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PARAGRASSMANN EXTENSIONS OF THE VIRASORO ALGEBRA

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1. Introduction

Different extensions of the Virasoro algebra are useful to formulate two-dimensional quantum conformal field theories with certain additional symmetries [1]. Such extensions are generated by the stress-energy tensor T and some currents corresponding to the additional symmetries. For example, if we add the Kac-Moody currents J of the conformal weight 1, the result will be the well-known semi-direct sum of the Virasoro and Kac-Moody algebras with Sugawara-type relations between J and T. Adding instead of J a fermionic current of the weight $3/2$ we get the Ramond-Neveu-Schwarz algebra (RNS). $SO(N)$- and $SU(N)$-invariant extensions and $N = 2, 3, 4$ super-extensions of the RNS algebra have been considered in the context of 2D conformal field theories in [2].

Introducing currents W_N of an integer weight $N \geq 3$ gives rise to the W_N-algebras of A. Zamolodchikov, which have demonstrated their usefulness in last years. So it seems that a most interesting new possibility is to look for extensions by trying to add currents with fractional conformal weights, in particular, parafermionic currents of the weight $(p+2)/(p+1)$ (p is a positive integer which is, in fact, the order of parastatistics or the degree of nilpotency of the underlying paragrassmann algebra). The construction of such an extension of the Virasoro algebra is the subject of the present paper.

Attempts to realize this possibility had already been made. First of all, we have to mention the work of V. Fateev and A. Zamolodchikov [3], where a system with $\mathbf{Z}_{p+1}$-symmetry had been explored and a certain associative operator 'algebra of parafermionic currents' have been constructed. That algebra, although, can hardly be regarded as a paragrassmann extension of the Virasoro-RNS algebra since it does not reproduce the RNS algebra for $p = 1$.

Another attempt, stimulated by the V. Rubakov and V. Spiridonov approach to a para-supersymmetric quantum mechanics [4], had been undertaken by S. Durand et.al.[5]. The formulation of Ref. [5] was based on a paragrassmann calculus defined in the frame of the Green representation for the paragrassmann algebra also known as the Green ansatz [6]. In a later paper [7] S. Durand is using the paragrassmann calculus developed [1] in Ref. [8] to derive interesting identities involving Virasoro-RNS - like generators. Some of these identities look like plausible ingredients of a para-extension of the superconformal algebra but their relation to any symmetry transformations remains unclear. A relation of similar identities to the conformal algebra in two dimensions, also hinting at possible generalizations of the superconformal symmetry, has been found by T.Nakanishi [10].

All this motivates our present attempt to find a systematic para-generalization of the superconformal and Virasoro-RNS algebras. To clarify the logic of our paper we first recall the classical cases $p = 0$ and $p = 1$ in a framework we are going to generalize.

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[1] As we have learned after submitting our paper [8] to the HEP-TH database, some ingredients of this calculus for one variable were earlier mentioned in Ref. [9].

$p=0$ (*Virasoro case*)
The space is simply the complex plane with the coordinate z. $F=F(z)$ is an analytic function. Let us consider an analytic map

$$z \mapsto \tilde{z}(z)\,, \quad F(z) \mapsto \tilde{F} \equiv F(\tilde{z}(z))\,. \tag{1.1}$$

The trivial identity

$$\frac{\partial}{\partial z}\tilde{F} = (\tilde{z}')\frac{\partial}{\partial \tilde{z}}\tilde{F} \tag{1.2}$$

(prime will always denote the derivative with respect to z) means that the old and new derivatives are proportional for any function $\tilde{z}$, or in other words, any transformation (1.1) will be conformal, i.e. 'preserving the form of the derivative'. The group of the conformal transformations will be denoted by CON.

Passing to the infinitesimal form of the transformation (1.1), $\tilde{z}=z+\lambda\omega(z)$ (λ is a small number), and defining its generator $T(\omega)$ by

$$\tilde{F} = (1+\lambda\, T(\omega))\, F\ , \tag{1.3}$$

one easily sees that $T(\omega)=\omega\partial_z$. The Lie algebra generated by $T(\omega)$ is defined by the commutators

$$[T(\omega), T(\psi)] = T(\omega\psi' - \omega'\psi) \tag{1.4}$$

and is called the conformal algebra, Con in our notation. It coincides with the whole algebra of vector fields on the complex plane and its (unique) central extension is the standard Virasoro algebra denoted by Vir.

There is a simple generalization of the construction for F being not ordinary functions but conformal fields of a weight Δ. In this case, the transformation rule is

$$F_\Delta(z) \mapsto \tilde{F}_\Delta \equiv (\tilde{z}')^\Delta F_\Delta(\tilde{z})\ , \tag{1.5}$$

and the generators have the form $T(\omega)=\omega\partial_z+\Delta\omega'$. Their Lie algebra coincides with (1.4).

$p=1$ (*RNS-case*)
Now the space is a complex superplane with coordinates z and θ, $\theta^2=0$. In fact, we need a Grassmann algebra of more than one variable. One of them will be specified as θ while the rest will be referred as 'other thetas'. A super-analytic map is

$$z \mapsto \tilde{z} \ = \ Z_0 + \theta Z_1\,, \tag{1.6}$$
$$\theta \mapsto \tilde{\theta} \ = \ \Theta_0 + \theta\Theta_1\,, \quad \tilde{\theta}^2 = 0\,, \tag{1.7}$$

where Z_i and Θ_i are functions of z and of 'other thetas' with needed Grassmann parity. This map transforms a function $F=F(z,\theta)$ into $\tilde{F}\equiv F(\tilde{z},\tilde{\theta})$.

The fractional derivative of order 1/2, $\mathcal{D}$, ($\mathcal{D}^2=\partial_z$) can be defined as

$$\mathcal{D} = \partial_\theta + \theta\partial_z\ , \tag{1.8}$$

due to the Grassmann relations $\theta^2=0=\partial_\theta^2$, $\{\partial_\theta,\theta\}_+=1$. The super-analytic map (1.6), (1.7) is called a superconformal transformation when the superderivative transforms homogeneously (see e.g. [11]). This requirement, similar to (1.2), looks like

$$\mathcal{D}\tilde{F} = \Phi\tilde{\mathcal{D}}\tilde{F}\ , \tag{1.9}$$

and leads to certain restrictions on the parameter functions, namely,

$$\Phi = \mathcal{D}\tilde{\theta}\,, \mathcal{D}\tilde{z} = \mathcal{D}(\tilde{\theta})\tilde{\theta}\,,$$

or, in the component notation,

$$\begin{aligned} Z_1 &= \Theta_1\Theta_0\,, \\ Z_0' &= \Theta_0'\Theta_0 + (\Theta_1)^2\,. \end{aligned} \tag{1.10}$$

These transformations form a group which is called superconformal, or CON_1 in our notation.

For the infinitesimal form of the mapping (1.6), (1.7),

$$\begin{aligned} \tilde{z} &= z + \lambda\Omega(z,\theta) = z + \lambda(\omega_0+\theta\omega_1)\ , \\ \tilde{\theta} &= \theta + \lambda\mathcal{E}(z,\theta) = \theta + \lambda(\epsilon_0+\theta\epsilon_1)\ , \end{aligned} \tag{1.11}$$

the conditions (1.10) read

$$\omega_1 = \epsilon_0\ , \quad \omega_0' = 2\epsilon_1\,. \tag{1.12}$$

Thus, defining the generators $\mathcal{T}(\omega_0)$ and $\mathcal{G}(\epsilon_0)$ by

$$\tilde{F} = (1+\lambda(\mathcal{T}(\omega_0)+\mathcal{G}(\epsilon_0)))\,F\ , \tag{1.13}$$

one can easily find that

$$\mathcal{T}(\omega) = \omega\partial_z + \frac{1}{2}\omega'\theta\partial_\theta\ , \quad \mathcal{G}(\epsilon) = \epsilon(\partial_\theta - \theta\partial_z)\ . \tag{1.14}$$

These generators close into the well-known Lie algebra

$$\begin{aligned} [\mathcal{T}(\omega),\mathcal{T}(\psi)] &= \mathcal{T}(\omega\psi'-\omega'\psi)\,, \\ [\mathcal{T}(\omega),\mathcal{G}(\epsilon)] &= \mathcal{G}(\omega\epsilon'-\tfrac{1}{2}\omega'\epsilon)\,, \\ [\mathcal{G}(\epsilon),\mathcal{G}(\zeta)] &= 2\mathcal{T}(\epsilon\,\zeta)\,. \end{aligned} \tag{1.15}$$

that we denote by $\mathcal{CON}_1$.

As in the previous example, it is possible to introduce similar generators (1.14) for the superconformal field F of an arbitrary weight Δ: $F(z,\theta)\mapsto(\mathcal{D}\tilde{\theta})^\Delta F(\tilde{z},\tilde{\theta})$ (cf. with (1.5)). The corresponding generators

$$\mathcal{T}(\omega) = \omega\partial_z + \frac{1}{2}\omega'\theta\partial_\theta + \frac{\Delta}{2}\omega'\ , \quad \mathcal{G}(\epsilon) = \epsilon(\partial_\theta - \theta\partial_z) - \Delta\epsilon'\theta\ . \tag{1.16}$$

obey the same algebra (1.15). The operator $Q = (\partial_\theta - \theta\partial_z)$ which appeared in the definition of $\mathcal{G}(\epsilon)$ anticommutes with $\mathcal{D}$ and is called the supersymmetry generator.

Up to now, ω and ϵ were respectively even and odd functions[2] of the variables z and 'other thetas', i.e. some polynomials in 'other thetas' with coefficients being analytic functions of z. Therefore the generators $\mathcal{T}$ and $\mathcal{G}$ are polynomials in 'other thetas' also, with coefficients being the 'bare' generators T and G whose arguments ω and ϵ are ordinary functions of z independent of 'other thetas'. Then the expressions for T and G coincide with (1.14) and they obey the algebra (1.15) with the last commutator replaced by anticommutator. This algebra, or more precisely superalgebra, is what is usually called the 'superconformal algebra'. We will denote it by Con_1 . Its unique central extension, the Ramond-Neveu-Schwarz algebra, is denoted by Vir_1.

The distinction between the algebras $\mathcal{CON}_1$ and Con_1 is usually ignored, being practically trivial. As we shall see in Sect.3, this is not the case for their para-analogues $\mathcal{CON}_p$ and Con_p since the paragrassmann algebra of 'other thetas' is a much more complicated object than the Grassmann one (and, probably, not uniquely defined, see [13]). In particular, there is no simple and general commutation rule between the elements of the paragrassmann algebra. As a result, closing the algebra $\mathcal{CON}_p$ becomes a rather non-trivial problem. On the contrary, the algebra Con_p closes quite easily, (even in several non-equivalent variants, if the number $p+1$ is rich in divisors). Each variant of closing defines an extended algebra, Vir_p, with a number of central charges (from 1 to $[(p+1)/2]$ of them).

The simplest variant of Vir_p looks as follows:

$$\begin{aligned} [L_n\,,\,L_m] &= (n-m)L_{n+m} + \frac{2}{p+1}\left(\sum_j c_j\right)(n^3-n)\delta_{n+m,0}\,, \\ [L_n\,,\,G_r] &= (\frac{n}{p+1} - r)G_{n+r}\,, \\ \{G_{r_0},\ldots,G_{r_p}\}_c &= (p+1)L_{\Sigma r} - \sum_j c_j\left(\sum_i r_i r_{i+j} + \frac{1}{p+1}\right)\delta_{\Sigma r,0}\,, \\ j &= 1\ldots\left[\frac{p+1}{2}\right]\,, \end{aligned} \tag{1.17}$$

where $L_n = T(z^{-n+1})$, $G_r = G(z^{-r+1/(p+1)})$, and $\{\ldots\}_c$ is the cyclic sum of the $(p+1)$-linear monomials:

$$\{G_0\,,\ldots,G_p\}_c = G_0\cdots G_p + G_p\cdot G_0\cdots G_{p-1} + \ldots + G_1\cdots G_p\cdot G_0\,.$$

Note, that a particular variant of this algebra Vir_p, with totally symmetric bracket in the third line and without central extensions, had been presented in [12] under

[2]Note that the commutation relations between ω, ϵ, and θ are fixed by Eq. (1.9) and by the condition $\tilde\theta^2 = 0$.

the name 'fractional Virasoro algebra'. Recently, a generalization that relates the algebras of Refs. [5] and [12], has been given in [7].

The rest of the paper is devoted to a generalization of the previous scheme to arbitrary integer p. In Sect.2, a necessary preliminary technique of the paragrassmann algebras is developed. It is interesting in itself as a consistent differential calculus on the paragrassmann algebras generalizing that of our previous work [8]. A somewhat surprising result of this analysis is that all apparently different realizations ('versions') of the paragrassmann calculus are, under certain restrictions, equivalent. This allows us to speak of the unique paragrassmann calculus generalizing the Grassmann one for one variable θ and one differentiation ∂_θ. Possible formulations of the paragrassmann calculus for many variables are discussed in a separate publication [13].

In Sect.3, we recollect the definition of the fractional derivative $\mathcal{D}$ ($\mathcal{D}^{p+1} = \partial_z$) in any version of the paragrassmann calculus. Then in the spirit of the above scheme, we introduce paraconformal transformations to construct a paraconformal group CON_p and corresponding algebras $\mathcal{CON}_p$, and Con_p. We show that a p-analogue of the infinitesimal transformations (1.11) must look as

$$\begin{aligned} \tilde\theta &= \theta + \lambda\mathcal{E}(z,\theta)\,, \\ \tilde z &= z + \lambda\Omega^{(1)} + \ldots + \lambda^p\Omega^{(p)}\,. \end{aligned} \tag{1.18}$$

Unlike the Lie algebras (and superalgebras) having only first order generators, here we have to retain 'higher-order generators' thus introducing into consideration a p-jet structure. This suggests that the algebra $\mathcal{CON}_p$ might be a p-filtered Lie algebra containing the generators of p 'generations' $\{\mathcal{L}^{(M)}\}$, so that an analogue of the formula (1.13) would look like

$$\tilde F = (1 + \lambda\{\mathcal{L}^{(1)}\} + \lambda^2\{\mathcal{L}^{(2)}\} + \ldots + \lambda^p\{\mathcal{L}^{(p)}\})\,F\,.$$

The algebra $\mathcal{CON}_p$ contains generators of a new type (we call them $\mathcal{H}$-generators) that do not act on z but are crucial in closing the algebra. The algebras Con_p can be closed without them.

In Sect.4, we briefly discuss the meaning of the construction in terms of algebraic geometry. This allows us to introduce the central charges in a straightforward way and so derive the algebras Vir_p.

The concluding Sect.5 presents a discussion of the properties of these algebras, possible generalizations, and unsolved problems.

Concluding this rather long introduction we would like to point out that the main formal results of this paper were known to us for some time and have been presented at seminars and workshops this spring. However, we refrained from publishing them prior to understanding their geometric meaning. We hope that we can now suggest a reliable geometric foundation for our formal construction in terms of the versions ('version covariance') and of jet-like structures.

2. Paragrassmann Algebra Π_{p+1}

In Ref. [8] we have considered paragrassmann algebras $\Gamma_{p+1}(N)$ with N nilpotent variables θ_n, $\theta_n^{p+1}=0$, $n=1,\ldots,N$. Some wider algebras $\Pi_{p+1}(N)$ generated by θ_n and additional nilpotent generators ∂_n have also been constructed. These additional generators served for defining a paragrassmann differentiation and paragrassmann calculus. The building block for this construction was the simplest algebra $\Pi_{p+1}(1)$. By applying a generalized Leibniz rule for differentiations in the paragrassmann algebra $\Gamma_{p+1}(N)$ we have found two distinct realizations for $\Pi_{p+1}(1)$ closely related to the q-deformed oscillators. We have mentioned in [8] that other realizations of the $\Pi_{p+1}(1)$ may be constructed. The aim of this section is to demonstrate this in detail. We shall also show that, under certain conditions, all these realizations are equivalent and one may choose those which are most convenient for particular problems.

Intuitively, paragrassmann algebra Π_{p+1} should be understood as some good p-generalization of the classical fermionic algebra Π_1

$$\theta^2 = 0 = \partial^2 , \tag{2.1}$$
$$\partial\theta + \theta\partial = 1 . \tag{2.2}$$

By 'p-generalization' we mean that (2.2) is to be replaced by

$$\theta^{p+1} = 0 = \partial^{p+1} , \tag{2.3}$$

(it is implied, of course, that $\theta^p \neq 0$ and the same for ∂). So the question is, which generalization of (2.2) might be called 'good'. Many variants have been tried already (see for example [6]). As a rule, they deal with certain symmetric multilinear combinations, like $\theta^2\partial + \theta\partial\theta + \partial\theta^2$ (for $p=2$), and meet with difficulties when commuting θ and ∂.

To find a correct generalization recall that (2.2) allows to define the Grassmann differential calculus. It shows how to push the differentiation operator ∂ to the right of the variable θ. On the other hand, representing ∂ and θ by 2×2 real matrices, we can make them Hermitian conjugate and thus interpret as annihilation and creation operators. Then (2.2) is the normal ordering rule. The second important feature of this relation is that it preserves the Grassmann grading, -1 for ∂ and $+1$ for θ. In physics terminology this means that the normal ordering is not changing the number of 'particles'.

Thus, to construct a generalization of the relation (2.2), we first define a natural grading in the associative algebra generated by θ and ∂ satisfying (2.3)

$$deg\ (\theta^{r_1}\partial^{s_1}\theta^{r_2}\partial^{s_2}\ldots\theta^{r_k}\partial^{s_k}) = \Sigma r_i - \Sigma s_i , \tag{2.4}$$

and denote by $\Pi_{p+1}(l)$ the linear shell of monomials of the degree l. Then our basic requirement is

$$a\ set\ L^{(l)} = \{\theta^r\partial^s\ ,\ r-s=l\}\ \ forms\ a\ basis\ of\ \Pi_{p+1}(l)\,. \tag{2.5}$$

This immediately reduces the range of possible degrees to $-p\le l\le p$ and makes all the subspaces $\Pi_{p+1}(l)$ and the entire algebra

$$\Pi_{p+1} = \bigoplus_{l=-p}^{p} \Pi_{p+1}(l) \tag{2.6}$$

finite-dimensional:

$$\pi^l = dim(\Pi_{p+1}(l)) = p+1-|l|\ ,\ \ dim(\Pi_{p+1}) = (p+1)^2\,.$$

Then, by applying the assumptions (2.4) and (2.5) to $\partial\theta$ we find that

$$\partial\theta = b_0 + b_1\theta\partial + b_2\theta^2\partial^2 + \ldots + b_p\theta^p\partial^p , \tag{2.7}$$

where b_i are complex numbers restricted by consistency of the conditions (2.4) and (2.7) and by further assumptions to be formulated below. With the aid of Eq. (2.7) any element of the algebra can be expressed in terms of the basis $\theta^r\partial^s$, i.e. in the normal-ordered form.

A useful alternative set of parameters, α_k , may be defined by

$$\partial\theta^k = \alpha_k\theta^{k-1} + (\ldots)\partial\,, \tag{2.8}$$

where dots denote a polynomial in θ and ∂. This relation is a generalization of the commutation relation for the standard derivative operator, $\partial_z z^k = kz^k + z^k\partial_z$, and we may define the differentiation of powers of θ by analogy,

$$\partial(\theta^k) = \alpha_k\theta^{k-1}, \alpha_0 \equiv 0\,, \tag{2.9}$$

to be justified later.

By applying Eq. (2.7) to Eq. (2.8) one may derive recurrent relations connecting these two sets of the parameters:

$$\begin{aligned}
\alpha_1 &= b_0\,,\\
\alpha_2 &= b_0 + b_1\alpha_1\,,\\
\alpha_3 &= b_0 + b_1\alpha_2 + b_2\alpha_1\alpha_2\,,\\
&\cdots\\
\alpha_{k+1} &= \sum_{i=0}^{k} b_i\frac{(\alpha_k)!}{(\alpha_{k-i})!}\,,\\
&\cdots
\end{aligned} \tag{2.10}$$

where $(\alpha_k)! = \alpha_1\alpha_2\cdots\alpha_k$. These relations enable us to express α_k as a function of the numbers b_i , $0\le i\le k-1$. The first few expressions are

$$\begin{aligned}
\alpha_1 &= b_0\,,\ \ \alpha_2 = b_0\frac{1-b_1^2}{1-b_1}\,,\ \ \alpha_3 = b_0\frac{1-b_1^3}{1-b_1} + b_2b_0^2(1+b_1),\\
\alpha_4 &= b_0\frac{1-b_1^4}{1-b_1} + b_2b_1b_0^2(1+b_1) + b_0(b_3+b_2b_0)(1+b_1)\alpha_3, \ldots
\end{aligned} \tag{2.11}$$

The inverse operation, deriving b_i in terms of α_k, is well-defined only if all $\alpha_k \neq 0$.

The consistency condition mentioned above is that the parameters must be chosen so as to satisfy the identity

$$0 \equiv \partial\theta^{p+1} \, .$$

Taking into account that the second term in Eq. (2.8) vanishes for $k = p+1$ we have $\alpha_{p+1} = 0$, with no other restrictions on the parameters α_k with $k \leq p$. The corresponding restriction on $p+1$ parameters b_i follow from Eq. (2.10),

$$\alpha_{p+1}(b_0, \dots, b_p) \equiv b_0 + b_1\alpha_p + b_2\alpha_p\alpha_{p-1} + \dots + b_p\alpha_p\alpha_{p-1}\cdots\alpha_2\alpha_1 = 0 \, , \qquad (2.12)$$

where the parameters α_i are expressed in terms of b_i. Any admissible set $\{b\}$ determines an algebra $\Pi_{p+1}^{\{b\}}$ with the defining relations (2.3), (2.7). To each algebra $\Pi_{p+1}^{\{b\}}$ there corresponds a set $\{\alpha\}$. *A priori*, there are no restrictions on $\{\alpha\}$, but, if we wish to treat ∂ as a non-degenerate derivative with respect to θ, it is reasonable to require, in addition to (2.5), that

$$\textit{all} \;\; \alpha_k \neq 0 \, . \qquad (2.13)$$

So let us call a set $\{b\}$ (and corresponding algebra$\Pi_{p+1}^{\{b\}}$) *non-degenerate*, if the condition (2.13) is fulfilled, and degenerate otherwise. As it was already mentioned, in the non-degenerate case the numbers b_i are completely determined by the numbers α_k, so we can use the symbol $\{\alpha\}$ as well as $\{b\}$.

At first sight, the algebras corresponding to different sets $\{b\}$ look very dissimilar. Indeed, different sets $\{b\}$ determine, in general, non-equivalent algebras $\Pi_{p+1}^{\{b\}}$. However, this is not true for the non-degenerate ones. In fact, all non-degenerate algebras $\Pi_{p+1}^{\{b\}}$ are isomorphic to the associative algebra $Mat(p+1)$ of the complex $(p+1)\times(p+1)$ matrices.

This isomorphism can be manifested by constructing an explicit exact ('fundamental') representation for $\Pi_{p+1}^{\{b\}}$. With this aim we treat θ and ∂ as creation and annihilation operators (in general, not Hermitian conjugate) and introduce the ladder of $p+1$ states $|k\rangle$, $k = 0, 1, \dots, p$ defined by

$$\partial|0\rangle = 0 \, , \; |k\rangle \sim \theta^k|0\rangle \, , \theta|k\rangle = \beta_{k+1}|k+1\rangle \, , \qquad (2.14)$$

where β's are some non-zero numbers, reflecting the freedom of the basis choice. As $|p+1\rangle = 0$, the linear shell of the vectors $|k\rangle$ is finite-dimensional and in the nondegenerate case, when all $\beta_k \neq 0 \;\; (k = 1, \dots, p)$, its dimension is $p+1$.

Using (2.14) and (2.8) we find

$$\partial|k\rangle = (\alpha_k/\beta_k)|k-1\rangle \; . \qquad (2.15)$$

Thus the fundamental (Fock-space) representations of the operators θ and ∂ is

$$\begin{aligned} \theta_{mn} &= \langle m|\theta|n\rangle = \beta_{n+1}\delta_{m,n+1} \; , & (2.16)\\ \partial_{mn} &= \langle m|\partial|n\rangle = (\alpha_n/\beta_n)\delta_{m,n-1} \; . & (2.17) \end{aligned}$$

It is not hard to see that for non-zero α's, the matrices corresponding to $\theta^m\partial^n$ $(m, n = 0\dots p)$, form a complete basis of the algebra $Mat(p+1)$. The isomorphism is established.

Nothing similar occurs for degenerate algebras. To show an evidence against using them in the paragrassmann calculus, consider an extremely degenerate algebra with $b_0 = b_2 = \dots = b_p = 0$, $b_1 \neq 0$, so that all $\alpha_k = 0$. This algebra has nothing to do with $Mat(p+1)$, and its properties essentially depend on the value of b_1. It is abelian if $b_1 = 1$; it is a paragrassmann algebra of the type $\Gamma_{p+1}(2)$ if b_1 is a primitive root of unity (see [8]), and so on. We hope this remark is not sounding like a death sentence on the degenerate algebras. At least, it has to be suspended until further investigation which will probably prove their usefulness in other contexts. However, if we wish to have paragrassmann calculus similar to the Grassmann one, we have to use the nondegenerate algebras.

Thus, two natural requirements (2.5) and (2.13) reduce the range of possible generalizations of the fermionic algebra Π_1 to the unique algebra Π_{p+1} that is isomorphic to $Mat(p+1)$[3]. The grading (2.4) in Π_{p+1} corresponds to 'along-diagonal' grading in $Mat(p+1)$. Different non-degenerate algebras $\Pi_{p+1}^{\{b\}}$ are nothing more than alternative ways of writing one and the same algebra Π_{p+1}. We will call them *versions* having in mind that fixing the b-parameters is analogous to a gauge-fixing (in H. Weyl's usage).

This implies that we will mainly be interested in 'version-covariant' results, i.e. independent on a version choice. Nevertheless, special versions may have certain nice individual features making them more convenient for concrete calculations (thus allowing for simpler derivations of covariant results by non-covariant methods). Several useful versions will be described below. Before turning to this task we end our general discussion with several remarks.

First. The existence of the exact matrix representation (2.16), (2.17) is very useful for deriving version-covariant identities in the algebra Π_{p+1}. For instance, it is easy to check that

$$\begin{aligned} \{\partial \, , \, \theta^{(p)}\} &= (\Sigma\, \alpha_k)\; \theta^{p-1} \, , \\ \{\partial^p \, , \, \theta^{(p)}\} &= \Pi\, \alpha_k \, , \end{aligned} \qquad (2.18)$$

and to find many other relations. Here we have introduced a useful notation

$$\{\Xi \, , \, \Psi^{(l)}\} = \Xi\Psi^l + \Psi\Xi\Psi^{l-1} + \dots + \Psi^l\Xi \, . \qquad (2.19)$$

The identities (2.18) generalize those known in the parasupersymmetric quantum mechanics [4].

Second. One may adjust the parameters β_k to get a convenient matrix representation for θ and ∂. As a rule, we take $\beta_k = 1$. Note that for the versions with real

[3] H. Weyl in his famous book on quantum mechanics had foreseen relevance of these algebras to physics problems. After detailed description of the spin algebras he discussed more general finite algebras and remarked that the finite algebras like those discussed here will possibly appear in future physics. We think it natural to call Π_{p+1} the 'finite Weyl algebra' or 'para-Weyl algebra'.

parameters α_k, it is possible to choose β_k so as to have $\theta^\dagger = \partial$. We also normalize θ and ∂ so that $\alpha_1 \equiv b_0 = 1$. This gives a more close correspondence with the Grassmann relation (2.2).

Third. In a given (non-degenerate) version $\Pi^{\{b\}}_{p+1}$ the components of the vector $R^{(l)}_{\{b\}} = col\{\partial^j\theta^i\}_{i-j=l}$ form a basis of the subspace $\Pi_{p+1}(l)$ that is completely equivalent to the original one having the components $L^{(l)}_{\{b\}} = col\{\theta^i\partial^j\}_{i-j=l}$, see (2.5). Hence, there must exist a non-degenerate matrix $C^{(l)}_{\{b\}} \in Mat(\pi^{(l)}, \mathbf{C})$ connecting these two bases,

$$R^{(l)}_{\{b\}} = C^{(l)}_{\{b\}} \cdot L^{(l)}_{\{b\}}\,, \quad l = -p, \ldots, p\,. \tag{2.20}$$

The elements of the C-matrix are certain functions of b_i which are usually not easy to calculate except simple versions. The original commutation relation (2.7) is also included in the system (2.20), for $l = 0$.

Quite similarly, two L-bases (R-bases) taken in different versions $\{b\}$ and $\{b'\}$ are connected by a non-degenerate matrix $M_{\{bb'\}}$ ($N_{\{bb'\}}$), i.e.

$$L_{\{b\}} = M_{\{bb'\}} L_{\{b'\}}\,, \tag{2.21}$$

$$R_{\{b\}} = N_{\{bb'\}} R_{\{b'\}}\,, \tag{2.22}$$

where the indices (l) are suppressed. The matrices $M^{(l)}_{\{bb'\}}$ (and $N^{(l)}_{\{bb'\}}$) belong to $Mat(\pi^{(l)})$ and obey the cocyclic relations:

$$M_{\{bb'\}} M_{\{b'b\}} = 1\,, \quad M_{\{bb'\}} M_{\{b'b''\}} M_{\{b''b\}} = 1\,.$$

By applying Eq. (2.20) we immediately get the relation

$$N_{\{bb'\}} = C_{\{b\}} M_{\{bb'\}} C^{-1}_{\{b'\}}\,. \tag{2.23}$$

that permits evaluating C-matrices for complicated versions once we know them in one version. In particular, Eq. (2.21) tells that the operator ∂ in any version can be represented as a linear combination of the operators ∂ , $\theta\partial^2$, ... , $\theta^{p-1}\partial^p$ of any other version. We shall see soon that this, for instance, permits to realize q-oscillators in terms of generators θ and ∂ of other versions and *vice versa*.

Now consider some special versions related to the simplest forms of Eq. (2.7).

(0): Primitive Version

Here $b_1 = \ldots = b_{p-1} = 0\,, \ b_p = -1\,,$ so that $\alpha_i = 1\,,$

$$(\partial_{(0)})_{mn} = \delta_{m,n-1}\,, \quad \partial_{(0)}\theta = 1 - \theta^p\partial^p_{(0)}\,. \tag{2.24}$$

This realization of ∂ may be called 'almost-inverse' to θ. In the matrix representation (2.16), (2.17) with $\beta_k = 1$ we have $\theta^T = \partial_{(0)}$. This version is the simplest possible but the differential calculus is a fancy-looking thing in this disguise and it is unsuitable for many applications. Still, it has been used in some cases. For example,

the operators θ and $\partial_{(0)}$ for $p = 2$ coincide with parafermions in the formulation of the parasupersymmetric quantum mechanics [4]. Realization of parafermions and parabosons [6] for $p \geq 2$ within our approach is possible in other versions discussed in Appendix.

(1): q - Version, or Fractional Version

Here $b_1 = q \neq 0\,, \ b_2 = b_3 = \ldots = b_p = 0\,,$ so that

$$\alpha_i = 1 + q + \ldots + q^{i-1} = \frac{1-q^i}{1-q}\,.$$

The condition $\alpha_{p+1} = 0$ tells that $q^{p+1} = 1\,,$ $(q \neq 1)$ while the assumption that all $\alpha_i \neq 0$ forces $q = b_1$ to be a primitive root, i.e. $q^{n+1} \neq 1\,,\ n < p$. Thus, in this version ($\partial = \partial_{(1)}$),

$$\partial_{(1)}\theta = 1 + q\theta\partial_{(1)}\,, \tag{2.25}$$

$$\partial_{(1)}(\theta^n) = (n)_q\theta^{n-1}\,, \quad (n)_q = \frac{1-q^n}{1-q}\,.$$

These relations were introduced in Ref. [8] by assuming that ∂ is a generalized differentiation operator, i.e. satisfying a generalized Leibniz rule (a further generalization is introduced below). The derivative $\partial_{(1)}$ is naturally related to the q-oscillators [15, 16] and to quantum algebras; Eq. (2.25) is also extremely convenient for generalizing to the Paragrassmann algebras with many θ's and ∂'s (see [8], [13] and references therein).

(2): Almost Bosonic Version

For this Version

$$b_1 = 1\,, \ b_2 = \ldots = b_{p-1} = 0\,, \ b_p \neq 0\,, \ \text{so that} \ \alpha_k = k$$

and $\alpha_{p+1} = 0$ gives $b_p = -\frac{p+1}{p!}$. Thus

$$(\partial_{(2)})_{mn} = n\,\delta_{m,n-1}\,, \quad \partial_{(2)}\theta = 1 + \theta\partial_{(2)} - \frac{p+1}{p!}\,\theta^p\partial^p_{(2)}\,, \tag{2.26}$$

suggesting that this derivative is 'almost bosonic' as $\partial_{(2)}(\theta^n) = n\theta^{n-1}$ $(n < p+1)$. This Version is convenient for rewriting the generators of para-extensions of the Virasoro algebra in the most concise form in the next section.

In agreement with the general formula (2.21), derivatives from different versions can be expressed one through the other by certain non-linear relations. For example,

$$\partial_{(1)} = \sum_{j=0}^{p-1} \frac{(q-1)^j}{(j+1)!}\,\theta^j\partial^{j+1}_{(2)}\,.$$

Similar formulas can be written for any couple of versions.

Let us now discuss the interrelations between θ and ∂. As we have already mentioned the notation itself hints at treating ∂ as a derivative with respect to θ (see (2.8). To be more precise, let us represent an arbitrary vector $|F\rangle = \sum_{k=0}^{p} f_k|k\rangle$ of the 'Fock space' (2.14) as a function of θ

$$F(\theta) = \sum_{k=0}^{p} f_k\theta^k.$$

The action of the derivative ∂ on this function is defined by (2.15) ($\beta_k = 1$),

$$\partial(1) = 0\,, \quad \partial(\theta^n) = \alpha_n\theta^{n-1} \quad (1 \le n \le p)\,. \tag{2.27}$$

It is clear, however, that this derivative does not obey standard Leibniz rule $\partial(ab) = \partial(a)b + a\partial(b)$.

So consider the following modification of the Leibniz rule [8], [14]

$$\partial(FG) = \partial(F)\bar{g}(G) + g(F)\partial(G)\,. \tag{2.28}$$

The associativity condition (for differentiating FGH) tells that g and $\bar{g}$ are homomorphisms, i.e.

$$g(FG) = g(F)g(G)\,, \quad \bar{g}(FG) = \bar{g}(F)\bar{g}(G)\,. \tag{2.29}$$

The simplest natural homomorphisms compatible with the relations (2.27), (2.28), and (2.29) are linear automorphisms of the algebra Γ_{p+1},

$$g(\theta) = \gamma\theta\,, \quad \bar{g}(\theta) = \bar{\gamma}\theta\,, \tag{2.30}$$

where γ, $\bar{\gamma}$ are arbitrary complex parameters and

$$\alpha_k = \frac{\bar{\gamma}^k - \gamma^k}{\bar{\gamma} - \gamma}. \tag{2.31}$$

Now the condition (2.12) yields the equation

$$\alpha_{p+1} = \frac{\bar{\gamma}^{p+1} - \gamma^{p+1}}{\bar{\gamma} - \gamma} = 0, \tag{2.32}$$

and assuming nondegeneracy requirements $\alpha_k \neq 0$ ($k < p+1$) we conclude that $\bar{\gamma}/\gamma$ must be a primitive $(p+1)$-root of unity. Thus we may formulate another interesting version of the paragrassmann algebra Π_{p+1} .

(3): g-$\bar{g}$ - **Version**
As the parameters α_k are given by Eq. (2.31), we have to calculate b_i by solving Eq. (2.10): $b_0 = 1$, $b_1 = \bar{\gamma} + \gamma - 1$, $b_2 = (\bar{\gamma} - \bar{\gamma}\gamma + \gamma - 1)/(\bar{\gamma} + \gamma)$, ...
Here γ and $\bar{\gamma}$ are complex numbers constrained by the condition that $q = \bar{\gamma}/\gamma$ is a primitive root of unity

$$\left(\frac{\bar{\gamma}}{\gamma}\right)^{p+1} = 1$$

From Eqs. (2.28) and (2.30) one can derive the following operator relations for the automorphisms g, $\bar{g}$

$$\partial\theta - \gamma\theta\partial = \bar{g}\,, \quad \partial\theta - \bar{\gamma}\theta\partial = g\,. \tag{2.33}$$

For the special case $\gamma = (\bar{\gamma})^{-1} = q^{1/2}$ redefining $\partial = a$, $\theta = a^\dagger$, allows to recognize in (2.33) the definitions of the q-deformed oscillators in the Biedenharn-MacFarlane form [15]). Note that the Version-(1) can be derived from the Version-(3) by putting $\bar{\gamma} = q$, $\gamma = 1$ (or $\bar{\gamma} = 1$, $\gamma = q$), So we may regard the Version-(3) as a generalization of the Version-(1). Moreover, it can be shown that for $p = 2$ both the Version-(0) and the Version-(2) are specializations of the Version-(3). However, it is not true for $p > 2$ and, in general, the Leibniz rule (2.28) has to be further modified. This modification is discussed in our accompanying paper [13] in which paragrassmann calculus with many variables is also considered in some detail. Note that a reasonably simple many-variable paragrassmann calculus with a generalized Leibniz rule can be formulated only for Versions (1) and (3). Nevertheless, Version-(2) is also useful in applications as will be demonstrated below.

3. Paraconformal Algebra Con_p

To start realizing the program outlined in Introduction consider a para-superplane $z = (z,\ \theta)$, where $z \in \mathbf{C}$ and θ is the generator of the paragrassmann algebra $\Gamma_{p+1}(1) = \Gamma_\theta$, i.e. $\theta^{p+1} = 0$. Any function defined on this plane has the form

$$F \equiv F(z,\ \theta) = F_0(z) + \theta F_1(z) + \theta^2 F_2(z) + \ldots + \theta^p F_p(z). \tag{3.1}$$

It is useful to define an analogue of the superderivative as a $(p+1)$-root of the derivative ∂_z [8], [9]

$$\mathcal{D} = \partial_\theta + \mu\frac{\theta^p}{(\alpha_p)!}\partial_z\,, \quad \mathcal{D}^{p+1} = \mu\partial_z\,. \tag{3.2}$$

We denote here the θ-derivative in arbitrary version by ∂_θ instead of ∂ and shall often use this notation to make some formulas more transparent. The number μ will be fixed later.

The action of this operator on the function (3.1) is

$$\mathcal{D}F(z,\ \theta) = F_1(z) + \alpha_2\theta F_2(z) + \ldots + \alpha_p\theta^{p-1}F_p(z) + \mu\frac{\theta^p}{(\alpha_p)!}F_0'(z)\,, \tag{3.3}$$

where $F' = \partial_z F$. In analogy with the super-calculus (see [11]) the inverse operator $\mathcal{D}^{-1}$ defined by

$$\mathcal{D}^{-1}F \equiv \int^z dz\ F = \frac{(\alpha_p)!}{\mu}\int^z dz'\ F_p(z') + \theta F_0(z) + \frac{\theta^2}{\alpha_2}F_1(z) + \ldots + \frac{\theta^p}{\alpha_p}F_{p-1}(z)$$

may be formally interpreted as an 'indefinite' integral.

Let $\Gamma_{p+1}(1)$ is embedded into some infinite dimensional paragrassmann algebra $\Gamma_{p+1}(1+\infty)$ with generators $\theta_0 = \theta,\ \theta_1, \theta_2, \ldots$. Then a 'definite' integral may be defined [4],

$$\int_{z_2}^{z_1} dz\ F = \int^{z_1} dz\ F - \int^{z_2} dz\ F, \tag{3.4}$$

naturally giving $p+1$ coordinates for para-super-translation invariant functions on the para-superplane are (cf. with [11]):

$$\theta_{12} = \theta_1 - \theta_2 = \int_{z_2}^{z_1} dz\ ,$$

$$\theta_{12}^{(2)} = \int_{z_2}^{z_1} dz \int_{z_2}^{z} dz^{(1)} = \frac{1}{\alpha_2}\theta_1^2 - \theta_1\theta_2 + (1 - \frac{1}{\alpha_2})\theta_2^2\ ,\ \ldots$$

$$\theta_{12}^{(p+1)} = z_{12} = \int_{z_2}^{z_1} dz \int_{z_2}^{z} dz^{(1)} \ldots \int_{z_2}^{z^{(p-1)}} dz^{(p)} =$$

$$\frac{1}{\mu}(z_1 - z_2) - \frac{1}{(\alpha_p)!}\theta_1^p\theta_2 + (1 - \frac{1}{\alpha_2})\frac{1}{(\alpha_{p-1})!}\theta_1^{p-1}\theta_2^2 + \ldots .$$

We think that these properties of $\mathcal{D}$ and $\mathcal{D}^{-1}$ justify regarding $\mathcal{D}$ as a correct generalization of the superderivative.

Now consider invertible transformations of the para-superplane

$$\begin{array}{lll} z & \rightarrow & \tilde{z}(z,\theta)\ ,\quad deg(\tilde{z}) = 0\ , \\ \theta & \rightarrow & \tilde{\theta}(z,\theta)\ ,\quad deg(\tilde{\theta}) = 1\ . \end{array} \tag{3.5}$$

$$\tilde{z} = Z_0(z) + \theta Z_1(z) + \ldots + \theta^p Z_p(z)\ , \tag{3.6}$$
$$\tilde{\theta} = \Theta_0(z) + \theta\Theta_1(z) + \ldots + \theta^p\Theta_p(z)\ , \tag{3.7}$$

where deg is a natural $\mathbf{Z}_{p+1}$-grading in $\Gamma_{p+1}(1+\infty)$ and Z_i and Θ_i are functions of z with values in $\Gamma_{p+1}(1+\infty)/\Gamma_\theta$ of needed grading. Here we assume that it is possible to move all θ's to the left-hand sides of the para-superfields (3.6) and (3.7) (for Version-(1) it is evident). We also require that

$$\tilde{\theta}^{p+1} = 0\ . \tag{3.8}$$

The corresponding transformation for the functions (3.1) is defined as

$$\tilde{F} \equiv \tilde{F}(z,\theta) = F(\tilde{z},\tilde{\theta}) = F_0(\tilde{z}) + \tilde{\theta}F_1(\tilde{z}) + \ldots + \tilde{\theta}^p F_p(\tilde{z})\ . \tag{3.9}$$

[4]The integral over z is to be understood as a contour integral in the complex z-plane, and it is contour-independent as far as the contours are not crossing singularities of $F_p(z)$. Thus the integral might be regarded as a 'contour' integral in a para-superplane.

In accord with (3.3), $\tilde{\mathcal{D}}\tilde{F}$ is

$$\tilde{\mathcal{D}}\tilde{F} = F_1(\tilde{z}) + \tilde{\theta}F_2(\tilde{z}) + \ldots + \frac{\mu}{(\alpha_p)!}\tilde{\theta}^p F_0'(\tilde{z})\ . \tag{3.10}$$

Following the route described in Section 1, let us consider the transformations obeying the requirement analogous to Eq. (1.9)

$$\mathcal{D}F(\tilde{\theta},\tilde{z}) = \Phi(\theta,z)\tilde{\mathcal{D}}F(\tilde{\theta},\tilde{z})\ , \tag{3.11}$$

or, in the operator form,

$$\mathcal{D} = \Phi\tilde{\mathcal{D}}\ . \tag{3.12}$$

Acting on $\tilde{\theta}$ we immediately get

$$\Phi = \mathcal{D}\tilde{\theta}\ . \tag{3.13}$$

Consider main properties of these transformations. If exist, they form a group since two subsequent transformations $z \mapsto \tilde{z}(z) \mapsto \tilde{\tilde{z}}(\tilde{z}(z))$ automatically yield (see (3.12), (3.13)) the needed multiplier

$$\mathcal{D} = (\mathcal{D}\tilde{\theta})\tilde{\mathcal{D}} = (\mathcal{D}\tilde{\theta})(\tilde{\mathcal{D}}\tilde{\tilde{\theta}})\tilde{\tilde{\mathcal{D}}} = (\mathcal{D}\tilde{\tilde{\theta}})\tilde{\tilde{\mathcal{D}}}\ . \tag{3.14}$$

This group will be called 'paraconformal' and referred to as $\mathbf{CON}_p$.

The condition (3.12) is a very strong restriction on possible form of transformation functions (3.5). It is not hard to see that all the restrictions can be derived by putting F to be $\tilde{\theta}^k$, $\tilde{z}$ and $\tilde{\theta}\tilde{z}$. These give

$$\mathcal{D}\tilde{\theta}^k = \alpha_k(\mathcal{D}\tilde{\theta})\tilde{\theta}^{k-1}\ , \tag{3.15}$$
$$\mathcal{D}\tilde{z} = \frac{\mu}{(\alpha_p)!}(\mathcal{D}\tilde{\theta})\tilde{\theta}^p\ , \tag{3.16}$$
$$\mathcal{D}(\tilde{\theta}\tilde{z}) = (\mathcal{D}\tilde{\theta})\tilde{z}\ . \tag{3.17}$$

Note that (3.16) allows us to interpret (3.12) as the rule for differentiation of composite functions

$$\mathcal{D} = (\mathcal{D}\tilde{\theta})(\tilde{\partial}_\theta + \mu\frac{\tilde{\theta}^p}{(\alpha_p)!}\tilde{\partial}_z) = (\mathcal{D}\tilde{\theta})\tilde{\partial}_\theta + (\mathcal{D}\tilde{z})\tilde{\partial}_z\ . \tag{3.18}$$

Summarizing the main properties of the transformations satisfying (3.12), we conclude that it is reasonable to regard them as the paraconformal transformations of the para-superplane.

The main restrictions on the parameters of a paraconformal transformations are coming from Eq. (3.16) that, taken in components, give rise to the following relations between Z_i and Θ_j (defined by (3.6), (3.7))

$$
\begin{aligned}
Z_1 &= \frac{\mu}{(\alpha_p)!}\Theta_1\Theta_0^p\,,\\
\theta Z_2 &= \frac{\mu}{(\alpha_p)!}(\theta\Theta_2\Theta_0^p + \frac{1}{\alpha_2}\Theta_1\{\theta\Theta_1\ ,\ \Theta_0^{(p-1)}\})\,,\\
&\dots\\
\theta^{p-1}Z_p &= \frac{\mu}{(\alpha_p)!}(\theta^{p-1}\Theta_p\Theta_0^p + \ldots + \frac{1}{\alpha_2}\Theta_1\{(\theta\Theta_1)^{(p-1)}\ ,\ \Theta_0\})\,,\\
\theta^p Z_0' &= \frac{\mu}{(\alpha_p)!}\theta^p\Theta_0'\Theta_0^p + \alpha_p\theta^{p-1}\Theta_p\{\theta\Theta_1\ ,\ \Theta_0^{(p-1)}\} + \ldots + \Theta_1\cdot(\theta\Theta_1)^p\,.
\end{aligned}
\tag{3.19}
$$

Here the notation introduced in (2.19) is used.

These relations generalize the much more simple Eq. (1.10) to any p . It is rather hard to push forward the analysis with complicated expressions (3.19) without having a well-established technique for handling many thetas. For this reason, we are forced to turn to the infinitesimal language. Then, introducing a small c-number parameter λ one may rewrite the transformations (3.5 – 3.7) as

$$\tilde{z}(z,\theta) = z + \lambda\Omega(z,\theta) + O(\lambda^2)\,,\ \Omega = \sum_{i=0}^{p}\theta^i\omega_i(z)\,, \tag{3.20}$$

$$\tilde{\theta}(z;\theta) = \theta + \lambda\mathcal{E}(z,\theta) + O(\lambda^2)\,,\ \mathcal{E} = \sum_{i=0}^{p}\theta^i\epsilon_i(z)\,. \tag{3.21}$$

where ω_i and ϵ_i are first coefficients in expansion of functions $Z_i(z)$ and $\Theta_i(z)$ in powers of λ. *A priori*, there is no reason to exclude the higher powers from consideration. The infinitesimal transformations (3.20), (3.21) satisfying the paraconformal conditions (3.15 – 3.17) define a space which we denote by $\mathcal{CON}_p$. This is an infinitesimal object corresponding to the paraconformal group CON_p and so it must carry some algebraic structure induced by the group structure of CON_p . In this sense, we will speak of 'the algebra $\mathcal{CON}_p$' though its algebraic properties will be discussed later. Here we briefly analyze a geometric meaning of $\mathcal{CON}_p$.

As it is evident from (3.21), the only component of $\tilde{\theta}$ containing a finite part is $\Theta_1 = 1 + \lambda\epsilon_1(z)$; the rest of Θ_i are of the first order in λ. Then (3.19) tells that only Z_0 and Z_p contain the terms of the first order in λ, while the other Z_i with $1 \le i \le p-1$ must be of the order λ^{p+1-i}. This suggests that all terms up to the order λ^p must be kept in (3.20), (3.21). So, since the transformed function is generally defined as $\tilde{F} = (1 + (general\ \ element\ \ of\ \ the\ \ algebra))F$, (cf. (1.13)) a general element of the algebra $\mathcal{CON}_p$ must be of the form

$$(general\ \ element) = \lambda\{\mathcal{L}^{(1)}\} + \lambda^2\{\mathcal{L}^{(2)}\} + \ldots + \lambda^p\{\mathcal{L}^{(p)}\}\,. \tag{3.22}$$

Here we denote by $\{\mathcal{L}^{(M)}\}$ a set of the generators of the M-th *generation*. Eq.(3.19) shows that M-th generation $\{\mathcal{L}^{(M)}\}$ must contain some new generators $\{\mathcal{X}^{(M)}\}$ that

are not present in $\{\mathcal{L}^{(M-1)}\}$. This would guarantee, in particular, the appearance of non-zero Z_{p+1-M}. If it were possible to put all Z_i $(i = 1\ldots p-1)$ to zero we would get rid of these subtleties. But as we will see below, this contradicts to the requirement of the bilinear closure of the algebra $\mathcal{CON}_p$. In fact, new generators naturally arise from certain bilinear brackets of the old, and this process stops only at the p-th generation. That is an algebraic question, though.

Returning to geometry, the formula (3.22) indicates that the algebra $\mathcal{CON}_p$ lives not in the tangent space of the group, as it is customary for Lie algebras, but rather in a space of p-jets. The p-jets are famous for that there must be generators of the form $(something)\partial^j$ $(j = 1\ldots p)$. This is right the case for the algebra $\mathcal{CON}_p$, as we will see below.

Now an explicit realization of the paraconformal algebra generators $\mathcal{L}^{(M)}$ is in order. We concentrate on the first generation because generators of the higher generations must be obtained through multilinear combinations of them. Substituting (3.20) (3.21) in (3.15) – (3.17) we get the (first order smallness) infinitesimal form of the paraconformal conditions

$$\mathcal{D}\{\mathcal{E},\theta^{(k)}\} = (k+1)((\mathcal{D}\mathcal{E})\theta^k + \{\mathcal{E},\theta^{(k-1)}\})\,,\ k = 1\ldots p-1\,, \tag{3.23}$$

$$\mathcal{D}\Omega = \frac{\mu}{(\alpha_p)!}((\mathcal{D}\mathcal{E})\theta^p + \{\mathcal{E},\theta^{(p-1)}\})\,, \tag{3.24}$$

$$\mathcal{D}(\mathcal{E}z + \theta\Omega) = (\mathcal{D}\mathcal{E})z + \Omega\,. \tag{3.25}$$

Eqs.(3.23 – 3.25) lead to certain restrictions on the functions ϵ_i , ω_i, (we assume ω_0 and ϵ_1 to be ordinary functions, free of any paragrassmann content). The condition (3.24) gives

$$\omega_1 = \omega_2 = \ldots = \omega_{p-1} = 0\,, \tag{3.26}$$

$$\epsilon_1 = \frac{1}{p+1}\omega_0'\,, \tag{3.27}$$

$$\alpha_p\theta^{p-1}\omega_p = \frac{\mu}{(\alpha_p)!}\{\epsilon_0,\theta^{(p-1)}\}\,, \tag{3.28}$$

wherefrom by virtue of $\{\epsilon_0\ ,\ \theta^{(p)}\} = 0$, coming from the nilpotency condition (3.8), we have

$$\theta^p\omega_p = -\frac{1}{\alpha_p}\frac{\mu}{(\alpha_p)!}\epsilon_0\theta^p\,. \tag{3.29}$$

From the third relation (3.25) we find that

$$\theta^p\omega_p = \frac{\mu}{(\alpha_p)!}\theta^p\epsilon_0\,,\ \text{ or }\ \omega_p = \frac{\mu}{(\alpha_p)!}\epsilon_0\,, \tag{3.30}$$

which, together with (3.29), gives the commutation relation, valid for any version:

$$\epsilon_0\theta^p + \alpha_p\theta^p\epsilon_0 = 0 \tag{3.31}$$

The condition (3.23) provides the commutation rules of ϵ_i with θ^k and ∂:

$$\partial(\theta^i\{\epsilon_i,\theta^{(k)}\}) = (k+1)(\alpha_i\theta^{i-1}\epsilon_i\theta^k + \theta^i\{\epsilon_i,\theta^{(k-1)}\}) \ , \quad k=1\dots p-1\,. \tag{3.32}$$

We emphasize that, in general, these rules do not require any ϵ_i to be zero.

Thus the resulting infinitesimal paraconformal transformations of the first order in λ look as follows

$$\begin{aligned}
\delta z &= \lambda(\omega_0(z) - \frac{1}{\alpha_p}\frac{1}{(\alpha_p)!}\epsilon_0(z)\theta^p) \ , \\
\delta\theta &= \lambda(\epsilon_0(z) + \frac{1}{p+1}\omega_0'(z)\theta + \theta^2\epsilon_2 + \dots + \theta^p\epsilon_p) \ , \\
\delta(\theta^k) &= \lambda(\{\epsilon_0,\theta^{(k-1)}\} + \frac{k}{p+1}\omega_0'\theta^k + \theta^2\{\epsilon_2,\theta^{(k-1)}\} + \dots + \theta^{p+1-k}\{\epsilon_{p+1-k},\theta^{(k-1)}\}\ ) \,.
\end{aligned} \tag{3.33}$$

To obtain generators of the transformations (3.33), it is convenient to define new operators $\mathcal{J}_0$ and $\bar\partial$ acting on θ^k in the following way

$$\mathcal{J}_0(\theta^k) = k\theta^k \ , \tag{3.34}$$
$$\epsilon_i\bar\partial(\theta^k) = \{\epsilon_i\,,\theta^{(k-1)}\} \ , \tag{3.35}$$

The first operator is a generator of the automorphism group of paragrassmann algebra (see Ref.[8]). The second one is wanted to be interpreted as differentiation in certain other version ($\bar N$), which is, in this sense, associated to the original version (N). So we require, in addition to (3.35), that there must exist a set of non-zero numbers $\bar\alpha_k$ such that

$$\bar\partial(\theta^k) = \bar\alpha_k\theta^{k-1}\,. \tag{3.36}$$

This requirement is not too strong but, together with (3.35), it leads to certain nontrivial restrictions on commutation of ϵ_i and θ. We will not specify the associated version in general. It is enough for us, that, as we will see soon, for the special versions $(1)_q$ and (2) the associated versions are $(1)_{q^{-1}}$ and (2) itself respectively.

Now it is convenient to introduce the operator

$$Q = \bar\partial - \frac{1}{\alpha_p}\frac{\mu}{(\alpha_p)!}\theta^p\partial_z \ ,$$

and assume $\mu = -\alpha_p\frac{(\alpha_p)!}{(\bar\alpha_p)!}$, so that $Q^{p+1} = \partial_z$ (Q is an analogue of the supersymmetry generator and might be called the para-supersymmetry generator).

By virtue of these operators we can establish the generators of the transformations (3.33) in any version (we omit zero index at ω_0 from now)

$$\begin{aligned}
T(\omega) &= \omega\partial_z + \frac{1}{p+1}\omega'\mathcal{J}_0 \ , \\
\mathcal{G}(\epsilon_0) &= \epsilon_0(\bar\partial_\theta + \frac{\theta^p}{(\bar\alpha_p)!}\partial_z) = \epsilon_0 Q \ , \\
\mathcal{H}_j(\epsilon_{j+1}) &= \theta^{j+1}\epsilon_{j+1}\bar\partial_\theta \,.
\end{aligned} \tag{3.37}$$

These are the generators of the first generation $\{\mathcal{L}^{(1)}\}$. They must generate, through correct bilinear and multilinear brackets, the entire algebra $\mathcal{CON}_p$ corresponding to the group CON_p.

Let us now describe these generators in the preferred versions.

Version-$(1)_q$.

Remember that this means $\partial\theta = 1 + q\theta\partial$, and therefore $\partial\theta^k = (k)_q\theta^{k-1} + q^k\theta^k\partial$. Then, assuming the commutation relations $\epsilon_i\theta = r_i\theta\epsilon_i$, where r_i are some numbers, we find from (3.32) that all these factors r_i must be equal to q (for ϵ_0 this might be seen from (3.31) since $\alpha_p = (p)_q = -q^p$). Now recalling the definition of $\bar\partial$ (3.35) we find that

$$\bar\partial(\theta^k) = (k)_{q^{-1}}\theta^{k-1} \equiv q^{1-k}(k)_q\theta^{k-1} \ ,$$

and so $\bar\partial$ may be represented as $\bar\partial = g^{-1}\partial$, where g^{-1} is an automorphism of the paragrassmann algebra Γ_θ defined by $g^{-1}(\theta^k) = q^{-k}\theta^k$. It is easy to prove (see [8]) that

$$\begin{aligned}
g &= q^{\mathcal{J}_0} = \partial\theta - \theta\partial \ , \\
g^{-1} &= q^{-\mathcal{J}_0} = \bar\partial\theta - \theta\bar\partial \ .
\end{aligned}$$

Thus, we see that $\bar\partial$ is in fact a differentiation in the associated Version $(1)_{q^{-1}}$. The generators $\mathcal{G}$ and $\mathcal{H}$ are represented as

$$\mathcal{G}(\epsilon) = \epsilon(\bar\partial_\theta + \frac{\theta^p}{(p)_{q^{-1}}!}\partial_z) = \epsilon Q \ , \tag{3.38}$$
$$\mathcal{H}_j(\epsilon_{j+1}) = q^{-(j+1)}\epsilon_{j+1}\theta^{j+1}\bar\partial_\theta \,. \tag{3.39}$$

while T-generator has the same form as in Eqs. (3.37).

We would like to recall that the operator $\mathcal{D}$ (3.2) in the Version-$(1)_q$ and the para-supersymmetry generator Q (3.38) have been introduced earlier in the context of the fractional supersymmetry [9].

Version-(2) .

The unique property of this version is that the θ-derivative acts as the standard one except the terms proportional to θ^p in the Leibniz rule and corresponding terms in the operator formulas. This departure from the standard rules is not important in considering infinitesimal paraconformal transformations (3.33), and this makes Version-(2) the most convenient to express the generators.

The basic formula of the Version-(2) required in this context is

$$\partial\theta^j = j\theta^{j-1} + \theta^j\partial - \frac{p+1}{(p+1-j)!}\theta^p\partial^{p+1-j} \ , \quad j=1\dots p+1\,. \tag{3.40}$$

Applying it to the relations

$$\partial(\{\epsilon_0,\theta^{(k)}\}) = (k+1)\{\epsilon_0,\theta^{(k-1)}\} \ , \quad k=1\dots p-1 \ , \tag{3.41}$$

which are just (3.32) for ϵ_0, one can easily get by induction that

$$\partial(\epsilon_0\theta^k) = k\ \epsilon_0\theta^{k-1}\ ,\ \ k = 1\ldots p-1\ . \tag{3.42}$$

On the other hand, (3.41) can be considered, by the definition of $\tilde{\partial}$ (3.35 – 3.36), as an equation on $\tilde{\alpha}_k$'s

$$\tilde{\alpha}_{k+1}\ \partial(\epsilon_0\theta^k) = (k+1)\ \tilde{\alpha}_k\ \epsilon_0\theta^{k-1}\ ,\ \ k = 1\ldots p-1\ . \tag{3.43}$$

In account of (3.42), we get $\tilde{\alpha}_k = k$ for all $k = 1\ldots p$ since $\tilde{\alpha}_1 = 1$. Therefore $\tilde{\partial} \equiv \partial$ and the associated version is identical to Version-(2) itself.

Note that the above paragraph could be considered as a simplest example of a general procedure for obtaining the associated version. Note also that the relations (3.42) together with $\partial(\epsilon_0\theta^k) = -p^2\theta^{p-1}\epsilon_0 \equiv -\frac{p}{(p-1)!}\theta^{p-1}\epsilon_0\partial^p\ (\theta^p)$ can be summarized in a single operator formula

$$\partial\ \epsilon_0 = \epsilon_0\partial - \frac{p}{(p-1)!}\theta^{p-1}\epsilon_0\partial^p - \frac{1}{(p-1)!}\epsilon_0\theta^{p-1}\partial^p\ .$$

This is an example of how the commutation relations of an algebra with many paragrassmann variables can look in a version other than Version-(1). In fact, they can look as monstrously as one wants.

The operator $\mathcal{J}_0$, standing in the T-generator, has a very simple expression in Version-(2): $\mathcal{J}_0 = \theta\partial$, and so the generators (3.37) look in this version in a quite vector-like fashion

$$\begin{aligned} T(\omega) &= \omega\partial_z + \frac{1}{p+1}\omega'\theta\partial_\theta\ , \\ \mathcal{G}(\epsilon_0) &= \epsilon_0(\partial_\theta + \frac{\theta^p}{p!}\partial_z)\ , \\ \mathcal{H}_j(\epsilon_{j+1}) &= \theta^{j+1}\epsilon_{j+1}\partial_\theta\ . \end{aligned} \tag{3.44}$$

Usually, to derive some identities about the generators, it is better to take them in Version-(2), in the form (3.44). Version-(1) is better adapted to the situations with many paragrassmann variables.

Now let us turn to the algebra of the generators (3.37). The commutators with T are simple as they should be due to the commutativity of ω

$$[T(\omega),\ T(\eta)] = T(\omega\eta' - \omega'\eta)\ , \tag{3.45}$$

$$[T(\omega),\ \mathcal{G}(\epsilon)] = \mathcal{G}(\omega\epsilon' - \frac{1}{p+1}\omega'\epsilon)\ , \tag{3.46}$$

$$[T(\omega),\ \mathcal{H}_j(\xi)] = \mathcal{H}_j(\omega\xi' + \frac{j}{p+1}\omega'\xi)\ . \tag{3.47}$$

The rest of the commutators promise some subtleties. For instance, the commutator of two $\mathcal{G}$-generators gives rise to a new generator $\mathcal{G}^{(2)}$

$$[\mathcal{G}(\epsilon),\ \mathcal{G}(\zeta)] \propto \mathcal{G}^{(2)}(\epsilon\zeta - \zeta\epsilon) + \frac{1}{2}\mathcal{H}_{p-1}(\epsilon\zeta' - \epsilon'\zeta + \zeta'\epsilon - \zeta\epsilon')\ . \tag{3.48}$$

This new generator has the form

$$\mathcal{G}^{(2)}(\phi) = \phi Q^2 + \frac{1}{2}\phi'\frac{\theta^p}{(\tilde{\alpha}_p)!}\tilde{\partial} \tag{3.49}$$

and contains a term $\phi\theta^{p-1}\partial_z$, modifying z on a quantity proportional to θ^{p-1}, that is forbidden by the condition (3.26). Note, however, that (3.26) is a condition on the first order smallness variation, while $\mathcal{G}^{(2)}$ appears only in the second order of smallness. Similar effects occur when considering commutators of two $\mathcal{H}$-generators or of $\mathcal{G}$ and $\mathcal{H}$, giving rise to the generators $\mathcal{H}_j^{(2)}$. The latter have the form $\mathcal{H}_j^{(2)}(\psi) \propto \psi\theta^{j+1}\tilde{\partial}^2$.

$\mathcal{G}^{(2)}$ and $\mathcal{H}_j^{(2)}$ are right those new generators of the second generation, $\{\mathcal{X}^{(2)}\}$, expected to appear in the term of order λ^2 in (3.22). They could be obtained straightforward by reproducing the reasoning (3.23 – 3.37) for $\tilde{z}$ and $\tilde{\theta}$ being taken from (3.20), (3.21) up to the terms of the order λ^2.

Similar procedure can be carried out (but the calculations become more and more complicated) for λ^M , $M = 3$, 4 , *etc.* , giving rise to the generators of M-th generation $\{\mathcal{G}^{(M)}\ ,\ \mathcal{H}_j^{(M)}\} \equiv \{\mathcal{X}^{(M)}\}$. Generators $\mathcal{G}^{(M)}$ are right those that contain a term $\sim \theta^{p+1-M}\partial_z$, which gives rise to non-zero Z_{p+1-M} in (3.19). Generators $\mathcal{H}_j^{(M)}$ are proportional to $\theta^{j+1}\tilde{\partial}^M$ and do not affect z-coordinate. They lead to a deviation of θ^M from $(\tilde{\theta})^M$ on a quantity of order λ^M. Actually, this could be interpreted as a shift of the version during a paraconformal transformation.

New generators stop appearing at the order λ^p. This fact could be explained by two circumstances. First: the resource of possible combinations of θ , $\tilde{\partial}$ and ∂_z, which are the building blocks for generators, is exhausted at the order λ^p. Second, (closely related to the first): the algebra of the generators

$$\{\mathcal{L}^{(p)}\} = \{\mathcal{L}^{(1)}\} \cup \{\mathcal{X}^{(2)}\} \cup \ldots \cup \{\mathcal{X}^{(p)}\} = \{T,\ \mathcal{G},\ \mathcal{H}_j\ ;\ \mathcal{G}^{(2)},\ \mathcal{H}_j^{(2)}\ ;\ldots;\ \mathcal{G}^{(p)},\ \mathcal{H}_j^{(p)}\}$$

closes bilinearly. So, if we denote $A^{(M)}$ the linear shell of all the generators of M-th generation, $\{\mathcal{L}^{(M)}\}$, then the entire algebra $\mathcal{CON}_p$ can be performed as a p-filtered algebra

$$\mathcal{CON}_p \equiv A^{(\infty)} = A^{(p)} \supset A^{(p-1)} \supset \ldots \supset A^{(2)} \supset A^{(1)}\ ;\ \ [A^{(M)}\ ,\ A^{(K)}]_{MK} \subset A^{(M+K)}\ .$$

Each coset $A^{(M)}/A^{(M-1)}$ is based on the generators $\{\mathcal{X}^{(M)}\}$.

An explicit expression of the bilinear brackets $[...]_{MK}$ is not known to us yet, except the $[...]_{11}$, which is simply the commutator. In general, it is not clear, which combination of the para-generators (this word applies to $\mathcal{G}$- and $\mathcal{H}$-type ones, in any generation) should be taken, because the true bracket should be determined by the analogue of the Hausdorff formula for the para-supergroup, and the latter has not been obtained so far. Though taking the commutator of two $\mathcal{G}$-generators seems to be more or less consistent with the first order calculation, the commutator of, say, $\mathcal{G}$ and $\mathcal{G}^{(2)}$ seems scarcely having any relation to the group $\mathbf{CON}_p$. We would not like also to care about commutation properties of ϵ_i, especially taking into account that for general elements of the paragrassmann algebra of many variables the commutation formulas are not specified.

So the best thing we can do in this situation is to extract paragrassmann multipliers out of all ϵ-s and omit them, in analogy to what is done in the supercase, leaving the arguments of all generators to be just ordinary functions commuting with everything. Then we have to investigate the identities of such 'deparagrassmannized' (or, as we prefer to say, 'bare') generators, keeping in mind the hope that they will contain some hints about the true structure of the true para-superalgebra and para-supergroup. The complete and rigorous construction of these objects would require a more sharpened technique of handling paragrassmann algebras of many variables than we actually have.

So let us proceed the dealing with 'bare' para-generators, which will be denoted by the same letters but in more modest print. We concentrate on G-generators, for the reason will be clarified below.

The identities generated by G's can be described by three following statements:

1. The cyclic bracket of *any* number of G-generators depends only on the product of their arguments:

$$\frac{1}{M}\{G(\epsilon_1),\ldots,G(\epsilon_M)\}_c \equiv G^{(M)}(\eta)\ ,\quad \eta=\epsilon_1\cdot\ldots\cdot\epsilon_M \tag{3.50}$$

2. The similar fact takes place for $G^{(M)}$:

$$\frac{1}{K}\{G^{(M)}(\eta_1),\ldots,G^{(M)}(\eta_K)\}_c = G^{(KM)}(\zeta)\ ,\quad \zeta=\eta_1\cdot\ldots\cdot\eta_K \tag{3.51}$$

3.

$$G^{(p+1)}(\omega)\equiv T(\omega) \tag{3.52}$$

These assertions can be proved straightforwardly by virtue of certain identities in the algebra Π_{p+1}. The third of them presents the simplest variant of the closure of the algebra generated by T and G by the cyclic bracket, or the *cyclator*, of $p+1$ G-generators.

The identities (3.50) serve the same time as the definition of the generators $G^{(M)}$ which must be treated as the 'bare' variants of $\mathcal{G}^{(M)}$. They have an elegant explicit form

$$G^{(M)}(\eta)=\eta\, Q^M+\frac{1}{M}\,\eta'\mathcal{J}_{p+1-M}\ \ (M=1,\ldots,p+1)\ , \tag{3.53}$$

where $\mathcal{J}_l$ are certain generators of paragrassmann algebra automorphisms acting as

$$\mathcal{J}_l\left(\frac{\theta^k}{(\bar\alpha_k)!}\right)=k\,\frac{\theta^{k+l}}{(\bar\alpha_{k+l})!}\ .$$

Another consequence of (3.50 – 3.52) is existence of a set of non-equivalent $(p+1)$-linear brackets for composite $p+1$. Really, if $p+1=\nu_1\cdot\ldots\cdot\nu_k$ for some integer numbers ν_i, one can replace $G^{(p+1)}$ in (3.52) by the ν_1-linear cyclator of the generators $G^{(\nu_2\cdots\nu_k)}$. Then, continue using (3.51), replace each of $G^{(\nu_2\cdots\nu_k)}$ by the ν_2-linear cyclator of the generators $G^{(\nu_3\cdots\nu_k)}$, and so on. As a result one yields a $(p+1)$-linear multi-cyclic bracket of the generators G. It is completely determined by the (ordered) sequence $\nu=\langle\nu_1,\ldots,\nu_k\rangle$ of the orders of sub-brackets (from outer to inner), and may be labelled by the subscript ν. The subgroup of permutations that leaves the bracket $\{\ldots\}_\nu$ invariant will be denoted by H_ν. Its order, which coincides with the number of monomials in the bracket, is

$$N_\nu=\nu_1(\ldots\nu_{k-1}(\nu_k)^{\nu_{k-1}}\ldots)^{\nu_1}$$

It is curious to note that this number is a divisor of $(p+1)!$.

Brackets corresponding to different sequences ν are linearly independent. For example, for $p=5$ one can find three multi-cyclic brackets, corresponding to the sequences $\langle 6\rangle$, $\langle 2,3\rangle$ and $\langle 3,2\rangle$, which we represent symbolically as $\{123456\}$, $\{\{123\}\{456\}\}$ and $\{\{12\}\{34\}\{56\}\}$ (here and below $\{\ldots\}\equiv\{\ldots\}_c$). They are evidently independent and containing 6, 18 and 24 terms respectively. One could also introduce some not so symmetrical brackets, of type $\{\{12\}\{34\}\{5678\}\}$ or even, of type $\{12\}3+\{23\}1+\{31\}2$, but they can be reduced to a sum of several classes of multi-cyclic brackets.

Thus the algebra generated by T and G can be established in general form as

$$\begin{aligned}
[T(\omega),\ T(\eta)] &= T(\omega\eta'-\omega'\eta)\ ,\\
[T(\omega),\ G(\epsilon)] &= G(\omega\epsilon'-\frac{1}{p+1}\omega'\epsilon)\ ,\\
\{G(\epsilon_0),\ G(\epsilon_1),\ \ldots,G(\epsilon_p)\}_\nu &= N_\nu T(\epsilon_0\epsilon_1\cdots\epsilon_p)\ .
\end{aligned} \tag{3.54}$$

With introducing the component generators

$$L_n=T(z^{-n+1})\ ,\ \ G_r=G(z^{-r+1/(p+1)}) \tag{3.55}$$

it can be performed as

$$\begin{aligned}
[L_n\ ,\ L_m] &= (n-m)L_{n+m}\ ,\\
[L_n\ ,\ G_r] &= (\frac{n}{p+1}-r)G_{n+r}\ ,\\
\{G_{r_0},\ldots,G_{r_p}\}_\nu &= N_\nu L_{\sum r_j}\ .
\end{aligned} \tag{3.56}$$

A particular case of this algebra when ν denotes the brackets with all permutations of G 's have been presented in the papers [7], [12] under the name 'fractional para-Virasoro' algebra.

It must be noted that the generators of the algebra (3.54) possess a general representation depending on an arbitrary 'para-conformal weight' Δ (cf. (1.16)). In the case of Version-(2), the generalization of the formulas (3.44) looks as

$$\begin{aligned} T(\omega) &= \omega\partial_x + \frac{1}{p+1}\omega'(\theta\partial_\theta + \Delta)\,, \\ G(\epsilon) &= \epsilon(\partial_\theta + \frac{\theta^p}{p!}\partial_x) - \frac{\Delta}{p}\epsilon'\frac{\theta^p}{p!}\,. \end{aligned} \tag{3.57}$$

The algebra (3.54), or (3.56), will be called below 'the paraconformal', or Con_p, and its central extensions will be present in the next section.

Before turning to this task we have to make some comments on the H-generators. Really, if the algebra Con_p pretends to be a bare form of $\mathcal{CON}_p$, it has to handle all the generators H_j and $H_j^{(M)}$, as well as T , G and $G^{(M)}$. The problem is in the following. In case of G-generators, the structure of the identities (3.50 – 3.51) and the form of $G^{(M)}$ is practically fixed by the requirement, that the argument of $G^{(M)}$ must be the product of the arguments of the correspondent G 's. In case of H-generators, the similar requirement is almost always fulfilled automatically, and, therefore, it is not clear, which combination should be considered as the right definition of $H^{(M)}$. Then, there exist $p-1$ generators $H_j^{(M)}$ in each of p generations M, so the number of different identities grows up drastically as p increases. All this makes writing correct relations with H-generators a rather subtle problem.

To illustrate the situation, consider the simplest case $p = 2$ with one generator $H(\xi)$. The complete paraconformal algebra Con_2 may be written as

$$\begin{aligned} [T(\omega),\, T(\phi)] &= T(\omega\phi' - \omega'\phi)\,, \\ [T(\omega),\, G(\epsilon)] &= G(\omega\epsilon' - \tfrac{1}{3}\omega'\epsilon)\,, \\ [T(\omega),\, H(\xi)] &= H(\omega\xi' + \tfrac{1}{3}\omega'\xi)\,, \\ \{G(\epsilon), G(\zeta), G(\eta)\}_c &= 3\,T(\epsilon\zeta\eta)\,, \\ \{G(\epsilon), G(\zeta), H(\xi)\}_c &= G(\epsilon\zeta\xi)\,, \\ \{G(\epsilon), H(\sigma), H(\xi)\}_c &= H(\epsilon\sigma\xi)\,. \end{aligned} \tag{3.58}$$

$G^{(2)}$-generator has the usual form (3.50). $H^{(2)}$-generator can be defined by

$$G(\eta)H(\xi) + q^{1/2}H(\xi)G(\eta) = H^{(2)}(\eta\xi)\,, \tag{3.59}$$

so that

$$G(\zeta)H^{(2)}(\tau) + q^{-1/2}H^{(2)}(\tau)G(\zeta) = G(\zeta\tau)\,. \tag{3.60}$$

Here q denotes a primitive cubic root of unity, but this has no connection to the q-version. Note that there are no particular reasons for defining $H^{(2)}$ as above, except the conciseness of the formulas (3.59) and (3.60).

The cyclic brackets, similar to those in (3.58), also exist for $p > 2$ but their number increases with p very fast due to the growing number of H-generators. So the problem of a correct description of the H-sector in the algebra Con_p looks rather messy. It can hardly be solved without using a Lie-type theory of para-supergroups. For this reason we have excluded the H-generators in our treatment of the algebras Con_p. The other reason is that the H-generators are irrelevant to constructing central extension, as will be clarified in the next section.

4. Central Extension of the Con_p Algebra

A geometric meaning of the algebra Con_p becomes practically obvious after noting that the arguments ω , ϵ of the generators (3.57) can be considered not as ordinary functions but as λ-differentials of a suitable weight. The general rule is that generators representing the currents of the spin s (conformal dimension s) must have the differentials of the weight $\lambda = 1-s$ as their arguments. So, for T of the dimension 2 we have $\omega \in \mathcal{F}^{-1}$ and for G of the dimension $\frac{p+2}{p+1}$ we have $\epsilon \in \mathcal{F}^{-\frac{1}{p+1}}$, ($\mathcal{F}^\lambda$ denotes the space of the λ-differentials: $\mathcal{F}^\lambda = \{\omega(z)\ :\ \omega(z) \mapsto (\tilde{z}')^\lambda\omega(\tilde{z})\,,\ \text{when}\ z \mapsto \tilde{z}\}$). The algebra of the generators is then determined by suitable differential operators relating the differentials of different weights. For the generators T and G we may write symbolically

$$\begin{aligned} [T(\omega_1), T(\omega_2)] &= T(l(\omega_1,\omega_2)) \\ [T(\omega), G(\epsilon)] &= G(m(\omega,\epsilon)) \\ \{G(\epsilon_0),\ldots,G(\epsilon_p)\}_\nu &= T(n(\epsilon_0,\ldots,\epsilon_p))\,, \end{aligned} \tag{4.1}$$

where the operators l, m, n act as follows:

$$\begin{array}{cccc} l : \mathcal{F}^{-1} & \wedge\ \ \mathcal{F}^{-1} & \rightarrow & \mathcal{F}^{-1} \\ (\ \omega_1 & ,\ \ \omega_2\ ) & \mapsto & \omega_1\omega_2' - \omega_1'\omega_2\,, \\ & l & = & d_2 - d_1 \end{array} \tag{4.2}$$

$$\begin{array}{cccc} m : \mathcal{F}^{-1} & \wedge\ \ \mathcal{F}^{-\frac{1}{p+1}} & \rightarrow & \mathcal{F}^{-\frac{1}{p+1}} \\ (\ \omega & ,\ \ \epsilon\ ) & \mapsto & \omega\epsilon' - \frac{1}{p+1}\omega'\epsilon\,, \\ & m & = & d_2 - \frac{1}{p+1}d_1 \end{array} \tag{4.3}$$

$$\begin{array}{ccc} n : (\mathcal{F}^{-\frac{1}{p+1}}\times\ \ldots\ \times\mathcal{F}^{-\frac{1}{p+1}})_\nu & \rightarrow & \mathcal{F}^{-1} \\ (\ \epsilon_0\,,\ \ \ldots\ \ ,\epsilon_p\ ) & \mapsto & \epsilon_0\cdot\ldots\cdot\epsilon_p\,. \end{array} \tag{4.4}$$

Here the symbol d_i means differentiating the i-th multiplier; the subscript ν reminds of the symmetry of the bracket.

Let us now turn to central extensions of Con_p. Being numbers, central charges can arise from λ-differentials only as residues of some 1-forms. Thus, all we have to do to get the central extensions is to find out differential operators acting from the left-hand sides of (4.2) and (4.4) to the space $\mathcal{F}^1/d\mathcal{F}^0$. The first is the well-known and unique (modulo total derivative) Gelfand-Fuks operator of differential order three: $\phi = d_1^3 - d_2^3$. The second must be of the order two and have the symmetry group of the bracket, H_ν. Thus for the simplest, cyclic bracket we can construct $[\frac{p+1}{2}]$ operators ψ_j:

$$\psi_j = \sum_{i=1}^{p+1} d_i\,d_{i+j}\ ,\ \ j = 1,\ldots,\left[\frac{p+1}{2}\right]\,. \tag{4.5}$$

It is not much more difficult to construct the operators $\psi_j^{(\nu)}$ corresponding to the bracket $\{\ldots\}_\nu$ with the symmetry H_ν. For instance, the bracket of the type $\{\{123\}\{456\}\}$ admits two operators

$$\begin{aligned}\psi_1 &= d_1d_2+d_2d_3+d_3d_1+d_4d_5+d_5d_6+d_6d_4\,,\\ \psi_2 &= d_1d_4+d_2d_4+d_3d_4+d_1d_5+d_2d_5+d_3d_5\\ &+ d_1d_6+d_2d_6+d_3d_6\,. \end{aligned} \tag{4.6}$$

And so on. The larger is the symmetry group of the bracket, H_ν , the smaller is the number of admissible central charges E_ν . For the multi-cyclic brackets

$$E_\nu = \sum_k \left[\frac{\nu_k}{2}\right]\,.$$

The resulting extended algebra is

$$\begin{aligned}[T(\omega_1),T(\omega_2)] &= T(\omega_1\omega_2'-\omega_1'\omega_2)+C\phi(\omega_1,\omega_2)\\ [T(\omega),G(\epsilon)] &= G(\omega\epsilon'-\tfrac{1}{p+1}\omega'\epsilon)\\ \{G(\epsilon_0),\ldots,G(\epsilon_p)\}_\nu &= N_\nu T(\epsilon_0\cdot\ldots\cdot\epsilon_p)+\textstyle\sum_{j=1}^{B_\nu} c_j\psi_j^{(\nu)}(\epsilon_0,\ldots,\epsilon_p)\,,\end{aligned} \tag{4.7}$$

where N_ν is the number of the terms in the bracket. The central charge C can be expressed in terms of c_j by commuting the third line of Eq. (4.7) with some $T(\eta)$ and then comparing both sides of the resulting identity. Let us apply this procedure to the cyclic bracket. Writing the generators in components we get the algebra announced in the Introduction as Vir_p :

$$\begin{aligned}[L_n\,,\,L_m] &= (n-m)L_{n+m}+\frac{2}{p+1}\left(\sum_j c_j\right)(n^3-n)\delta_{n+m,0}\,,\\ [L_n\,,\,G_r] &= (\frac{n}{p+1}-r)G_{n+r}\,,\\ \{G_{r_0},\ldots,G_{r_p}\}_c &= (p+1)L_{\Sigma r}-\sum_j c_j\left(\sum_i r_ir_{i+j}+\frac{1}{p+1}\right)\delta_{\Sigma r,0}\,,\\ j &= 1\ldots\left[\frac{p+1}{2}\right]\,.\end{aligned} \tag{4.8}$$

Note that the symmetry of the extension operators may be taken wider than that of the bracket. That would be equivalent to constraining some of the charges c_j. For example, there exists a unique totally symmetric operator $\Psi=\sum_{i<j} d_id_j$ that can be used with all kinds of brackets. Unfortunately, this simple extension seems to be unsuitable for constructing a non-trivial analogue of a Verma module.

5. Discussion and Conclusion

Let us summarize the results and problems beginning with the results.

1. The first one concerns the paragrassmann algebra Π_{p+1} generated by the nilpotent variables θ and ∂. We have shown that the requirements:

a) all ∂'s can be moved to the right of all θ's with preserving the natural grading;

b) ∂ can be interpreted as a non-degenerate derivative with respect to θ ;

uniquely determine Π_{p+1} as the algebra isomorphic to $Mat(p+1)$ with the 'along-diagonal' grading.

Different admissible forms of the commutation relations represent different versions of the same algebra, and are connected by certain non-linear transformation.

2. Transformations of the para-superplane preserving the form of the fractional derivative $\mathcal{D}$ obey the transitivity condition and form a group $\mathbf{CON}_p$ that is called the paraconformal group.

3. The corresponding infinitesimal object, a 'true' paraconformal algebra $\mathcal{CON}_p$, is related to the space of p-jets rather than to the tangent space. $\mathcal{CON}_p$ is a p-filtered algebra with generators in p generations. Generators of M-th generation do not occur in the order of smallness less than M.

The generators of the first generation are: the usual conformal generator T with the conformal weight 2, the paraconformal generator G with the weight $\frac{p+2}{p+1}$, and the paragrassmann generators $\mathcal{H}_j$, $(j=1\ldots p-1)$ with the weights $\frac{p+1-j}{p+1}$. The generators of the M-th generation are: $\mathcal{G}^{(M)}$ with the weight $1+\frac{M}{p+1}$ and $\mathcal{H}_j^{(M)}$ with the weight $\frac{p+M-j}{p+1}$. The $\mathcal{H}$-type generators do not affect z-coordinate of the para-superplane (z,θ) but they are required by self-consistency of the algebra.

4. Algebra $\mathcal{CON}_p$ can be considered as a paragrassmann shell of a 'bare' ('skeleton') algebra Con_p also called paraconformal. Its generators T, $G^{(M)}$, $H_j^{(M)}$ have as their arguments ordinary functions (in fact, λ-differentials). The connection between $\mathcal{CON}_p$ and Con_p is trivial in the supercase ($p=1$) but it is not so clear for $p>1$. We have systematically derived identities in the algebra Con_p that must encode some information about the structure of the algebra $\mathcal{CON}_p$ but understanding of exact relations between these two algebras is still lacking [5] .

5. The algebra Con_p can be closed in terms of T- and G-generators only (unlike $\mathcal{CON}_p$). There exist many multilinear identities with G-generators based on the cyclic brackets of arbitrary order. For composite $p+1$, they give rise to a set of non-equivalent $(p+1)$-linear brackets of G closing to T. This, by the way, makes evidence that the algebra Con_p contains as a subalgebra Con_r for each $(r+1)$ being a divisor of $(p+1)$.

6. To each of these brackets there corresponds a set of basic central extension operators having the same symmetry as the bracket. The wider is the symmetry,

[5] The matter may be illustrated by the following example: *two* identities in the algebra Con_p , $\{G(\epsilon),G(\zeta)\}=2G^{(2)}(\epsilon\zeta)$ and $[G(\epsilon),G(\zeta)]=H_{p-1}(\epsilon\zeta'-\epsilon'\zeta)$, are the reflections of *one* relation (3.48) in the algebra $\mathcal{CON}_p$.

the smaller is the number of extensions. Constraining the coefficients (the central charges), one can enlarge the symmetry of the central term as compared to the bracket.

Here emerges a branching point for future development.

One way is to consider different brackets (and the identities of smaller order as well) just as the identities in the same algebra Con_p, and then to deal with the unique central extension that suits all of them. This corresponds to the extension generated by the totally symmetric operator Ψ.

The other way is to forget all preliminaries about the infinitesimal generators and paragrassmann algebras and to consider the algebras of T and G with different brackets as independent infinite-dimensional algebras, each having its own central extension. This approach might appear fruitful for simple brackets, like the cyclic ones. A right way is probably somewhere in between and hence the problem of a lucky choice of the symmetry breaking arises.

We think that 'the right way' is that leads to a nontrivial Verma module. In fact, a natural program to develop the theory is to define a suitable analogue of a Verma module over the algebra Vir_p and to search for degenerate modules, Kac determinant and rational models. Unfortunately, even first steps appear to be nontrivial. Let us illustrate the problem by a simple example.

Consider the algebra Vir_2 with the third line being taken with a totally symmetric bracket:

$$\{G_r, G_s, G_t\}_{sym} = 6L_{r+s+t} + C(r^2 + s^2 + t^2)\delta_{r+s+t,0} \,.$$

Assume that the constraint $F_{r+s} = \{G_r, G_s\}$ of the algebra Con_2 is preserved. Then for any acceptable positive k we can write two strings

$$F_{\frac{2k}{3}} G_{-\frac{2k}{3}} + 2F_{-\frac{k}{3}} G_{\frac{k}{3}} = 6L_0 + \frac{2}{3}C(k^2 - 1) \,,$$

$$2F_{\frac{2k}{3}} G_{-\frac{2k}{3}} + F_{-\frac{4k}{3}} G_{\frac{4k}{3}} = 6L_0 + \frac{2}{3}C(4k^2 - 1) \,.$$

Now defining a vacuum so that

$$G_{>0} \approx 0 \;, \quad L_0 \approx \Delta$$

(we write $X \approx Y$ for $X|vac) = Y|vac)$) we immediately get a contradiction,

$$F_{\frac{2k}{3}} G_{-\frac{2k}{3}} \approx 6\Delta + \frac{2}{3}C(k^2 - 1) \approx \frac{1}{2}(6\Delta + \frac{2}{3}C(4k^2 - 1)) \,,$$

unless both Δ and C are zero. This is an evidence of a rather general phenomenon, namely: preserving constraints while keeping to a naive definition of the vacuum is, as a rule, inconsistent with a nontrivial central charge (and often with a nontrivial highest weight, as in the example). To bypass this disaster one might either try to select an appropriate set of the constraints to be preserved or to redefine the vacuum

in a more skilful way. Our attempts in this direction have not produced anything valuable so far.

There might be another variant of salvation, based on extending all order constraints in a consistent way, but this would require introducing fractional order differential operators in central terms which would increase dramatically the range of possibilities.

One might also suspect the H-generators might play a role in defining the vacuum and the module. However, the above example gives little support to this suspicion.

Some light on the topic might be thrown by investigating concrete physical systems possessing paraconformal symmetry. But the algebraic results of the present paper are hinting that quantizing such systems will be rather ambiguous.

Thus, the current problems may be summarized in the following list:

1. The main theoretical problem is to find a rigorous connection between the three constructed paraconformal objects: CON_p, $\mathcal{CON}_p$ and Con_p. This problem requires further developing the calculus for many paragrassmann variables.

2. It is not clear how to correctly include the H-generators into the algebras Con_p and Vir_p.

3. The main practical question is to find non-trivial Verma modules over Vir_p .

Ending, we would like to note that paraconformal algebras are not of a pure aesthetic interest. For example, on a Riemann surface of genus g the natural scale of conformal dimension is $\frac{1}{2(g-1)}$, as a consequence of the Gauss-Bonnet theorem, and thus the natural fractional derivative is of the same order, and the natural conformal algebra would be Vir_{2g-3} rather than $Vir_1 = RNS$. One may also speculate that using paraconformal algebras might drastically change the critical dimensions in the string theory.

6. Appendix

Here we describe parafermions and parabosons [6] in the framework of our approach to the paragrassmann algebras.

1. Parafermionic Version

Parafermionic generators θ and ∂ ($\theta^{p+1} = \partial^{p+1} = 0$) satisfy the commutation relations [6]

$$[[\partial,\, \theta]\theta] = -\rho\theta \;, \quad [[\partial,\, \theta]\partial] = \rho\partial \,. \tag{6.1}$$

It is hard to extract the basis for the algebra with these generators, because we can not move all ∂'s to the right-hand side of any monomial $\ldots\partial^i\theta^j\partial^k\theta^l\ldots$. Thus, our aim is to find a structure relation (2.7) which is in agreement with (6.1). To do this, we apply the relations (6.1) to the vector $|k) = (\theta^k)$. Then taking into account Eqs.(2.8) we derive the following recurrent equations

$$\begin{array}{c} \rho = 2\alpha_n - \alpha_{n+1} - \alpha_{n-1} \quad , \quad n = 1, \ldots, p \,, \\ \alpha_0 \;=\; \alpha_{p+1} \;=\; 0 \,. \end{array} \tag{6.2}$$

The solution is

$$\alpha_k = k\alpha_1 - \frac{k(k-1)}{2}\rho\,, \quad \rho = \frac{2\alpha_1}{p}\,. \tag{6.3}$$

Choosing the normalization condition $\alpha_1 = 1$ we have

$$\rho = 2/p\,, \quad \alpha_k = k(p+1-k)/p\,. \tag{6.4}$$

From Eqs.(2.10) one can find the parameters b_i specifying the commutation relation for ∂ and θ (2.7). For first few b_i we obtain

$$b_0 = 1\,,\ b_1 = \frac{p-2}{p}\,,\ b_2 = -\frac{2}{p(p-1)}\,,\ b_3 = -\frac{4}{p(p-1)(p-2)}\,,\ \ldots\,. \tag{6.5}$$

2. **Parabosonic Version**

This Version is specified by the commutation relations [6]

$$[\{\partial,\,\theta\}\theta] \equiv \partial\theta^2 - \theta^2\partial = -\rho\theta\,, \quad [\{\partial,\,\theta\}\partial] \equiv \theta\partial^2 - \partial^2\theta = \rho\partial\,. \tag{6.6}$$

Now the recurrence equations are

$$\begin{aligned} \alpha_{k+1} &= \alpha_{k-1} - \rho\,, \quad k = 1,\ldots,p\,, \\ \alpha_0 &= \alpha_{p+1} = 0\,. \end{aligned} \tag{6.7}$$

A solution of these equations (for $\rho \neq 0$) exists for even p only. With $\alpha_1 = 1$ we obtain

$$\begin{aligned} \rho &= 2/p\,, \\ \alpha_k &= \begin{cases} -k/p & \text{for even } k\,, \\ (p+1-k)/p & \text{for odd } k\,. \end{cases} \end{aligned} \tag{6.8}$$

As above, b_i are derived by Eq.(2.7)

$$b_0 = 1\,,\ b_1 = -\frac{p+2}{2}\,,\ b_2 = 2\frac{p+1}{p}\,,\ b_3 = 4\frac{p+1}{p(p-2)}\,,\ \ldots\,. \tag{6.9}$$

Thus, we see that parafermionic and parabosonic algebras can be defined by the relations (2.3) and (2.7) with an appropriate choice of the parameters b_i.

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Параграссмановы расширения алгебры Вирасоро

Дается дальнейшее развитие параграссманова дифференциального исчисления. Построены алгебры преобразований парасуперплоскости, сохраняющие форму парасуперпроизводной, и обсужден их геометрический смысл. Новая черта этих алгебр состоит в том, что они включают в себя генераторы автоморфизмов параграссмановой алгебры (наряду с конформными генераторами типа Рамона - Невё - Шварца). В качестве первого шага в исследовании этих алгебр мы вводим более трактуемые мультилинейные алгебры, не содержащие генераторов нового типа. В таких алгебрах существует множество мультилинейных тождеств, основывающихся на циклических поликоммутаторах. Вследствие этого появляются различные возможности для замыкания. Построены центральные расширения этих алгебр. Их число варьируется от 1 до [(p+1)/2] в зависимости от выбранной формы замыкания.

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Paragrassmann Extensions of the Virasoro Algebra

The paragrassmann differential calculus is further developed. Algebras of the transformations of the para-superplane preserving the form of the para-superderivative are constructed and their geometric meaning is discussed. A new feature of these algebras is that they contain generators of the automorphisms of the haragrassmann algebra (in addition to Ramond - Neveu - Schwarz-like conformal generators). As a first step in analyzing these algebras we introduce more tractable multilinear algebras not including the new generators. In these algebras there exists a set of multilinear identities based on the cyclic polycommutators. Different possibilities of the closure are therefore admissible. The central extensions of the algebras are given. Their number varies from 1 to [(p+1)/2] depending on the form of the closure chosen.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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