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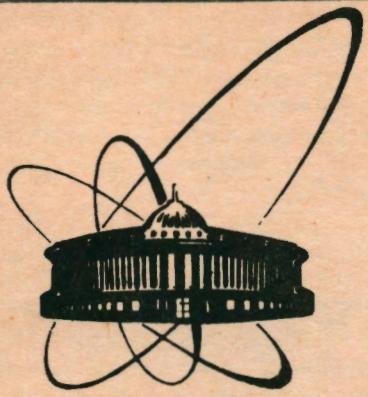

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ON PARAGRASSMANN DIFFERENTIAL CALCULUS

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Dedication

This paper is in memory of Mikhail Konstantinovich Polivanov. One of the authors (A.T.F.) had a privilege to be a friend of him for many and many years. He was not only a distinguished scientist but a true Russian intellectual having deep roots in Russian culture. It is a great sorrow that we can no more have a talk with him on science, poetry, religion...

1. Introduction

Paragrassmann algebras (PGA) are interesting for several reasons. First, they are relevant to conformal field theories [2]. Second, studies of anyons and of topological field theories show the necessity of unusual statistics. These include not only the well-known parastatistics but fractional statistics as well [3]. There were also known some hints that PGA have a connection to quantum groups [4] and this connection has been demonstrated in detail in Ref. [1]. Finally, it looks aesthetically appealing to find a generalization of the Grassmann analysis [5] that proved to be so successful in describing supersymmetry.

Recently, some applications of PGA have been discussed in literature. In Ref. [6] that inspired many other investigations, a parasupersymmetric generalization of quantum mechanics had been proposed. Ref. [7] has attempted at a more systematic consideration of the algebraic aspects of PGA based on the Green ansatz [8] and introduced, in that frame, a sort of paragrassmann generalization of the conformal algebra. Applications to the relativistic theory of the first-quantized spinning particles have been discussed in [9]. Further references can be found in Refs. [1], [10].

The aim of this paper is to construct a consistent generalization of the Grassmann algebra (GA) to a paragrassmann one preserving, as much as possible, those features of GA that were useful in physics applications. A crucial point of our approach is defining generalized derivatives in the paragrassmann variables satisfying natural restriction allowing to construct a differential calculus. As in the previous paper [1], here we mainly concentrate on the algebraic aspects leaving the applications to future publications.

Section 2 treats algebras generated by one paragrassmann variable θ, $\theta^{p+1} = 0$ and a differentiation operator ∂. This generalized differentiation coincides with the Grassmann one for $p = 1$ and with the standard differentiation when $p \to \infty$. We construct a most general realization of these algebras and identify a set of nondegenerate ones which proved to be equivalent. The different (but equivalent) realizations are presented in Section 3.

In Section 4 the simplest PGA generated by many variables θ_i and corresponding differentiations ∂_i are defined. They obey the nilpotency condition $\theta^{p+1} = 0$ ($\partial^{p+1} = 0$), where θ (∂) is any linear combination of θ_i (∂_i), and appear to be naturally related to the non-commutative spaces satisfying the commutation relations $\theta_i\theta_j =$

$q_{ij}\theta_j\theta_i$, $i < j$ (and similar relations for $\partial_i\partial_j$), where $q_{ij}^{p+1} = 1$. These relations once more demonstrate a deep connection between PGA and quantum groups with deformation parameters q being roots of unity.

Section 5 summarizes the results and presents one more relation of our algebras to quantum groups.

2. Differential Calculus with One Variable

In Ref. [1] we have considered paragrassmann algebras $\Gamma_{p+1}(N)$ with N nilpotent variables θ_n, $\theta_n^{p+1} = 0$, $n = 1, \ldots, N$. Some wider algebras $\Pi_{p+1}(N)$ generated by θ_n and additional nilpotent generators ∂_n have also been constructed. These additional generators served for defining a paragrassmann differentiation and paragrassmann calculus. The building block for this construction was the simplest algebra $\Pi_{p+1}(1)$. By applying a generalized Leibniz rule for differentiations in the paragrassmann algebra $\Gamma_{p+1}(N)$ we have found two distinct realizations for $\Pi_{p+1}(1)$ closely related to the q-deformed oscillators. We have mentioned in [1] that other realizations of the $\Pi_{p+1}(1)$ may be constructed. The aim of this section is to demonstrate this in detail. We shall also show that, under certain conditions, all these realizations are equivalent and one may choose those which are most convenient for particular problems.

Intuitively, paragrassmann algebra Π_{p+1} should be understood as some good p-generalization of the classical fermionic algebra Π_1

$$\theta^2 = 0 \;=\; \partial^2 \;, \tag{1}$$

$$\partial\theta + \theta\partial \;=\; 1 \;. \tag{2}$$

By 'p-generalization' we mean that (1) is to be replaced by

$$\theta^{p+1} = 0 = \partial^{p+1} \;, \tag{3}$$

(it is implied, of course, that $\theta^p \neq 0$ and the same for ∂). So the question is, which generalization of (2) might be called 'good'. Many variants have been tried already (see for example [8]). As a rule, they deal with certain symmetric multilinear combinations, like $\theta^2\partial + \theta\partial\theta + \partial\theta^2$ (for $p = 2$), and meet with difficulties when commuting θ and ∂.

To find a correct generalization recall that (2) allows to define the Grassmann differential calculus. It shows how to push the differentiation operator ∂ to the right of the variable θ. On the other hand, representing ∂ and θ by 2×2 real matrices, we can make them Hermitian conjugate and thus interpret as annihilation and creation operators. Then Eq. (2) is the normal-ordering rule. The second important feature of this relation is that it preserves the Grassmann grading, -1 for ∂ and $+1$ for θ. In physics terminology this means that the normal-ordering is not changing the number of 'particles'.

Thus, to construct a generalization of the relation (2), we first define a natural grading in the associative algebra generated by θ and ∂ obeying Eq. (3)

$$deg\ (\theta^{r_1}\partial^{s_1}\theta^{r_2}\partial^{s_2}\ldots\theta^{r_k}\partial^{s_k}) = \Sigma r_i - \Sigma s_i \;, \tag{4}$$

and denote by $\Pi_{p+1}(l)$ the linear shell of monomials of the degree l. Then our basic requirement is

$$a\ set\ L^{(l)} = \{\theta^r\partial^s \;,\; r - s = l\}\ \ forms\ a\ basis\ of\ \Pi_{p+1}(l) \;. \tag{5}$$

This immediately reduces the range of possible degrees to $-p \leq l \leq p$ and makes all the subspaces $\Pi_{p+1}(l)$ and the entire algebra

$$\Pi_{p+1} = \oplus_{l=-p}^{p}\Pi_{p+1}(l) \tag{6}$$

finite-dimensional:

$$\pi^l = dim(\Pi_{p+1}(l)) = p + 1 - |l| \;\;,\;\; dim(\Pi_{p+1}) = (p+1)^2 \;.$$

Then, by applying the assumptions (4) and (5) to $\partial\theta$ we find that

$$\partial\theta = b_0 + b_1\theta\partial + b_2\theta^2\partial^2 + \ldots + b_p\theta^p\partial^p, \tag{7}$$

where b_i are complex numbers restricted by certain consistency conditions to be formulated below. With the aid of Eq. (7) any element of the algebra can be expressed in terms of the basis $\theta^r\partial^s$, i.e. in the normal-ordered form.

A useful alternative set of parameters, α_k, may be defined by

$$\partial\theta^k = \alpha_k\theta^{k-1} + (\ldots)\partial \;, \tag{8}$$

where dots denote a polynomial in θ and ∂. This relation is a generalization of the commutation relation for the standard derivative operator, $\partial_z z^k = kz^k + z^k\partial_z$, and we may define the differentiation of powers of θ by analogy,

$$\partial(\theta^k) = \alpha_k\theta^{k-1},\ \alpha_0 \equiv 0 \;, \tag{9}$$

to be justified later.

By applying Eq. (7) to Eq. (8) one may derive recurrent relations connecting these two sets of the parameters:

$$\begin{aligned} \alpha_1 &= b_0 \;, \\ \alpha_2 &= b_0 + b_1\alpha_1 \;, \\ \alpha_3 &= b_0 + b_1\alpha_2 + b_2\alpha_1\alpha_2 \;, \\ &\cdots\cdots \;, \\ \alpha_{k+1} &= \sum_{i=0}^{k} b_i\frac{(\alpha_k)!}{(\alpha_{k-i})!} \;, \\ &\cdots\cdots \;, \end{aligned} \tag{10}$$

where $(\alpha_k)! = \alpha_1\alpha_2\cdots\alpha_k$. These relations enable us to express α_k as a function of the numbers b_i , $0 \le i \le k-1$. The first few expressions are

$$\begin{aligned} \alpha_1 &= b_0\,,\ \alpha_2 = b_0\frac{1-b_1^2}{1-b_1}\,,\ \alpha_3 = b_0\frac{1-b_1^3}{1-b_1} + b_2b_0^2(1+b_1)\,, \\ \alpha_4 &= b_0\frac{1-b_1^4}{1-b_1} + b_2b_1b_0^2(1+b_1) + b_0(b_3+b_2b_0)(1+b_1)\alpha_3\,, \ \cdots \end{aligned} \tag{11}$$

The inverse operation, deriving b_i in terms of α_k, is well-defined only if all $\alpha_k \neq 0$.

The consistency condition mentioned above is that the parameters must be chosen so as to satisfy the identity

$$0 \equiv \partial\theta^{p+1}\,.$$

Taking into account that the second term in Eq. (8) vanishes for $k = p+1$ we have $\alpha_{p+1} = 0$, with no other restrictions on the parameters α_k with $k \le p$. The corresponding restriction on $p+1$ parameters b_i follows from Eq. (10),

$$\alpha_{p+1}(b_0,\ldots,b_p) \equiv b_0 + b_1\alpha_p + b_2\alpha_p\alpha_{p-1} + \ldots + b_p\alpha_p\alpha_{p-1}\cdots\alpha_2\alpha_1 = 0\,, \tag{12}$$

where the parameters α_i are expressed in terms of b_i. Any admissible set $\{b\}$ determines an algebra $\Pi_{p+1}^{\{b\}}$ with the defining relations (3), (7). To each algebra $\Pi_{p+1}^{\{b\}}$ there corresponds a set $\{\alpha\}$. *A priori*, there are no restrictions on $\{\alpha\}$ but, if we wish to treat ∂ as a non-degenerate derivative with respect to θ, it is reasonable to require, in addition to (5), that

$$all\ \ \alpha_k \neq 0\,. \tag{13}$$

So let us call a set $\{b\}$ (and corresponding algebra$\Pi_{p+1}^{\{b\}}$) *non-degenerate*, if the condition (13) is fulfilled, and degenerate otherwise. As it was already mentioned, in the non-degenerate case the numbers b_i are completely determined by the numbers α_k, so we can use the symbol $\{\alpha\}$ as well as $\{b\}$.

At first sight, the algebras corresponding to different sets $\{b\}$ look very dissimilar. In general, different sets $\{b\}$ determine non-equivalent algebras $\Pi_{p+1}^{\{b\}}$. However, this is not true for the non-degenerate ones. In fact, all non-degenerate algebras $\Pi_{p+1}^{\{b\}}$ are isomorphic to the associative algebra $Mat(p+1)$ of the complex $(p+1)\times(p+1)$ matrices.

This isomorphism can be manifested by constructing an explicit exact ('fundamental') representation for $\Pi_{p+1}^{\{b\}}$. With this aim, we treat θ and ∂ as creation and annihilation operators (in general, not Hermitian conjugate) and introduce the ladder of $p+1$ states $|k\rangle$, $k = 0,1,\ldots,p$ defined by

$$\partial|0\rangle = 0\,,\ |k\rangle \sim \theta^k|0\rangle\,, \theta|k\rangle = \beta_{k+1}|k+1\rangle\,, \tag{14}$$

where β's are some non-zero numbers, reflecting the freedom of the basis choice. As $|p+1\rangle = 0$, the linear shell of the vectors $|k\rangle$ is finite-dimensional and in the nondegenerate case, when all $\beta_k \neq 0$ $(k = 1,\ldots,p)$, its dimension is $p+1$.

Using (14) and (8) we find

$$\partial|k\rangle = (\alpha_k/\beta_k)|k-1\rangle\ . \tag{15}$$

Thus the fundamental (Fock-space) representations of the operators θ and ∂ are

$$\theta_{mn} = \langle m|\theta|n\rangle = \beta_{n+1}\delta_{m,n+1}\ , \tag{16}$$

$$\partial_{mn} = \langle m|\partial|n\rangle = (\alpha_n/\beta_n)\delta_{m,n-1}\ . \tag{17}$$

It is not hard to see that, for non-zero α's, the matrices corresponding to $\theta^m\partial^n$ $(m,n = 0\ldots p)$ form a complete basis of the algebra $Mat(p+1)$. The isomorphism is established.

Nothing similar occurs for degenerate algebras. To show an evidence against using them in the paragrassmann calculus, consider an extremely degenerate algebra with $b_0 = b_2 = \ldots = b_p = 0$, $b_1 \neq 0$, so that all $\alpha_k = 0$. This algebra has nothing to do with $Mat(p+1)$, and its properties essentially depend on the value of b_1. It is abelian if $b_1 = 1$; it is a paragrassmann algebra of the type $\Gamma_{p+1}(2)$ if b_1 is a primitive root of unity (see [1]), and so on. We hope this remark is not sounding like a death sentence on the degenerate algebras. At least, it has to be suspended until further investigation which will probably prove their usefulness in other contexts. However, if we wish to have paragrassmann calculus similar to the Grassmann one, we have to use the nondegenerate algebras.

Thus, two natural requirements (5) and (13) reduce the range of possible generalizations of the fermionic algebra Π_1 to the unique algebra Π_{p+1} that is isomorphic to $Mat(p+1)$ [1]. The grading (4) in Π_{p+1} corresponds to 'along-diagonal' grading in $Mat(p+1)$. Different non-degenerate algebras $\Pi_{p+1}^{\{b\}}$ are nothing more than alternative ways of writing one and the same algebra Π_{p+1}. We will call them *versions* having in mind that fixing the b-parameters is analogous to a gauge-fixing (in H. Weyl's usage).

This implies that we will mainly be interested in 'version-covariant' results, i.e. independent on a version choice. Nevertheless, special versions may have certain nice individual features making them more convenient for concrete calculations (thus allowing for simpler derivations of covariant results by non-covariant methods). Several useful versions will be described below. Before turning to this task we end our general discussion with several remarks.

First. The existence of the exact matrix representation (16), (17) is very useful for deriving version-covariant identities in the algebra Π_{p+1}. For instance, it is easy to check that

$$\{\partial\,,\ \theta^{(p)}\} = (\Sigma\alpha_k)\ \theta^{p-1}\,, \tag{18}$$

$$\{\partial^p\,,\ \theta^{(p)}\} = \Pi\alpha_k\,, \tag{19}$$

[1] H. Weyl in his famous book on quantum mechanics had foreseen relevance of these algebras to physics problems. After detailed description of the spin algebras he discussed more general finite algebras and remarked that the finite algebras like those discussed here will possibly appear in future physics. We think it natural to call Π_{p+1} the 'finite Weyl algebra' or 'para-Weyl algebra'.

and to find many other relations. Here we have introduced a useful notation

$$\{\Xi\,,\ \Psi^{(l)}\} = \Xi\Psi^l + \Psi\Xi\Psi^{l-1} + \ldots + \Psi^l\Xi\,. \tag{20}$$

The identities (18) generalize those known in the parasupersymmetric quantum mechanics [6].

Second. One may adjust the parameters β_k to get a convenient matrix representation for θ and ∂. As a rule, we take $\beta_k = 1$. Note that for the versions with real parameters α_k, it is possible to choose β_k so as to have $\theta^\dagger = \partial$. We also normalize θ and ∂ so that $\alpha_1 \equiv b_0 = 1$. This gives a more close correspondence with the Grassmann relation (2).

Third. In a given (non-degenerate) version $\Pi^{\{b\}}_{p+1}$ the components of the vector $R^{(l)}_{\{b\}} = col\{\partial^j\theta^i\}_{i-j=l}$ form a basis of the subspace $\Pi_{p+1}(l)$ that is completely equivalent to the original one (5) having the components $L^{(l)}_{\{b\}} = col\{\theta^i\partial^j\}_{i-j=l}$. Hence, there must exist a non-degenerate matrix $C^{(l)}_{\{b\}} \in Mat(\pi^{(l)}, \mathbf{C})$ connecting these two bases,

$$R^{(l)}_{\{b\}} = C^{(l)}_{\{b\}} \cdot L^{(l)}_{\{b\}}\,,\quad l = -p, \ldots, p\,. \tag{21}$$

The elements of the C-matrix are certain functions of b_i which are usually not easy to calculate except simple versions. The original commutation relation (7) is also included in the system (21), for $l = 0$.

Quite similarly, two L-bases (R-bases) taken in different versions $\{b\}$ and $\{b'\}$ are connected by a non-degenerate matrix $M_{\{bb'\}}$ ($N_{\{bb'\}}$), i.e.

$$L_{\{b\}} = M_{\{bb'\}}L_{\{b'\}}\,, \tag{22}$$

$$R_{\{b\}} = N_{\{bb'\}}R_{\{b'\}}\,, \tag{23}$$

where the indices (l) are suppressed. The matrices $M^{(l)}_{\{bb'\}}$ (and $N^{(l)}_{\{bb'\}}$) belong to $Mat(\pi^{(l)})$ and obey cocycle relations:

$$M_{\{bb'\}}M_{\{b'b\}} = 1\,,\quad M_{\{bb'\}}M_{\{b'b''\}}M_{\{b''b\}} = 1\,.$$

By applying Eq. (21) we immediately get the relation

$$N_{\{bb'\}} = C_{\{b\}}M_{\{bb'\}}C^{-1}_{\{b'\}}\,, \tag{24}$$

that permits evaluating C-matrices for complicated versions once we know them in one version. In particular, Eq. (22) tells that the operator ∂ in any version can be represented as a linear combination of the operators ∂ , $\theta\partial^2$, ..., $\theta^{p-1}\partial^p$ of any other version. We shall see soon that this, for instance, permits to realize q-oscillators in terms of generators θ and ∂ of other versions and *vice versa*.

3. Versions of the Paragrassmann Calculus

Now consider some special versions related to the simplest forms of Eq. (7).

(0): **Primitive Version**

Here $b_1 = \ldots = b_{p-1} = 0$, $b_p = -1$, so that $\alpha_i = 1$,

$$(\partial_{(0)})_{mn} = \delta_{m,n-1}\,,\quad \partial_{(0)}\theta = 1 - \theta^p\partial^p_{(0)}\,. \tag{25}$$

This realization of ∂ may be called 'almost-inverse' to θ. In the matrix representation (16), (17) with $\beta_k = 1$ we have $\theta^T = \partial_{(0)}$. This version is the simplest possible but the differential calculus is a fancy-looking thing in this disguise and it is unsuitable for many applications. Still, it has been used in some applications. For example, the operators θ and $\partial_{(0)}$ for $p = 2$ coincide with parafermions in the formulation of the parasupersymmetric quantum mechanics [6].

(1): **q-Version or Fractional Version**

Here $b_1 = q \neq 0$, $b_2 = b_3 = \ldots = b_p = 0$, so that

$$\alpha_i = 1 + q + \ldots + q^{i-1} = \frac{1-q^i}{1-q}\,.$$

The condition $\alpha_{p+1} = 0$ tells that $q^{p+1} = 1$, $(q \neq 1)$ while the assumption that all $\alpha_i \neq 0$ forces $q = b_1$ to be a *primitive* root, i.e. $q^{n+1} \neq 1$, $n < p$. Thus, in this version $(\partial = \partial_{(1)})$,

$$\partial_{(1)}\theta = 1 + q\theta\partial_{(1)}\,, \tag{26}$$

$$\partial_{(1)}(\theta^n) = (n)_q\theta^{n-1}\,,\quad (n)_q = \frac{1-q^n}{1-q}\,.$$

These relations were introduced in Ref. [1] by assuming that ∂ is a generalized differentiation operator, i.e. satisfying a generalized Leibniz rule (a further generalization is introduced below). The derivative $\partial_{(1)}$ is naturally related to the q-oscillators (q-derivative) and to quantum algebras (see [1] and references therein). Eq. (26) is also extremely convenient for generalizing to the Paragrassmann algebras with many θ's and ∂'s.

(2): **Almost Bosonic Version**

For this Version

$$b_1 = 1\,,\quad b_2 = \ldots = b_{p-1} = 0\,,\quad b_p \neq 0\,,\quad \text{so that}\quad \alpha_k = k$$

and $\alpha_{p+1} = 0$ gives $b_p = -\frac{p+1}{p!}$. Thus

$$(\partial_{(2)})_{mn} = n\,\delta_{m,n-1}\,,\quad \partial_{(2)}\theta = 1 + \theta\partial_{(2)} - \frac{p+1}{p!}\,\theta^p\partial^p_{(2)}\,, \tag{27}$$

suggesting that this derivative is 'almost bosonic' as $\partial_{(2)}(\theta^n) = n\theta^{n-1}$ $(n \neq p+1)$.

Let us now discuss the interrelations between θ and ∂. As we have already mentioned, the notation itself hints at treating ∂ as a derivative with respect to θ

(see (8)). To be more precise, let us represent vectors of the 'Fock space' (14) as polynomials in θ

$$|F\rangle = \sum_{k=0}^{p} f_k|k\rangle \quad \Leftrightarrow F(\theta) = \sum_{k=0}^{p} f_k\theta^k.$$

The action of the derivative ∂ on this function is defined by (14) and (15) ($\beta_k = 1$),

$$\partial(1) = 0\,, \quad \partial(\theta^n) = \alpha_n\theta^{n-1} \quad (1 \le n \le p)\,. \tag{28}$$

It is however clear that this derivative does not obey the standard Leibniz rule $\partial(ab) = \partial(a)b + a\partial(b)$.

So consider the following modification of the Leibniz rule [1], [11]

$$\partial(FG) = \partial(F)\bar{\mathrm{g}}(G) + \mathrm{g}(F)\partial(G)\,. \tag{29}$$

The associativity condition (for differentiating FGH) tells that g and $\bar{\mathrm{g}}$ are homomorphisms, i.e.

$$\mathrm{g}(FG) = \mathrm{g}(F)\mathrm{g}(G)\,, \quad \bar{\mathrm{g}}(FG) = \bar{\mathrm{g}}(F)\bar{\mathrm{g}}(G)\,. \tag{30}$$

The simplest natural homomorphisms compatible with the relations (28), (29), and (30) are linear automorphisms of the algebra Γ_{p+1},

$$\mathrm{g}(\theta) = \gamma\theta\,, \quad \bar{\mathrm{g}}(\theta) = \bar{\gamma}\theta\,, \tag{31}$$

where γ, $\bar{\gamma}$ are arbitrary complex parameters and

$$\alpha_k = \frac{\bar{\gamma}^k - \gamma^k}{\bar{\gamma} - \gamma}. \tag{32}$$

Now the condition (12) yields the equation

$$\alpha_{p+1} = \frac{\bar{\gamma}^{p+1} - \gamma^{p+1}}{\bar{\gamma} - \gamma} = 0, \tag{33}$$

and assuming nondegeneracy requirements $\alpha_n \neq 0$ ($n < p+1$) we conclude that $\bar{\gamma}/\gamma$ must be a primitive $(p+1)$-root of unity. Thus we may formulate another interesting version of the paragrassmann algebra Π_{p+1}

(3):$\mathrm{g} - \bar{\mathrm{g}}$–Version

As the parameters α_k are given by Eq. (32), we have to calculate b_i by solving Eqs.(10):

$$b_0 = 1,\ b_1 = \bar{\gamma} + \gamma - 1,\ b_2 = (\bar{\gamma} - \bar{\gamma}\gamma + \gamma - 1)/(\bar{\gamma} + \gamma),\ \ldots$$

Here γ and $\bar{\gamma}$ are complex numbers constrained by the condition that $q = \bar{\gamma}/\gamma$ is a primitive root of unity

$$\left(\frac{\bar{\gamma}}{\gamma}\right)^{p+1} = 1.$$

From Eqs. (29) and (31) one can derive the following operator relations for the automorphisms g, $\bar{\mathrm{g}}$

$$\partial\theta - \gamma\theta\partial = \bar{\mathrm{g}}\,, \quad \partial\theta - \bar{\gamma}\theta\partial = \mathrm{g}. \tag{34}$$

For the special case $\gamma = (\bar{\gamma})^{-1} = q^{1/2}$ redefining $\partial = a$, $\theta = a^\dagger$ allows to recognize in (34) the definitions of the q-deformed oscillators in the Biedenharn-MacFarlane form [12]. Note that Version-(1) can be derived from Version-(3) by putting $\bar{\gamma} = q$, $\gamma = 1$ (or $\bar{\gamma} = 1$, $\gamma = q$). So we may regard Version-(3) as a generalization of the Version-(1). Moreover, it can be shown that for $p = 2$ both the Version-(0) and the Version-(2) are specializations of the Version-(3). However, it is not true for $p > 2$ and, in general, the Leibniz rule (29) has to be further modified. To find a most general deformed Leibniz rule we slightly change the definition of the $(\mathrm{g} - \bar{\mathrm{g}})$-Version.

(4): **Generalized Version**

Namely, leaving untouched the equations (28) and (32) assume that γ and $\bar{\gamma}$ are arbitrary parameters not constrained by Eq. (33), i.e.

$$\tilde{\alpha}_{p+1} \equiv \frac{\bar{\gamma}^{p+1} - \gamma^{p+1}}{\bar{\gamma} - \gamma} \neq 0 \ \text{ but } \ \alpha_{p+1} = 0. \tag{35}$$

Then the conditions (12), (28), and (32) are only fulfilled if the equations (34) are modified as follows

$$\begin{aligned} \partial\theta - \gamma\theta\partial &= \bar{\mathrm{g}} - \tfrac{\tilde{\alpha}_{p+1}}{(\alpha_p)!}\theta^p\partial^p\,, \\ \partial\theta - \bar{\gamma}\theta\partial &= \mathrm{g} - \tfrac{\tilde{\alpha}_{p+1}}{(\alpha_p)!}\theta^p\partial^p\,. \end{aligned} \tag{36}$$

Note that Version-(0) may be derived from this version by substituting $\mathrm{g}(\theta^k) = \delta_{k,0}$, $\bar{\mathrm{g}} = 1$ that means $\gamma = 0$, $\bar{\gamma} = 1$ (or, equivalently, $\mathrm{g} = 1$, $\bar{\mathrm{g}}(\theta^k) = \delta_{k,0}$; $\gamma = 1$, $\bar{\gamma} = 0$). Versions-(1) and -(3) are reproduced if we put $\tilde{\alpha}_{p+1} = 0$, while Version-(2) may be obtained in the limit $\gamma = \bar{\gamma} \to 1$. Thus, Version-(4) generalizes all previous Versions.

The relations (36) dictate a more general modification of the Leibniz rule

$$\partial(FG) = \partial(F)\bar{\mathrm{g}}(G) + \mathrm{g}(F)\partial(G) + Lz(F,G)\,. \tag{37}$$

As follows from Eqs. (36), the additional term $Lz(F,G)$ belongs to the one dimensional space $\{|p\rangle\}$ and we suggest to call this term the 'Leibnizean'. Note that the associativity condition for the rule (37) requires Eqs. (30) and the additional relation

$$Lz(FG,H) + Lz(F,G)\bar{\mathrm{g}}(H) = Lz(F,GH) + \mathrm{g}(F)Lz(G,H)\,.$$

Versions (1) and (2) evidently reproduce the Grassmann calculus for $p = 1$ while the limit $p \to \infty$ gives $\mathrm{g} = \bar{\mathrm{g}} = 1$ and $Lz = 0$, thus reproducing the standard calculus in dimension one. Other possible versions obeying the conditions $\lim_{p\to\infty}(Lz) = 0$ and $\lim_{p\to\infty}(\mathrm{g},\ \bar{\mathrm{g}}) = 1$ are much more complicated (e.g. $b_1 = 1$, $b_2 = \ldots = b_{p-k} = 0$, $b_{p-k+1} \neq 0, \ldots,\ b_p \neq 0$, for some fixed $k \ge 2$).

Summarizing this discussion we note that in constructing a paragrassmann calculus for many variables we wish to have a generalized Leibniz rule. A most natural generalization must look like

$$\partial_i(FG) = \partial_j(F)\bar{\mathrm{g}}^j_i(G) + \mathrm{g}^j_i(F)\partial_j(G)\,, \tag{38}$$

where $\bar{\mathrm{g}}$ and g are some automorphisms and the summation over j is understood. Only Versions -(1) and -(3) are suitable in this context.

4. Paragrassmann Algebras with Many Variables

Here we present explicit realization of some paragrassmann algebras $\Pi_{p+1}(N)$ generated by N coordinates θ_i $(i = 1, \ldots, N)$: $\theta_i^{p+1} = 0$ and N corresponding derivatives ∂_i , $\partial_i^{p+1} = 0$. The simplest (bilinear) algebras can be constructed in Version-(1). Thus consider the algebra $\Pi_{p+1}(1)$ defined by

$$\partial\theta - q\theta\partial = 1\,,\quad \partial^{p+1} = \theta^{p+1} = 0\,, \tag{39}$$

where q is any primitive $(p+1)$-root of unity The algebra (39) was the starting point for considering the fractional para-supersymmetry [13]. Our motivation for using this version is its extreme simplicity. Furthermore, it gives bilinear commutation relations for generators of $\Pi_{p+1}(N)$ that are closely related to the definitions of the quantum hyper-plane [14], covariant q-deformed oscillators [15] and differential calculus on the quantum hyperplane [16]. The other versions (7) can be considered similarly but they yield non-bilinear multi-paragrassmann algebras (a generic example will be given below).

The generator of automorphisms in the Leibniz rule can be expressed as

$$\mathrm{g} = \partial\theta - \theta\partial. \tag{40}$$

It is easy to check that

$$\theta\partial = (\mathrm{g} - 1)/(q - 1)\,,\quad \partial\theta = (q\mathrm{g} - 1)/(q - 1)\,,$$

and

$$\mathrm{g}\theta = q\theta\mathrm{g}\,,\quad \mathrm{g}\partial = q^{-1}\partial\mathrm{g}\,. \tag{41}$$

Using this operator we define N paragrassmann variables

$$\theta_i = \mathrm{g}^{\rho^i_N} \otimes \mathrm{g}^{\rho^i_{N-1}} \otimes \cdots \otimes \mathrm{g}^{\rho^i_{i+1}} \otimes \theta\mathrm{g}^{\rho^i_i} \otimes \mathrm{g}^{\rho^i_{i-1}} \otimes \cdots \otimes \mathrm{g}^{\rho^i_1} \tag{42}$$

with the obvious commutation relations

$$\theta_i\theta_j = q^{\rho_{ij}}\theta_j\theta_i\,,\quad i < j\,. \tag{43}$$

We wish to restrict $N(N-1)/2$ numbers $\rho_{ij} = \rho^i_j - \rho^j_i$ so as any linear combination of θ_i is nilpotent,

$$(\sum_{i=0}^{p} c_i\theta_i)^{p+1} = 0\,, \tag{44}$$

and hence θ_i generate a paragrassmann algebra $\Gamma_{p+1}(N)$. One simple choice is

$$\rho_{ij} = a_j\ (i < j)\,, \tag{45}$$

where all q^{a_i} are primitive roots of unity. With this choice, all θ_i for $i < j$ acquire the same multiplier q^{a_j} in commuting through θ_j. So, if Eq. (44) is valid for the linear combinations of the first $(j-1)$ theta's, we may apply Eqs. (35), (36) of Ref. [1] and thus prove it to be valid for any number of theta's (provided that all q^{a_i} are primitive roots of unity).

The ansatz (42) generalizes the expressions for many theta's obtained in [1] by certain recurrent procedure. It is natural to define the derivatives ∂_i by

$$\partial_i = \mathrm{g}^{\sigma^i_N} \otimes \mathrm{g}^{\sigma^i_{N-1}} \otimes \cdots \otimes \mathrm{g}^{\sigma^i_{i+1}} \otimes \mathrm{g}^{\sigma^i_i}\partial \otimes \mathrm{g}^{\sigma^i_{i-1}} \otimes \cdots \mathrm{g}^{\sigma^i_1}. \tag{46}$$

Then the commutation relations for ∂_i are $(i < j)$

$$\partial_i\partial j = q^{\sigma_{ij}}\partial_j\partial_i\,,\quad \sigma_{ij} = \sigma^j_i - \sigma^i_j \tag{47}$$

$$\theta_i\partial_j = q^{-\sigma^j_i - \rho^i_j}\partial_j\theta_i\,, \tag{48}$$

$$\theta_j\partial_i = q^{-\sigma^i_j - \rho^j_i}\partial_i\theta_j\,. \tag{49}$$

Here the parameters σ^j_i are to be chosen so that any linear combinations of ∂'s is also nilpotent

$$(\sum_{i=0}^{p} c_i\partial_i)^{p+1} = 0. \tag{50}$$

Now, to obtain a closed algebra with quadratic commutation relations we have to solve the following *problem* $(\star)$:

to express $\partial_i\theta_i$ as a linear combination of 1 and $\theta_j\partial_j$, $i, j = 1, \ldots, N$.

It is more convenient to deal with the expressions

$$\partial_i\theta_i - q^{\tau^i_i+1}\theta_i\partial_i = \mathrm{g}^{\tau^i_N} \otimes \cdots \otimes \mathrm{g}^{\tau^i_i} \otimes \cdots \otimes \mathrm{g}^{\tau^i_1} \equiv \mathrm{g}^{[\tau^i]}\,, \tag{51}$$

where $\tau^i_j = \rho^i_j + \sigma^i_j$, or, as well, with

$$\partial_i\theta_i - q^{\tau^i_i}\theta_i\partial_i = \mathrm{g}^{\tau^i_N} \otimes \cdots \otimes \mathrm{g}^{\tau^i_i+1} \otimes \cdots \otimes \mathrm{g}^{\tau^i_1} \equiv \mathrm{g}^{[\tau^i]_+}\,. \tag{52}$$

The terms $\theta_j\partial_j$ are better represented by

$$E_j \equiv (q-1)q^{\tau^j_j}\theta_j\partial_j = \mathrm{g}^{\tau^j_N} \otimes \cdots \otimes \mathrm{g}^{\tau^j_j}(g-1) \otimes \cdots \otimes \mathrm{g}^{\tau^j_1} = \mathrm{g}^{[\tau^i]_+} - \mathrm{g}^{[\tau^i]}\,. \tag{53}$$

It is not hard to realize that the *problem* $(\star)$ is solvable if and only if for any i there exist a sequence of operators (53), producing (51) (or (52) as well) from $1 \equiv \mathrm{g}^{[0]}$. To formalize this idea more rigorously, consider in an N-dimensional space $2N$ points $\{ [\tau^i] , [\tau^i]_+ , i = 1 \dots N \}$ and N oriented segments $[\tau^i] \longrightarrow [\tau^i]_+$. This set of data composes an oriented graph $\mathcal{G}$, which obviously does not contain cycles since all the segments are mutually orthogonal. After these preliminaries, we can formulate the following

C r i t e r i o n : *The problem* $(\star)$ *is solvable if and only if the correspondent graph* $\mathcal{G}$ *is connected (and therefore an oriented tree) and contains the point* [0].

In other words, this means that one can define an equivalence relation $\sim$ on the set $\mathcal{T} = \{[0] , [\tau^i] , [\tau^i]_+ , i = 1 \dots N \}$ so that

$$\begin{array}{ll} a). & [\tau^i] \sim [\tau^i]_+ , \quad i = 1 \dots N , \\ b). & u = v \Rightarrow u \sim v , \quad \forall u, v \in \mathcal{T} . \end{array} \tag{54}$$

Then the criterion tells that the entire $\mathcal{T}$ must be a single equivalency class.

This criterion gives a simple procedure for getting the commutation relations of ∂_i and θ_j :

1. Draw an oriented tree with a root [0] and N edges;
2. Label the edges by the numbers from 1 to N;
3. Find a path from [0] to the beginning of the i-th edge;
4. Moving along this path, write

$$\mathrm{g}^{[\tau^i]} = 1 \pm E_{j_1} \pm E_{j_2} \dots ,$$

taking '+', if the move agrees with the orientation of the edge j_a and '−'otherwise;

5. Use the expressions (51) and (53).

This algorithm exhausts all admissible possibilities. In particular, it proves that all the numbers τ^i_j can be only 0 or ± 1. Thus, it brings some restrictions on the exponentials ρ^i_j and σ^i_j, though not too strong. In fact, the most hard thing can happen is the requirement that both ρ_{ij} and $\rho_{ij} + 1$ should be prime, which leads p to be even number and restricts a choice of ρ_{ij} in a rather soft manner. There are no direct restrictions on the values of ρ_{ij} or relations between them, coming from the criterion. So the last string of the algebra, the commutation relations of ∂_i and θ_i, is almost independent of the first four ones, (43), (47 – 49). Note that algebras corresponding to the different graphs are non-equivalent, at least at the level of linear combinations.

Let us present now two simplest examples of the paragrassmann algebras $\Pi_{p+1}(N)$.

$$\begin{array}{ll} 1). & \tau^i_j = \rho^i_j + \sigma^i_j \ = 0 \ \ (i \neq j) , \\ & \tau^i_i = 0 , -1 , \end{array} \tag{55}$$

or shortly: $[\tau^i] = [0]$ or $[\tau^i]_+ = [0]$. With this choice the algebra has the form $(i < j)$

$$\begin{array}{rcl} \theta_i\theta_j &=& q^{\rho_{ij}}\theta_j\theta_i , \\ \partial_i\partial_j &=& q^{\rho_{ij}}\partial_j\partial_i , \\ \theta_i\partial_j &=& q^{-\rho_{ij}}\partial_j\theta_i , \\ \theta_j\partial_i &=& q^{\rho_{ij}}\partial_i\theta_j , \\ \partial_i\theta_i - q^{2\tau^i_i+1}\theta_i\partial_i &=& 1 . \end{array} \tag{56}$$

This algebra has been described in [1], [17]. The correspondent graph $\mathcal{G}$ is a bunch of N segments coming from (or to) zero point.

$$\begin{array}{ll} 2). & \tau^i_j = 1 , \ j < i ; \\ & \tau^i_j = 0 , \ j \geq i , \end{array} \tag{57}$$

or shortly, $[\tau^1] = [0]$, $[\tau^{i+1}] = [\tau^i]_+$. Here we obtain $(i < j)$

$$\begin{array}{rcl} \theta_i\theta_j &=& q^{\rho_{ij}}\theta_j\theta_i , \\ \partial_i\partial_j &=& q^{\rho_{ij}+1}\partial_j\partial_i , \\ \theta_i\partial_j &=& q^{-1-\rho_{ij}}\partial_j\theta_i , \\ \theta_j\partial_i &=& q^{\rho_{ij}}\partial_i\theta_j , \\ \partial_i\theta_i - q\theta_i\partial_i &=& 1 + (q-1)\sum_{j=1}^{i-1}\theta_j\partial_j . \end{array} \tag{58}$$

This algebra resembles the differential calculus on the quantum hyperplane [16] (see also [18]). The correspondent graph is a chain of N arrows. Algebras of this kind can exist only for even p, as it was mentioned.

Concluding this discussion, we would like to formulate some problems related to complete classifying paragrassmann algebras.

1. It is clear that algebras $\Gamma_{p+1}(N)$ with different sets $\{a_i\}$ (see (45)) are not equivalent (unless the two sets are proportional). The question is how fully the ansatz (45) exhausts all admissible matrices ρ_{ij} in (43)? (We suspect that for N large enough it is exhaustive while for smaller N it is not.)

2. An algebra $\Pi_{p+1}(N)$ can be determined by an oriented tree $\mathcal{G}$ together with a suitable set $\{a_i\}$ (or, more generally, $\{\rho_{ij}\}$). Different trees and sets define non-equivalent algebras that cannot be related by any linear transformation of θ's and ∂'s. The question is: can they be related by a *non*-linear transformation like that connecting the versions of the algebra $\Pi_{p+1}(1)$? In other words, can different $\Pi^{\{\mathcal{G},a\}}_{p+1}(N)$ be considered as versions of the unique algebra $\Pi_{p+1}(N)$ or they are distinct as enveloping algebras?

The final remark concerns possible non-bilinear algebras. Our approach can be generalized to arbitrary version, with commutation relations (7). With this aim, we first introduce the linear automorphism operator g in the algebra (7) with the commutation relations (41) but considering q as an arbitrary parameter. Then, for the multi-paragrassmann generators defined as in (42), (46) one can derive the

following algebra ($i < j$)

$$\begin{aligned}
\partial_i\theta_i &= b_0 + b_1\theta_i\partial_i + b_2\theta_i^2\partial_i^2 + \ldots + b_p\theta_i^p\partial_i^p\,, \\
\theta_i\theta_j &= q^{a_i}\theta_j\theta_i\,, \\
\partial_i\partial_j &= q^{a_i}\partial_j\partial_i\,, \\
\partial_i\theta_j &= q^{-a_i}\theta_j\partial_i\,, \\
\partial_j\theta_i &= q^{-a_i}\theta_i\partial_j\,.
\end{aligned} \tag{59}$$

To satisfy the equations (44), (50) we have chosen the parameters $\sigma_{ij} = \rho_{ij} = a_j$ for $i < j$ and $\sigma_{ij} = \rho_{ij} = a_i$ for $i > j$. The integer numbers a_i are restricted by the condition that all q^{a_i} are primitive $(p+1)$-roots of unity. The most important feature of this construction is its independence of the version ('version covariance'). This property is of utmost importance in some applications, e.g. in constructing para-Virasoro algebras to be treated in our next paper. Note, however, that the generalized Leibniz rule (38) is only satisfied if the b-parameters correspond to the $\mathbf{g} - \bar{\mathbf{g}}$–Version. The algebra (59) may be further generalized but we will not present these generalizations here. Non-bilinear algebras deserve a separate thorough investigation.

5. Conclusion

In this paper we have given a general construction of the paragrassmann calculus with one variable and have shown that all nondegenerate algebras $\Pi_{p+1}^{\{b\}}$ are equivalent. Still, different versions may be useful in different applications. As has been shown in the last section, constructing algebras with many variables requires simplest versions. There is another reason for a separate consideration of different equivalent versions. Our approach to constructing paragrassmann calculus with many variables was to preserve the nilpotency property for linear combinations of θ_i (and of ∂_i). Then the commutation relations between different elements are just calculation tools not having any fundamental meaning. However, we may choose a different viewpoint considering the algebra of commutation relations as a prime object. Then it would be natural to look for transformations preserving the commutation relations.

Let us discuss this viewpoint. It is clear that transformations

$$\partial_i \to \partial_i' = t_{ij}\partial_j\,,\quad \theta_i \to \theta_i' = t_{ij}\theta_j\,. \tag{60}$$

do not preserve the commutation relations (43) and(47). To preserve these commutation relations (quantum hyperplane relations) we have to consider t_{ij} as generators of the multiparametric quantum group $GL_{q,\rho_{ij}}$. In particular, we have to require

$$t_{ik}t_{ij} = q^{\rho_{kj}}t_{ij}t_{ik}\,,$$

$$\theta_k t_{ij} = t_{ij}\theta_k\,.$$

The main paragrassmann identity now looks as

$$(\theta_i')^{p+1} = (\sum_{j=1}^{N} t_{ij}\theta_j)^{p+1} = 0 \tag{61}$$

It is clear that

$$(t_{ik}\theta_k)(t_{ij}\theta_j) = q^{2\rho_{kj}}(t_{ij}\theta_j)(t_{ik}\theta_k)$$

and eq. (61) is fulfilled only if $q^{2\rho_{ij}}$ are primitive roots of unity. As an example we present the following paragrassmann quantum plane $(\theta_i)^{p+1} = 0$ where $\theta_i = \mathrm{g}^{1/2} \otimes \cdots \otimes \mathrm{g}^{1/2} \otimes \underbrace{\theta \otimes 1 \otimes \cdots \otimes 1}_{i}$ The commutation relations have the form

$$\theta_i\theta_j = q^{1/2}\theta_j\theta_i\,,\quad (i < j)\,. \tag{62}$$

and these relations are not changed under the transformations (60) with $t_{ij} \in GL_{q^{1/2}}(N)$. Then it is clear that

$$(t_{ik}\theta_k)(t_{ij}\theta_j) = q(t_{ij}\theta_j)(t_{ik}\theta_k)$$

and, if q is primitive root of unity, we obtain that $(\theta_i')^{p+1} = 0$. Thus, the paragrassmann quantum plane (62) may be regarded as a linear space under rotations of the quantum group $GL_{q^{1/2}}$.

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О параграссмановом дифференциальном исчислении

Эта работа расширяет и обобщает нашу предыдущую работу [1] .Здесь мы обсуждаем общие явные конструкции для параграссманова исчисления одной и многих переменных. Для одной переменной показано, что невырожденные дифференциальные алгебры совпадают и эквивалентны алгебре комплексных (p+1)×(p+1) - матриц. Для многих переменных дана общая конструкция дифференциальных алгебр. Несколько частных примеров связаны с многопараметрическими квантовыми деформациями гармонического осциллятора.

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On Paragrassmann Differential Calculus

The paper significantly extends and generalizes our previous paper [1]. Here we discuss explicit general constructions for paragrassmann calculus with one and many variables. For one variable nondegenerate differentiation algebras are identified and shown to be equivalent to the algebra of (p+1)×(p+1) complex matrices. For many variables we give a general construction of the differentiation algebras. Some particular examples are related to the multiparametric quantum deformations of the harmonic oscillators.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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