have by Lemma 4: $\mu(\Omega) \geq \mu(A_k)$, hence

$$\mu(\Omega) \geq \mu(A) \geq \mu(\Omega_1).$$

The opposite inequality may be proved by analogy. For an arbitrary measurable set $\Omega \subset X$ we get the same equality as at the end of Theorem 1. Thus, Theorem 2 is proved.

5. Applications

As it is remarked in Section 1, the first point that leads the author to the consideration of invariant measures is the Poincare recurrence theorem (see [18]).

Theorem (Poincare)
Let f be a DS on a phase space X with a finite invariant measure $\mu: \mu(X) < \infty$. Then, almost all points of X lie on the trajectories stable according to Poisson.

According to theorems 1 and 2 we have constructed the invariant measure for our DS. As we will see further, it is not difficult to formulate conditions ensuring for the measure being finite. Unfortunately, we have to remark that Assumptions 1-5 are rigorously proved only for concrete partial differential equations in some partial situations. Of course, this is the problem for the theory of (nonlinear) partial differential equations. Assumptions 1, 2, 4 seem to be sufficiently natural but Assumption 3 is very strong (now it is proved only in some simple situations). Assumption 5 is very natural, too. Despite the mentioned difficulties we are able to prove the invariance of our abstract measure in several cases for the concrete nonlinear partial differential equations.

5.1. A nonlinear Schrödinger equation

Consider the problem

$$iu_t + u_{xx} + f(x, |u|^2)u = 0, \; x \in (0, A), \; t \in R, \tag{10}$$

$$u(0,t) = u(A,t) = 0, \tag{11}$$

$$u(x, t_0) = u_0(x). \tag{12}$$

Our basic hypothesis is the following:

(f1) Let f be a smooth real function and let there exist $C > 0$ such that

$$|f(x,s) + |(1+s)\frac{\partial}{\partial s}f(x,s)| < C$$

for all x,s.

We remark that Assumption (f1) is more weak than in paper [3].
We rewrite the problem (10)-(12) for the functions $u^1 = Re\; u$ and $u^2 = Im\; u$:

$$u_t^1 + u_{xx}^2 + f(x, (u^1)^2 + (u^2)^2)u^2 = 0, \; x \in (0, A), \; t \in R, \tag{13}$$

$$u_t^2 - u_{xx}^1 - f(x, (u^1)^2 + (u^2)^2)u^1 = 0, \tag{14}$$

$$u^i(0,t) = u^i(A,t) = 0, \quad i = 1,2, \tag{15}$$

$$u^i(x,t_0) = \phi_i(x). \tag{16}$$

We introduce the following definitions. Let $X = L_2(0,A) \otimes L_2(0,A)$, $Y = H_0^1(0,A) \otimes H_0^1(0,A)$. Let Q be the operator mapping $u^* \in (H_0^1)^*$ into $u \in H^{-1}$ such that $u^*(g) = -(u,g)_{L_2}$ for any $g \in H_0^1(0,A)$ and let $J = \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix}$. It is clear that the operator J maps a dense set $D \subset X^*$ into X. Then, let Δ be the closure of the operator $-\frac{d^2}{dx^2}$ in $L_2(0,A)$ defined first on $C_0^\infty(0,A)$ and let $S = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$.
Let $F(x,s) = \frac{1}{2}\int_0^s f(x,p)\,dp$ and

$$H(u^1,u^2) = \int_0^A \{\frac{1}{2}((u_x^1(x))^2 + (u_x^2(x))^2) - F(x, (u^1(x))^2 + (u^2(x))^2)\}\,dx.$$

In this notation one gets the representation of the system (13)-(16) in the form (1)-(2).
Later, let $\{e_n\}$ be the orthonormal basis of eigenvectors of the operator Δ with corresponding eigenvalues $\{\lambda_n\}$. We set $X_n = span\{e_1,...,e_n\} \otimes span\{e_1,...,e_n\}$ and let P_n be the orthogonal projector onto $span\{e_1,...,e_n\}$ in $L_2(0,A)$. Then, the approximate problem (3)-(4) takes the following form:

$$u_t^{1n} + u_{xx}^{2n} + P_n[f(x,(u^{1n})^2 + (u^{2n})^2)u^{2n}] = 0,$$

$$u_t^{2n} - u_{xx}^{1n} - P_n[f(x,(u^{1n})^2 + (u^{2n})^2)u^{1n}] = 0,$$

$$(u^{in})(x,t_0) = P_n\phi^i(x), \quad \phi^1 = Re\;\phi, \; \phi^2 = Im\;\phi, \; i = 1,2.$$

We can now present

Teorem 3
Let the hypothesis (f1) be valid. Then, NSE (13)-(16) satisfies Assumptions 1-4. Hence, the Borel measure

$$\mu(\Omega) = \int_\Omega e^{\int_0^A F(x,(u^1)^2+(u^2)^2)\,dx}\, w(du^1\,du^2)$$

is invariant for DS defined on the phase space X by this problem. (Here w is the centred Gaussian measure with the correlation operator S^{-1} on X.)

Example 1
The hypothesis (f1) is valid for two physical nonlinearities: $f(x,s) = \frac{\alpha s}{1+s}$ and $f(x,s) = e^{-\alpha s}$ with $\alpha > 0$ in the second case.

Remark 5

It may be proved that any ball $B_R = \{u \in X \mid ||u||_X \leq R\}$ is the invariant set for our DS. So, the ball B_R may be taken for a new phase space. It is clear that μ is finite on any such ball for each of nonlinearities presented in Example 1.

The verification of the validity of Assumptions 1-4 is not presented. In fact, the similar statement is proved for a nonlinear wave equation under hypotheses similar to (f1) in paper [5] (see also the following section). In our case one can prove this fact by analogy.

Remark 6

For the system (13)-(16) a result similar to theorem 2 is presented in paper [4] for the power nonlinearity $f(x, |u|^2)u = \lambda|u|^p u$, where $p \in (0,4)$ if $\lambda < 0$ and $p \in (0,2)$ if $\lambda > 0$. This paper is based on paper [19] where the correctness of the Cauchy problem for NSE with $\phi \in L_2$ is proved (in fact, this result was adapted to the system (13)-(16)). In this paper the non-triviality and the finiteness of the constructed invariant measure μ on any ball in X are demonstrated, too.

Remark 7

The described approach is applicable also to the problem periodic with respect to x for NSE without any essential modifications.

5.2. A nonlinear wave equation

Consider a nonlinear wave equation

$$u_{tt} - u_{xx} + f(x,u) = 0, \qquad x \in (0,A),\ t \in R, \tag{17}$$

$$u(0,t) = u(A,t) = 0, \tag{18}$$

$$u(x,t_0) = \phi(x),\ u_t'(x,t_0) = \psi(x). \tag{19}$$

Here all variables are real.

Since this problem is considered in paper [5] and since the result of the present paper is identical to the above result, we only demonstrate the possibility of the application of our abstract scheme to this problem. We take

$X = L_2(0,A) \otimes H^{-1}(0,A)$, $Y = H_0^1(0,A) \otimes L_2(0,A)$, $(u, u_t') \in X$, $F(x,u) = \int_0^u f(x,s)\,ds$,

$$H(u) = \int_0^A \{\frac{1}{2}(u_t^2 + u_x^2) + F(x,u)\}\,dx, \quad J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} Q_1,$$

where E is the unit operator and Q_1 maps $v^* = (u^*, u_1^*) \in Y^*$ into $v \in Z = L_2(0,A) \otimes L_2(0,A)$ such that $v^*(g) = (v,g)_Z$ for any $g \in Y$. In this notation one gets the problem (17)-(19) in the form (1)-(2), again. The basic hypothesis is as follows:

(f2) Let the function f be continuously differentiable and let there exist $C > 0$ such that

$$|f(x,u)| + |\frac{\partial}{\partial u} f(x,u)| < C, \quad \text{for all } x, u.$$

Finally, we take spaces X_n from section 3.1.

As in paper [5] one can verify that Assumptions 1-4 are valid. So, the measure

$$\mu(\Omega) = \int_\Omega e^{-\int_0^A F(x,u)\,dx}\, w(du\,dv)$$

where w is the centred Gaussian measure on X with the correlation operator $\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & \Delta^{-1} \end{pmatrix}$ is invariant.

Remark 8

Unfortunately, the author does not know any results verifying Assumptions 1-4 or 5 on the space X for a wider class of nonlinearities to make possible to apply Theorems 1 and 2.

Remark 9

In particular, the nonlinearities $f(x,u) = \frac{\alpha u^2}{1+u^2}u$ and $f(x,u) = ue^{-\alpha u^2}$ satisfy the hypothesis (f2). Since the integral of the functional $e^{\alpha||u||_X^2}$ over the measure w is finite for small $\alpha > 0$ and , our measure μ is finite for small $\alpha_0 < 0$ and $\alpha > \alpha_0$ for the first function and for all $\alpha > 0$ for the second one.

5.3 A generalized Korteweg-de Vries equation

Consider the problem

$$u_t + (a(x)u)_x + u_{xxx} = 0, \qquad x, t \in R, \tag{20}$$

$$u(x,t_0) = \phi(x), \tag{21}$$

where $a(x)$, $\phi(x)$ and $u(x,t)$ are periodic real functions of x with a period A. We assume that $a \in C^\infty$. Using the method of paper [20] one can easily prove

Theorem 4

For any periodic $\phi \in C^\infty$ there exists a unique solution of the problem $u(x,t)$ of the class C^∞ defined for all x, t which is periodic in x with the same period.

We take $H(u) = \int_R \frac{1}{2}(u_x^2 - a(x)u^2)\,dx$, the spaces of periodic real functions from $L_2(0,A)$ and $H^1(0,A)$ for X and Y, respectively, with the norms $||g||_X = \int_0^A g^2(x)\,dx$ and $||g||_Y = ||g'||_X + ||g||_X$. Finally, let $J = \frac{\partial}{\partial x}Q$ where the operator Q maps $v^* \in Y^*$ into $v \in X$ such that $v^*(g) = (v,g)_X$ for any $g \in Y$. Using the trivial estimate

$$\frac{d}{dt}\int_0^A (u(x,t) - v(x,t))^2\,dx \leq C \int_0^A (u(x,t) - v(x,t))^2\,dx,$$

where $C = const > 0$ and u and v are arbitrary solutions of the problem (20)-(21), one proves the existence and uniqueness of a solution in the space $C(I;X)$. Let $S = \Delta + E$, where Δ is taken from section 3.1 and E is the unit operator. Then, we take $e_{2n-1}(x) = (\frac{2}{A})^{\frac{1}{2}} sin(\frac{\pi n x}{A})$, $e_{2n}(x) = (\frac{2}{A})^{\frac{1}{2}} cos(\frac{\pi n x}{A})$ $(n = 1, 2, 3, ...)$, $e_0 = (\frac{1}{A})^{\frac{1}{2}}$ and let $X_n = span\{e_0, e_1, ..., e_{2n}\}$. In addition to the above inequality one can prove the following:

$$\frac{d}{dt} \int_0^A (u - u^n)^2 \, dx \leq C \int_0^A (u - u^n)^2 \, dx.$$

where u^n is the approximate solution introduced in Section 2. It is easy to verify that Assumptions 1-4 follow from these two inequalities and the inequality similar to the first one written for u^n. Then, the Borel measure

$$\mu(\Omega) = \int_\Omega e^{\int_0^A \frac{1}{2}\{u^2(x) + a(x) u^2(x)\} \, dx} \, w(du)$$

is invariant for our equation.

For the usual (generalized) Korteweg-de Vries equation

$$u_t + f(u) u_x + u_{xxx} = = 0$$

(we consider the periodic problem, again) any results on DS defined by this equation on the suitable phase space are unknown. In the unrigorous way, one can take the measure of the type

$$\mu(\Omega) = \int_\Omega e^{\int_0^A [F(u(x)) + \frac{1}{2} u^2(x)] \, dx} \, w(du) \, ,$$

where $F(u) = \int_0^u \int_0^s f(p) \, dp \, ds$ and w is the above Gaussian measure. However, one must be careful because the choice of the correlation operator depends on X. This operator may be nuclear with respect to one space X but may be found to be non-nuclear for another. So, it is very important to take a suitable phase space.

5.4. Conclusion

Formally, a wide class of "soliton" equations may be represented in the form (1)-(2). So, there is a possibility to write formulae for measures of the type μ which are probably invariant. Unfortunately, there is a principal difficulty of the rigorous mathematical treatment since the corresponding initial-boundary value problems for these equations are not sufficiently investigated.

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Жидков П.Е. E5-92-395

Об инвариантных мерах для некоторых бесконечномерных динамических систем

Рассматривается абстрактная бесконечномерная гамильтонова система, недавно предложенная М.Гриллакисом, Ж.Шата и В.Страусом. Многие нелинейные эволюционные уравнения математической физики могут быть представлены в этой форме. Цель статьи состоит в построении инвариантной меры для этой системы. В частности, получены условия конечности построенной меры, что позволяет применить теорему о возвращении Пуанкаре, которая объясняет явление Ферми-Паста-Улама, состоящее в возвращении любого решения к своей начальной точке с любой точностью спустя достаточное время. Результат использован для исследования конкретных физических задач.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Zhidkov P.E. E5-92-395

On Invariant Measures for Some Infinite-Dimensional Dynamical Systems

We consider an abstract infinite-dimensional Hamiltonian system recently introduced by M.Grillakis, J.Shatah and W.A.Strauss. A lot of nonlinear evolution equations of the mathematical physics may be represented in that form. The aim of the paper is the construction of an invariant measure for this system. This measure has many applications in the theory of dynamical systems. In particular, the conditions for the finiteness of the constructed measure are presented. It makes possible to apply the Poincare recurrence theorem which explains the well-known Fermi-Past-Ulam phenomenon of the return of any solution to its initial data with time with an arbitrary accuracy. The result is used to investigate concrete physical problems.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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