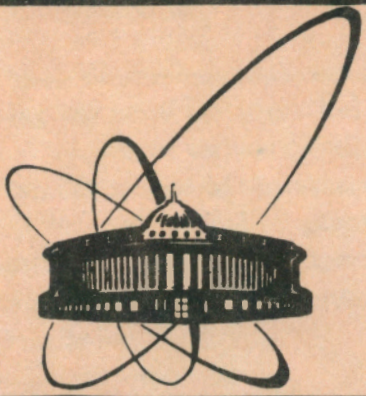

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ONE-DIMENSIONAL ANHARMONIC OSCILLATOR IN SELF-SIMILAR APPROXIMATION

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1. Introduction

The one - dimensional anharmonic oscillator is a touchstone for demonstrating any novel mathematical method. This is for three reasons. First, a great number of various physical problems has the mathematical structure similar to that of the anharmonic oscillator; among them e.g. is the so - called φ^4 model of quantum field theory. Second, the standard perturbation theory with respect to the anharmonicity parameter yields series with a zero radius of convergence [1]. Third, there are exact numerical results for the eigenvalues of the anharmonic oscillator [2-4] together with the corresponding asymptotic expansions in the weak coupling [1-3] and strong coupling [5-7] limits, which allows a direct estimation of accuracy for new methods.

When many first terms, about ten of them, of perturbation theory are known, one can find an effective sum of divergent series invoking some of the existing resummation techniques like the Padé approximation or Borel transformation. However, for the overhelming majority of realistic problems one is able to calculate only a few terms. Then the usual resummation techniques cannot be applied. In a situation like that there was a necessity to invent a method which could allow one to find, using a minimal number of first terms, an effective sum of a divergent series with a reasonable accuracy.

Such a method has been formulated in Ref.[8]. This method is based on constructing a convergent sequence of approximations using the property of functional self - similarity, because of which it has been called the method of self - similar approximations. The advantage of the method is in the use of only two first terms of perturbation theory in conjugation with a high accuracy. Using only two terms, the method [8] makes it possible to reach a better accuracy than either the Padé or Borel summations emploing ten terms. The method of self - similar approximations [8] was applied for finding out the ground - state energy of the anharmonic oscillator [8,9] and for considering several problems of statistical mechanics [10].

The aim of the present paper is to show that the method of self - similar approximations [8-10] retains its high accuracy not solely for the ground - state energy but for the whole spectrum of the anharmonic

oscillator. All energy levels as well as all anharmonicity parameters are considered. We also analyse several variants of the method in order to choose the most accurate among them.

2. Formulation of method

Here we present the scheme of the method of self - similar approximations whose detailed description is given in Ref.[8]. Let us have a sequence of functions $f_k(g)$, where $k = 0, 1, 2, \ldots$ is a number of approximation and g, a parameter, which we shall call the coupling constant. First of all, we should organize a convergent sequence of functions $f_k(g, z_k)$ by introducing a set of governing functions $z_k(g)$ whose role is to govern the convergence of the sequence of renormalized functions

$$f_k(g) = f_k(g, z_k(g)), \tag{1}$$

in which the notation

$$z(g) \equiv z_0(g) \equiv z_1(g)$$

will be used. The governing functions are to be defined from a fixed - point condition [8-10].

Define a coupling function $g(f)$ by the equation

$$f_0(g, z(g)) = f; \quad g = g(f). \tag{2}$$

Introduce the function

$$y_{sk}(f) = [f_s(g(f), z_k(g(f))) - f_k(g(f), z_k(g(f)))]^{-1} \tag{3}$$

satisfying the normalization

$$\int_{f_k(g)}^{f_*(g)} y_{sk}(f) df = 1, \tag{4}$$

in which $f_*(g)$ is just the sought self - similar approximation. The latter, generally, can depend on any number of other parameters.

Given exact numerical values for the function of interest $f(g)$, the accuracy of the method will be found be calculating the maximal error

$$\epsilon_* = \sup_g \left| 1 - \frac{f_*(g)}{f(g)} \right|.$$

When considering the spectra of Schrödinger operators we shall deal, in the place of the function $f_*(g)$, with the energy levels $e_*(n, g)$, where $n = 0, 1, 2 \ldots$ is a level number. Then the maximal error will be defined as

$$\epsilon_* = \sup_n \sup_g \left| \frac{e_*(n, g)}{e(n, g)} - 1 \right|, \tag{5}$$

where $e(n, g)$ is an exact value.

3. Anharmonic oscillator

Consider the eigenvalue problem for the operator

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 + \lambda m^2 x^4, \tag{6}$$

in which $m, \omega, \lambda > 0$ and $x \in (-\infty, +\infty)$. The eigenvalues, or energy levels, $E_n(g)$ are enumerated with the index $n = 0, 1, 2, \ldots$ and depend on the dimensionless coupling, or anharmonicity, parameter

$$g \equiv \frac{\lambda}{\omega^3}. \tag{7}$$

For convenience, we introduce the dimensionless energy levels

$$e(n, g) \equiv \frac{E_n(g)}{\omega}; \quad n = 0, 1, 2, \ldots \tag{8}$$

Enumerating the number of approximation by $k = 0, 1, 2, \ldots$, we write

$$e_k(n, g) \equiv e_k(n, g, z_k(n, g)). \tag{9}$$

Take as a zero approximation the harmonic oscillator Hamiltonian

$$H_0 = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega_0^2}{2}x^2, \tag{10}$$

and define the trial parameter

$$z \equiv \frac{\omega_0}{\omega}. \tag{11}$$

Starting with (10), we have to choose a particular sequence of approximations together with a fixed - point condition for the governing functions.

Let us use the Rayleigh - Schrödinger perturbation theory for constructing the corresponding approximations

$$e_k(n,g,z) \equiv \frac{E_n^{(k)}(g)}{\omega}; \quad k = 0,1,2,\ldots, \tag{12}$$

where $E_n^{(k)}(g)$ is the k -th approximation for the n -th energy level.

The initial approximation, connected with (10), is

$$e_0(n,g,z) = \left(n + \frac{1}{2}\right) z. \tag{13}$$

It is not too difficult to find the first

$$e_1(n,g,z) = e_0(n,g,z) + \left(n + \frac{1}{2}\right)\frac{1-z^2}{2z} + \left(n^2 + n + \frac{1}{2}\right)\frac{3g}{2z^2} \tag{14}$$

and second

$$e_2(n,g,z) = e_1(n,g,z) - \left(n + \frac{1}{2}\right)\frac{(1-z^2)^2}{8z^3} - \left(n^2 + n + \frac{1}{2}\right)\frac{3g(1-z^2)}{2z^4} -$$

$$- \left(n + \frac{1}{2}\right)(17n^2 + 17n + 21)\frac{g^2}{4z^5} \tag{15}$$

approximations.

A fixed - point condition for the governing functions can be written in one of the forms, either as a minimal - difference condition [11-15] or as a minimal - sensitivity condition [16-20]. Let us begin with the minimal - sensitivity condition

$$\frac{\partial}{\partial z} e_1(n,g,z) = 0; \quad z = z(n,g), \tag{16}$$

which gives the equation

$$z^3 - z - 6\gamma_n g = 0 \tag{17}$$

for the governing function $z(n,g)$. In Eq.(17) the notation

$$\gamma_n \equiv \frac{n^2 + n + 1/2}{n + 1/2} \tag{18}$$

is used. The solution to Eq.(17) supplies the governing functions

$$z(n,g) = \begin{cases} \frac{2}{\sqrt{3}}\cos(\frac{\alpha_n}{3}); & g \le g_n \\ A_n^+ + A_n^-; & g \ge g_n \end{cases} \tag{19}$$

in which

$$\alpha_n = \arccos(\frac{g}{g_n}),$$

$$A_n^{\pm} = (3g)^{1/3}\left[1 \pm \sqrt{1 - (g_n/g)^2}\right]^{1/3},$$

$$g_n = (9\sqrt{3}\gamma_n)^{-1} = 0.064150\gamma_n^{-1}.$$

Eqs.(14) and (15), taking account of (17), can be simplified becoming

$$e_1(n,g,z) = \left(n + \frac{1}{2}\right)\frac{3z^2 + 1}{4z}, \tag{20}$$

$$e_2(n,g,z) = e_1(n,g,z) - \left(n + \frac{1}{2}\right)\frac{a_n(1-z^2)^2}{48z^3}, \tag{21}$$

where

$$a_n \equiv \left(n + \frac{1}{2}\right)^2 \frac{34(n^2 + n + 1/2) + 25}{6(n^2 + n + 1/2)^2} - 6. \tag{22}$$

Eq.(2) for the coupling function $g(n,f)$ reads

$$e_0(n,g) = f; \quad g = g(n,f). \tag{23}$$

The function (3), defined as

$$y_{21}(n,f) = [e_2(n,g(n,f)) - e_1(n,g(n,f))]^{-1}, \tag{24}$$

takes the form

$$y_{21}(n,f) = -\frac{48f^3}{a_n[f^2-(n+1/2)^2]^2}. \qquad (25)$$

Finally, the normalization (4) yields the equation

$$\frac{e_*^2(n,g)-(n+\frac{1}{2})^2}{e_1^2(n,g)-(n+\frac{1}{2})^2} =$$

$$= \exp\left\{\frac{(n+\frac{1}{2})^2}{e_*^2(n,g)-(n+\frac{1}{2})^2} - \frac{(n+\frac{1}{2})^2}{e_1^2(n,g)-(n+\frac{1}{2})^2} - \frac{a_n}{24}\right\} \qquad (26)$$

for the self - similar approximation $e_*(n,g)$ of the anharmonic - oscillator spectrum. The comparison of the self - similar approximation given by (26) with exact numerical results [2,3] shows that the error very rarely exceeds 0.3% the maximal error (5) being 0.4% for any energy level $n = 0,1,2,\ldots\infty$ and all anharmonicity parameters $g \in (0,\infty)$. The corresponding values are shown in Table 1.

It is instructive to consider the strong anharmonicity limit $g \to \infty$. Then, the governing function (19) becomes

$$z(n,g) \simeq (6\gamma_n g)^{1/3}; \quad g \to \infty. \qquad (27)$$

The first and second terms (20) and (21) are

$$e_1(n,g) \simeq \frac{3}{4}\left(n+\frac{1}{2}\right)(6\gamma_n g)^{1/3}, \quad (g \to \infty),$$

$$e_2(n,g) \simeq \frac{3}{4}\left(n+\frac{1}{2}\right)\left(1-\frac{a_n}{36}\right)(6\gamma_n g)^{1/3}. \qquad (28)$$

The self - similar approximation defined by (26) is

$$e_*(n,g) \simeq e_1(n,g)\exp\left(-\frac{a_n}{48}\right); \quad g \to \infty. \qquad (29)$$

Consider in addition the asymptotic expressions as $n \to \infty$. Using the notation (18), (22) can be presented as

$$a_n = \left(n+\frac{1}{2}\right)\frac{17}{3\gamma_n} + \frac{25}{6\gamma_n^2} - 6.$$

In the case of high energy levels

$$\gamma_n \simeq n, \quad a_n \simeq -\frac{1}{3}; \quad n \gg 1.$$

With these properties (27) gives

$$z(n,g) \simeq (6ng)^{1/3}; \quad n,g \gg 1,$$

and (28) yields

$$e_1(n,g) \simeq \frac{3}{4}n^{4/3}(6g)^{1/3},$$

$$e_2(n,g) \simeq \frac{109}{144}n^{4/3}(6g)^{1/3}, \quad (g,n \gg 1) \qquad (30)$$

For the self - similar approximation (29) we obtain

$$e_*(n,g) \simeq \frac{3}{4}\exp\left(\frac{1}{144}\right)n^{4/3}(6g)^{1/3}; \quad g,n \gg 1. \qquad (31)$$

The latter formula can be compared with the exact asymptotic form [3]

$$e(n,g) \simeq \pi^2\left[\frac{3}{\Gamma^2(1/4)}\right]^{4/3} n^{4/3}g^{1/3}; \quad g,n \gg 1. \qquad (32)$$

For a more visual comparison let us rewrite (30) as

$$e_1(n,g) \simeq 1.362840n^{4/3}g^{1/3},$$

$$e_2(n,g) \simeq 1.375459n^{4/3}g^{1/3}, \quad (g,n \gg 1) \qquad (33)$$

the self - similar approximation (31) as

$$e_*(n,g) \simeq 1.372338n^{4/3}g^{1/3}; \quad g,n \gg 1, \qquad (34)$$

and the exact asymptotic limit (32) as

$$e(n,g) \simeq 1.376507n^{4/3}g^{1/3}; \quad g,n \gg 1. \qquad (35)$$

Eqs.(33) - (35) show that in the asymptotic region $g,n \to \infty$ the accuracy of $e_1(n,g)$ is 1% ; of $e_2(n,g)$ is 0.08% and that of (34) is 0.3% . Remind that the maximal error of $e_2(n,g)$ in the whole range of

$g \in (0,\infty)$ and $n = 0,1,2,\ldots\infty$ is about 1% while that of the self - similar approximation is 0.4% . The fact that in the asymptotic limit $g,n \to \infty$ the self - similar approximation gives smaller accuracy than the second order renormalized perturbation theory is purely accidental. This is clear from the Table 1 which shows that the accuracy oscillates with changing n and g . Thus, only the maximal error can characterize a method as a whole.

4. Analysis of procedure

To check the optimum of the procedure followed in the previous section, it is useful to try some other variants.

Let us take a fixed - point condition not in the form of the minimal - sensitivity condition (16) but in the form of the minimal - difference condition [11 - 13] defining the governing function from the equation

$$e_2(n,g,z) - e_1(n,g,z) = 0. \tag{36}$$

The distribution of approximations (3) will be taken as

$$y_{20}(f) = [e_2(n,g(n,f)) - e_0(n,g(n,f))]^{-1}. \tag{37}$$

Eq.(36) yields the equation

$$z^6 - 2z^4 - 12g\gamma_n z^3 + z^2 + 12g\gamma_n z + 2g^2(17n^2 + 17n + 21) = 0,$$

for the governing function $z = z(n,g)$. This equation for $n = 0,1$ has no real solution and for $n \geq 2$ has two solutions $z = z_{\pm}(n,g)$ whose asymptotic expression is

$$z_{\pm}(n,g) \simeq (6 \pm \sqrt{2})^{1/3}(ng)^{1/3}; \quad n,g \gg 1.$$

Constructing a self - similar approximation in this case and analysing its accuracy for the whole range of g and n including the strong coupling and high level limits, we find the maximal error (5) to be $\epsilon_* \sim 10\%$. Therefore, the minimal - difference condition produces less accurate results than the minimal - sensitivity condition (16), which is in agreement

with a similar analysis made by Stevenson [17] for modified perturbation theory.

Another possible change which we can make in our procedure is to use some other iterative scheme for obtaining a sequence of approximations in place of Rayleigh - Schrödinger perturbation approach. For instance, based on the spectral equation $H\psi_n = E_n\psi_n$ we can define the eigenvalues as

$$E_n = (\psi_n, H\psi_n) = \int_{-\infty}^{+\infty} \psi_n^*(x) H\psi_n(x)dx, \tag{38}$$

and the eigenfunctions as $\psi_n = C_n H\psi_n$, where the constant C_n is to be defined from the normalization condition $(\psi_n,\psi_n) = 1$ which gives $C_n = (\psi_n, H^2\psi_n)^{-1/2}$. Thus, the eigenfunctions are

$$\psi_n = H\psi_n/(\psi_n, H^2\psi_n)^{1/2} = H\psi_n/||H\psi_n||. \tag{39}$$

Considering (38) and (39) as equations to be solved by iterative process, we may define the latter by the recurrence relations

$$E_n^{(k+1)} = (\psi_n^{(k)}, H\psi_n^{(k)}),$$

$$\psi_n^{(k+1)} = H\psi_n^{(k)}/||H\psi_n^{(k)}||. \tag{40}$$

Starting with the harmonic oscillator Hamiltonian (10) we can find all further approximations by means of relations (40) and the integral

$$\frac{1}{2^n\sqrt{\pi}n!}\int_{-\infty}^{+\infty} x^{2m}e^{-x^2}H_n^2(x)dx =$$

$$= \frac{1}{2^{2m}m!}\sum_{k=0}^{m}(-1)^k C_m^k \frac{(n+2m-2k)!}{(n-2k)!},$$

in which $H_n(x)$ is the Hermite polinomial. For example, for the ground - state level, emploing the notation

$$e_k(g,z) \equiv e_k(0,g,z); \quad n = 0,$$

and defining the governing function $z = z(0, g)$ from equation (17), we get

$$e_0(g, z) = \frac{1}{2}z,$$

$$e_1(g, z) = \frac{3z^2 + 1}{8z},$$

$$e_2(g, z) = \frac{857z^6 - 1639z^4 + 1139z^2 - 165}{8z(35z^4 + 2z^2 + 11)}.$$

However, the use of the iterative procedure (40) supplies the results whose accuracy is much poorer than the accuracy of the self - similar approximations obtained by invoking the Rayleigh - Schrödinger perturbation scheme.

The reason why the accuracy of the procedure of Section 3 is sufficiently higher than that of the variants analysed in the present section can be easily understood if one considers the corresponding mapping multipliers

$$M_k(n, g) = \lim_{f \to e_*(n,g)} \left| \frac{\partial}{\partial f} e_k(n, g(n, f)) \right|. \qquad (41)$$

Calculating the latter it is not too difficult to make it sure that the mapping of Section 4 are not contracting, since $M_k(n, g)$ becomes equal or surpasses unity for some of parameters, while the mapping of Section 3 is contracting.

5. Discussion

We have shown that the method of self - similar approximations [8-10] can be applied not only for finding out the ground - state energy of one - dimensional anharmonic oscillator but for calculating the whole spectrum. From an analysis it follows that to achieve a high accuracy one has to deal with contracting mappings. Then the spectrum of the anharmonic oscillator can be found with a very good accuracy notwithstanding the use of only two terms of the Rayleigh - Schrödinger perturbation theory. The maximal error is 0.4% for any energy level $n = 0, 1, 2, \ldots, \infty$ and any anharmonicity parameter $g \in (0, \infty)$. For a better comparison of our method with the renormalized perturbation theory we adduce Table

2 presenting the results of the first and second order perturbation terms given by Eqs.(14) and (15), respectively.

In order to offer an additional proof for the above mentioned statement that it is by pure accident the asymptotic limit $n, g \to \infty$ of the self - similar approximations has smaller accuracy than the second order renormalized perturbation theory, we have considered the next step for the self - similar approximation. That is, we have calculate the self - similar approximation $e^*_{32}(n, g)$ given by the equation

$$\int_{e_2(n,g)}^{e^*_{32}(n,g)} y_{32}(n, f) df = 1, \qquad (42)$$

in which

$$y_{32}(n, f) = \{x_3(n, f) - x_2(n, f)\}^{-1},$$

$$x_k(n, f) \equiv e_k(n, g(n, f), z(n, g(n, f))),$$

and the governing function $z(n, g)$ is defined by (19). In the limit $n, g \to \infty$ this gives

$$e^*_{32}(n, g) \simeq 1.376653 n^{4/3} g^{1/3}. \qquad (43)$$

The term $e_3(n, g, z)$ is the third order approximation obtained by the Rayleigh - Schrödinger perturbation theory in analogy with (14) and (15). Substituting here $z = z_1(n, g) \equiv z(n, g)$ given by (19) we have

$$e_3(n, g, z_1(n, g)) \simeq 1.377037 n^{4/3} g^{1/3}, \qquad (44)$$

as $n, g \to \infty$. While if we substitute the governing function $z_3(n, g)$ defined by the equation

$$\frac{\partial}{\partial z} e_3(n, g, z) = 0, \quad z = z_3(n, g),$$

then in the limit $n, g \to \infty$

$$e_3(n, g, z_3(n, g)) \simeq 1.377545 n^{4/3} g^{1/3}. \qquad (45)$$

The errors of (43) as compared with the exact expansion (35) is 0.01% ; the error of (44) is 0.04% ; and that of (45) is 0.08% . As we see, the

Table 1.

The energy lelels of the one - dimensional anharmonic oscillator in the self - similar approximation defined by eq. (26) compared with the exact numerical values.

g	n	$e(n,g)$	$e_*(n,g)$	$error\%$
	0	0.50726	0.50728	0.004
	1	1.5357	1.5358	0.007
	2	2.5909	2.5909	0.000
	3	3.6711	3.6709	0.006
0.001	4	4.7749	4.7743	0.013
	5	5.9010	5.8998	0.021
	6	7.0483	7.0463	0.028
	7	8.2158	8.2129	0.035
	8	9.4030	9.3987	0.046
	0	0.63799	0.63959	0.25
	1	2.0946	2.0987	0.19
	2	3.8448	3.8407	0.11
	3	5.7970	5.7853	0.20
0.3	4	7.9100	7.8933	0.21
	5	10.167	10.141	0.26
	6	12.540	12.511	0.23
	7	15.030	14.992	0.26
	8	17.620	17.573	0.27
	0	0.80377	0.80606	0.28
	1	2.7379	2.7429	0.18
	2	5.1780	5.1683	0.19
	3	7.9400	7.9207	0.24
1	4	10.960	10.932	0.26
	5	14.203	14.161	0.30
	6	17.630	17.581	0.28
	7	21.240	21.172	0.32
	8	25.000	24.919	0.32
	0	3.9309	3.9284	0.06
	1	14.059	14.061	0.01
	2	27.550	27.447	0.37
	3	43.010	42.852	0.37
200	4	60.030	59.830	0.33
	5	78.400	78.127	0.35
	6	97.900	97.575	0.33
	7	118.40	118.05	0.29
	8	139.90	139.46	0.31
	0	18.137	18.121	0.09
	1	64.987	64.987	0.00
	2	127.51	127.02	0.39
	3	199.20	198.43	0.39
20000	4	278.10	277.15	0.34
	5	363.20	362.00	0.33
	6	454.00	452.20	0.40
	7	548.90	547.17	0.32
	8	648.50	646.47	0.31

Table 2.

The energy spectrum of the one - dimensional anharmonic oscillator given by the renormalized perturbation theory of first and second orders with the corresponding errors.

g	n	$e(n,g)$	$e_1(n,g)$	$error\%$	$e_2(n,g)$	$error\%$
	0	0.50726	0.50729	0.006	0.50725	0.000
	1	1.5357	1.5358	0.009	1.5356	0.000
	2	2.5909	2.5909	0.001	2.5908	0.001
	3	3.6711	3.6708	0.008	3.6711	0.001
0.001	4	4.7749	4.7740	0.019	4.7748	0.001
	5	5.9010	5.8993	0.030	5.9009	0.002
	6	7.0483	7.0455	0.040	7.0482	0.002
	7	8.2158	8.2116	0.051	8.2156	0.002
	8	9.4030	9.3968	0.066	9.4024	0.006
	0	0.63799	0.64163	0.57	0.63698	0.16
	1	2.0946	2.1050	0.49	2.0921	0.12
	2	3.8448	3.8424	0.06	3.8392	0.15
	3	5.7970	5.7795	0.30	5.7898	0.12
0.3	4	7.9100	7.8782	0.40	7.9040	0.08
	5	10.167	10.115	0.51	10.158	0.09
	6	12.540	12.474	0.53	12.534	0.05
	7	15.030	14.942	0.59	15.022	0.06
	8	17.620	17.510	0.62	17.610	0.06
	0	0.80377	0.81250	1.10	8.0078	0.37
	1	2.7379	2.7599	0.81	2.7314	0.24
	2	5.1780	5.1724	0.11	5.1658	0.23
	3	7.9400	7.9079	0.40	7.9276	0.16
1	4	10.960	10.900	0.55	10.948	0.11
	5	14.203	14.109	0.66	14.186	0.12
	6	17.630	17.508	0.69	17.615	0.08
	7	21.240	21.076	0.77	21.216	0.11
	8	25.000	24.800	0.80	24.972	0.11
	0	3.9309	4.0085	1.97	3.8994	0.80
	1	14.059	14.234	1.24	14.000	0.42
	2	27.550	27.484	0.24	27.434	0.42
	3	43.010	42.746	0.61	42.888	0.28
200	4	60.030	59.582	0.75	59.913	0.19
	5	78.400	77.735	0.85	78.259	0.18
	6	97.900	97.034	0.88	97.756	0.15
	7	118.40	117.36	0.88	118.28	0.10
	8	139.90	138.61	0.92	139.74	0.11
	0	18.137	18.502	2.01	17.988	0.82
	1	64.987	65.803	1.26	64.707	0.43
	2	127.51	127.19	0.25	126.96	0.43
	3	199.20	197.93	0.64	198.59	0.30
20000	4	278.10	275.99	0.76	277.53	0.20
	5	363.20	360.16	0.84	362.61	0.16
	6	454.00	449.67	0.95	453.03	0.21
	7	548.90	543.93	0.91	548.23	0.12
	8	648.50	642.50	0.92	647.78	0.11

self - similar approximation (43) has better accuracy than both variants of the renormalized perturbation theory, either (44) or (45). A more detailed analysis of higher order terms of the renormalized perturbation theory as well as of the method of self - similar approximations involves quite complicated formulae and needs a separate consideration which will be given in another paper.

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Одномерный ангармонический осциллятор
в автомодельном приближении

Новый метод, названный методом автомодельных приближений, применен для вычисления спектра одномерного ангармонического осциллятора. Преимущество данного метода состоит в возможности восстановления любой искомой функции на основе всего лишь двух первых членов теории возмущений. Несмотря на столь ограниченную информацию удается найти полный спектр ангармонического осциллятора с очень хорошей точностью для всех энергетических уровней и всех констант ангармоничности с максимальной ошибкой порядка 10^{-3}.

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Yukalova E.P., Yukalov V.I. E5-92-326

One-Dimensional Anharmonic Oscillator
in Self-Similar Approximation

A new method, called the method of self-similar approximations, is applied here for calculating the spectrum of the one-dimensional anharmonic oscillator. The advantage of this method is the possibility of reconstructing any sought function using only two first terms of perturbation theory. Notwithstanding such a limited information, the whole spectrum of the anharmonic oscillator can be found with a very good accuracy for all energy levels and all anharmonicity constants with the maximal error of an order 10^{-3}.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1992