92-305

объединенный
институт
ядерных
исследований
дубна

E5-92-305

P.E.Zhidkov

AN INVARIANT MEASURE FOR A NONLINEAR WAVE EQUATION

Submitted to "Nonl. Anal.; Theory, Meth. and Appl."

1992

1. Introduction

In this paper the invariant measure for a dynamical system defined by the nonlinear wave equation

$$u_{tt} - u_{xx} + f(x,u) = 0, \quad x \in (0,A), \; t \in R, \qquad (1)$$

$$u(x,t_0) = u_0(x), \quad u_t'(x,t_0) = u_1(x) \qquad (2)$$

with vanishing boundary conditions

$$u(0,t) = u(A,t) = 0 \qquad (3)$$

is constructed. Here f is a smooth function satisfying some conditions of growth. In the same way one can simply construct an invariant measure for the periodic problem when $u(x+A,t) = u(x,t)$ for any x, t.

There are several papers on this matter for various partial differential equations of the mathematical physics /1-5/. In the paper /1/ the invariant measure for some abstract equation is constructed, and in /2,3/ the same measures are constructed for two physical systems. Unfortunately, in the paper /3/ some important steps of the proof are omitted. In the papers /4,5/ the invariant measure is introduced for the one-dimensional nonlinear Schrödinger equation with the polynomial nonlinearity. Measures similar to those were considered in the papers /6-9/ but with other aims and without the proof of the invariance. The paper is organised as follows. In § 2 the basic notation is introduced and the basic results are formulated. In § 3 the problem (1)-(3) is investigated. In addition, the convergence of the solutions of the finite-dimensional problem arising in the approximation of (1)-(3) to the solution of (1)-(3) is proved. In § 4 the invariant measure for the dynamical system defined by (1)-(3) is constructed. Section 5 contains some generalizations and applications to physics.

2. Notation. Basic results

Let L^2 be the real space of quadratic integrable functions defined on $[0,A]$ with the scalar product $(g,h) = \int_0^A g(x)h(x)\,dx$ and the norm $\|g\| = (g,g)^{1/2}$. We denote by Δ the closure in L^2 of the operator $-\frac{d^2}{dx^2}$ defined first on the space $C_0^\infty(0,A)$ of infinitely differentiable functions satisfying $h(0) = h(A) = 0$. Then, Δ is a self-adjoint operator on L^2.

Let $s < 0$, H^s be the supplement of L^2 according to Hausdorff with respect to the norm $\|g\|_s = \|\Delta^{s/2} g\|$. Then, H^s is a Hilbert space with the scalar product $(g,h)_s = \frac{1}{4}\{\|g+h\|_s^2 - \|g-h\|_s^2\}$. If $s > 0$, we define the space H^s on the usual way.

Let $\{e_n\}_{n=1,2,3,\dots}$ be the basis of the orthogonal normed in L^2 eigenfunctions of Δ corresponding to the eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$. Let $X_n = \mathrm{span}\{e_1, \dots, e_n\}$ and let P_n be the orthogonal projector onto X_n in L^2. In what follows we denote by $C, C_1, C_2, C', C'', \dots$ arbitrary positive constants.

Finally, we denote by $C^k(\bar{I}; \bar{X})$ (k=0,1,2,...) the Banach space of k times continuously differentiable functions $u: I \to X$, where $I \subset X$ is an interval and X is a Banach space and let $\|u(\cdot)\|_{C^k(I;X)} = \sum_{m=0}^{k} \sup_{t \in I} \left\| \frac{d^m u(t)}{dt^m} \right\|_X$, where $\|\cdot\|_X$ is the norm in X. Then, $C^k(I;X)$ is a Banach space.

The hypothesis on f consists in the following:

(f) f is a real continuously differentiable function and there exists $A > 0$ such that

$$\left| f(x,u)/(1+u^2)^{1/2} \right| + \left| \frac{\partial}{\partial u} f(x,u) \right| \le A$$

for all x, u.

To investigate the problem (1)-(3), consider the equation

$$\varphi(t) = \dot{K}(t-t_0)u_0 + K(t-t_0)u_1 - \int_{t_0}^{t} d\tau\, K(t-\tau) f(\cdot, \varphi(\tau)), \qquad (4)$$

where $K(t) = \Delta^{-1/2} \sin \Delta^{1/2} t$ and φ is the unknown function of the real argument t with values in some space of functions of x. Also let us introduce the finite-dimensional problem

$$u_{tt}^n - u_{xx}^n + P_n[f(\cdot, u^n)] = 0, \quad x \in (0,A), \; t \in R, \qquad (5)$$

$$u^n(x,t_0) = P_n u_0, \quad \frac{\partial}{\partial t} u^n(x,t_0) = \sum_{k=1}^{n} \lambda_k (u_1, e_k)_{-1} e_k. \qquad (6)$$

One can easily write the equation similar to (4) for (5)-(6).

It may be simply seen that (5)-(6) define the system of ordinary differential equations

$$\frac{d^2}{dt^2} a_k + \lambda_k a_k + f_k(\bar{a}) = 0, \quad t \in R, \tag{7}$$

$$a_k(t_0) = (u_0, e_k), \quad \frac{d a_k(t_0)}{dt} = \lambda_k (u_1, e_k)_{-1}, \quad (k = \overline{1,n}), \tag{8}$$

where $\bar{a} = (a_1, \dots, a_n)$, $f_k(\bar{a}) = \int_0^A f(x, u^n(x,t)) e_k(x)\, dx$ and $u^n = \sum_{k=1}^n a_k e_k$. Hence, (5)-(6) has a unique solution $u^n(x,t)$ which is defined for all $t \in R$ by the hypothesis (f). The first result of the paper consists in

Theorem 1

Let the hypothesis (f) be valid. Then

(a) the problem (4) has a unique solution which belongs to $C(I; L^2) \cap C^1(I; H^{-1})$ for any $T > 0$ and $I = [t_0 - T, t_0 + T]$ and for any $u_0 \in L^2$, $u_1 \in H^{-1}$;

(b) for any finite $I \subset R$, $u_0 \in L^2$, $u_1 \in H^{-1}$

$$\lim_{n \to \infty} \left\{ \sup_{t \in I} \|u^n(\cdot,t) - \varphi(t)\| + \sup_{t \in I} \left\| \frac{d}{dt}[u^n(\cdot,t) - \varphi(t)] \right\|_{-1} \right\} = 0;$$

(c) for any finite $I \subset R$, $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{t \in I} \|u_1^n(\cdot,t) - u_2^n(\cdot,t)\| + \sup_{t \in I} \left\| \frac{d}{dt}[u_1^n(\cdot,t) - u_2^n(\cdot,t)] \right\|_{-1} < \varepsilon$$

and

$$\sup_{t \in I} \|\varphi_1(t) - \varphi_2(t)\| + \sup_{t \in I} \left\| \frac{d}{dt}[\varphi_1(t) - \varphi_2(t)] \right\|_{-1} < \varepsilon$$

for any two solutions u_1^n, u_2^n of the problem (5)-(6) or for any two solutions of the problem (4) for which

$$\|u_1^n(\cdot,t_0) - u_2^n(\cdot,t_0)\| + \left\| \frac{d}{dt}[u_1^n(\cdot,t_0) - u_2^n(\cdot,t_0)] \right\|_{-1} < \delta$$

and

$$\|\varphi_1(t_0) - \varphi_2(t_0)\| + \left\| \frac{d}{dt}[\varphi_1(t_0) - \varphi_2(t_0)] \right\|_{-1} < \delta;$$

(d) the problem (4) defines the dynamical system on the phase space $(\varphi, \varphi_t') \in X = L^2 \times H^{-1}$.

Remark 1

In what follows we call the solutions of equation (4) as the generalized solutions of (1)-(3). For the reason of this definition see /10,11/, for example. Formally the connection between (1)-(3) and (4) is obvious.

Let $F(x,u) = \int_0^u f(x,p)\,dp$, $\Phi(u) = \int_0^A F(x, u(x))\, dx$. Let ν and w be the centred gaussian measures on L^2 and H^{-1} with identical correlation operators Δ^{-1}. Since this operator is nuclear ν and w are σ-additive Borel measures. Let $\rho = \nu \otimes w$ be a direct product of the measures ν and w which is the measure on X. Let

$$\mu(\Omega) = \int_\Omega e^{-\Phi(g)} \rho(dg)$$

for any Borel set $\Omega \subset X$, where $\Phi(g) = \Phi(g_1)$ if $g = (g_1, g_2)$ where $g_1 \in L^2$, $g_2 \in H^{-1}$. The basic result of the paper consists in

Theorem 2

μ is an invariant measure for the dynamical system defined by (1)-(3).

3. Proof of theorem 1

We only sketch the proof because the methods of investigation of the problems (1)-(3) and (4) are well-known (see, for example, /10,11/. One can easily see that for small $T > 0$ and $I = [t_0 - T, t_0 + T]$ the operator on the right-hand side of (4) is the contraction of the complete metric space $C(I;X)$ and the local result (a) is valid. The global existence follows from the estimates

$$\|\varphi(t)\| \le C_1 \|u_0\| + C_2 \|u_1\|_{-1} + C_3 \int_{t_0}^t \|\varphi(\tau)\|\, d\tau$$

and

$$\left\| \frac{d}{dt} \varphi(t) \right\|_{-1} \le C_1' \|u_0\| + C_2' \|u_1\|_{-1} + C_3 \int_{t_0}^t \|\varphi(\tau)\|\, d\tau.$$

Then, using the hypothesis (f) one has the inequality

$$\|f(x,\varphi_1) - f(x,\varphi_2)\| \le C \|\varphi_1 - \varphi_2\| \tag{9}$$

with $C = \text{const} > 0$ independing of $\varphi_1, \varphi_2 \in L^2$. Hence, for any $T > 0$, $I = [t_0 - T, t_0 + T]$ and for any two solutions

Remark 1

In what follows we call the solutions of equation (4) as the generalized solutions of (I)-(3). For the reason of this definition see /10,11/, for example. Formally the connection between (I)-(3) and (4) is obvious.

Let $F(x,u)=\int_0^u f(x,p)\,dp$, $\Phi(u)=\int_0^A F(x,u(x))\,dx$.

Let ν and ω be the centred gaussian measures on L^2 and H^{-1} with identical correlation operators Δ^{-1}. Since this operator is nuclear ν and ω are σ-additive Borel measures. Let $\rho=\nu\otimes\omega$ be a direct product of the measures ν and ω which is the measure on X. Let

$$\mu(\Omega)=\int_\Omega e^{-\Phi(g)}\,\rho(dg)$$

for any Borel set $\Omega\subset X$, where $\Phi(g)=\Phi(g_1)$ if $g=(g_1,g_2)$ where $g_1\in L^2$, $g_2\in H^{-1}$. The basic result of the paper consists in

Theorem 2

μ is an invariant measure for the dynamical system defined by (I)-(3).

3. Proof of theorem 1

We only sketch the proof because the methods of investigation of the problems (I)-(3) and (4) are well-known (see, for example, /10,11/. One can easily see that for small $T>0$ and $I=[t_0-T,t_0+T]$ the operator on the right-hand side of (4) is the contraction of the complete metric space $C(I;X)$ and the local result (a) is valid. The global existence follows from the estimates

$$\|\varphi(t)\|\le C_1\|u_0\|+C_2\|u_1\|_{-1}+C_3\int_{t_0}^{t}\|\varphi(\tau)\|\,d\tau$$

and

$$\left\|\frac{d}{dt}\varphi(t)\right\|_{-1}\le C_1'\|u_0\|+C_2'\|u_1\|_{-1}+C_3\int_{t_0}^{t}\|\varphi(\tau)\|\,d\tau.$$

Then, using the hypothesis (f) one has the inequality

$$\|f(x,\varphi_1)-f(x,\varphi_2)\|\le C\|\varphi_1-\varphi_2\| \qquad (9)$$

with $C=\mathrm{Const}>0$ independing of $\varphi_1,\varphi_2\in L^2$. Hence, for any $T>0$, $I=[t_0-T,t_0+T]$ and for any two solutions

φ_1 and φ_2 of (4) of the class $C(I;L^2)$ the inequality

$$\|\varphi_1(t)-\varphi_2(t)\|\le C_1\|\varphi_1(t_0)-\varphi_2(t_0)\|+C_2\left\|\frac{d}{dt}[\varphi_1(t_0)-\varphi_2(t_0)]\right\|_{-1}+C_3\int_{t_0}^{t}\|\varphi_1(\tau)-\varphi_2(\tau)\|\,d\tau$$

is valid and the statement (c) is proved. (For the problem (5)-(6) and for first derivatives the proof may be hold by analogy).

Let us prove (b). For $u^n(\cdot,t),\ \varphi(t)\in C(I;L^2)\cap C^1(I;H^{-1})$ we get by (9):

$$\|u^n(\cdot,t)-\varphi(t)\|\le C_1\|u^n(\cdot,t_0)-\varphi(t_0)\|+C_2\left\|\frac{d}{dt}[u^n(\cdot,t_0)-\varphi(t_0)]\right\|_{-1}+$$
$$+C_3\int_{t_0}^{t}\|\varphi(\tau)-u^n(\cdot,\tau)\|\,d\tau+C_4\int_{t_0}^{t}\|f(\cdot,\varphi(\tau))-P_n[f(\cdot,\varphi(\tau))]\|\,d\tau$$
$$\le a_n+C_3\int_{t_0}^{t}\|\varphi(\tau)-u^n(\cdot,\tau)\|\,d\tau,$$

where $a_n>0$ and a_n tends to 0 when $n\to\infty$, and the first result of (b) is proved. The second one follows by analogy.

Finally, (d) is valid by the proved statements, and theorem 1 is proved.

4. An invariant measure

The system (5)-(6) is hamiltonian by (7)-(8). For Borel's $A\subset X_n\times X_n$ let

$$\rho_n(A)=(2\pi)^{-n}\prod_{k=1}^{n}\lambda_k^{1/2}\int_F e^{-\frac{1}{2}\sum_{k=1}^{n}(\lambda_k x_k^2+y_k^2)}\,dx\,dy, \qquad (10)$$

where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in R^n$, dx, dy are the Lebesque measures in R^n, $F=\{(x,y)\mid(\sum_{k=1}^{n}x_ke_k,\sum_{k=1}^{n}y_ke_k)\in A\}$, and let $\mu_n(A)=\int_A e^{-\Phi(g)}\,\rho_n(dg)$.

Then, using (7)-(8) one can easily prove that μ_n is the invariant measure for (5)-(6) and that $\rho_n(X_n\times X_n)=1$.

One can define measures ρ_n,μ_n on the Borel's σ-algebra in X using the rule

$$\rho_n(A)=\rho_n(A\cap[X_n\times X_n]),\quad \mu_n(A)=\mu_n(A\cap[X_n\times X_n]).$$

φ_1 and φ_2 of (4) of the class $C(I;L^2)$ the inequality

$$\|\varphi_1(t)-\varphi_2(t)\| \le C_1\|\varphi_1(t_0)-\varphi_2(t_0)\|+C_2\left\|\frac{d}{dt}[\varphi_1(t_0)-\varphi_2(t_0)]\right\|_{-1}+C_3\int_{t_0}^{t}\|\varphi_1(\tau)-\varphi_2(\tau)\|d\tau$$

is valid and the statement (c) is proved. (For the problem (5)-(6) and for first derivatives the proof may be hold by analogy).

Let us prove (b). For $u^n(\cdot,t),\ \varphi(t)\in C(I;L^2)\cap C^1(I;H^{-1})$ we get by (9):

$$\|u^n(\cdot,t)-\varphi(t)\| \le C_1\|u^n(\cdot,t_0)-\varphi(t_0)\|+C_2\left\|\frac{d}{dt}[u^n(\cdot,t_0)-\varphi(t_0)]\right\|_{-1}+$$

$$+C_3\int_{t_0}^{t}\|\varphi(\tau)-u^n(\cdot,\tau)\|d\tau+C_4\int_{t_0}^{t}\|f(\cdot,\varphi(\tau))-P_n[f(\cdot,\varphi(\tau))]\|d\tau$$

$$\le a_n+C_3\int_{t_0}^{t}\|\varphi(\tau)-u^n(\cdot,\tau)\|d\tau,$$

where $a_n>0$ and a_n tends to 0 when $n\to\infty$, and the first result of (b) is proved. The second one follows by analogy.

Finally, (d) is valid by the proved statements, and theorem 1 is proved.

4. An invariant measure

The system (5)-(6) is hamiltonian by (7)-(8). For Borel's $A\subset X_n\times X_n$ let

$$\rho_n(A)=(2\pi)^{-n}\prod_{k=1}^{n}\lambda_k^{1/2}\int_F e^{-\frac{1}{2}\sum_{k=1}^{n}(\lambda_k x_k^2+y_k^2)}\,dx\,dy, \qquad (10)$$

where $x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in R^n$, $dx,\ dy$ are the Lebesgue measures in R^n, $F=\{(x,y)\,|\,(\sum_{k=1}^{n}x_k e_k,\sum_{k=1}^{n}y_k e_k)\in A\}$, and let $\mu_n(A)=\int_A e^{-\Phi(g)}\rho_n(dg)$.

Then, using (7)-(8) one can easily prove that μ_n is the invariant measure for (5)-(6) and that $\rho_n(X_n\times X_n)=1$.

One can define measures ρ_n,μ_n on the Borel's σ-algebra in X using the rule

$$\rho_n(A)=\rho_n(A\cap[X_n\times X_n]),\quad \mu_n(A)=\mu_n(A\cap[X_n\times X_n]).$$

Since $A\cap[X_n\times X_n]$ is open, if $A\subset X$ is open, this is correct.

Lemma 1

Let the hypothesis (f) be valid. Then, the sequence $\{\rho_n\}$ weakly covergesto ρ.

Proof

First, let us prove the weak compactness of $\{\rho_n\}$.

Let $s_1\in(0,\frac{1}{2})$, $B_R=\{(g,h)\in H^{s_1}\times H^{s_1-1}\,|\,\|g\|_{s_1}\le R,\|h\|_{s_1-1}\le R\}$ and let $\bar B_R$ be the closure of B_R in X. Then, $\bar B_R$ is a compact. By lemma II.1.1 from /12/ one has

$$\rho_n(X\setminus\bar B_R)\le\left[\frac{\mathrm{Tr}\,\Delta^{-1+s_1}}{R^2}\right]^2.$$

Hence, by the Prokhorov's theorem the sequence ρ_n is weakly compact.

Later, let $M=\{(g,h)\in X\,|\,[(g,e_{j_1}),\dots,(g,e_{j_m}),(h,\lambda_{j_{m+1}}^{1/2}e_{j_{m+1}}),\dots,(h,\lambda_{j_\ell}^{1/2}e_{j_\ell})]\in F\}$ be the cylindrical set in X where F is a Borel's set in R^ℓ and $j_k\ne j_i$ if $k,i\le m$ or $k,i>m$ and $k\ne i$. By the definition (10)

$$\rho_n(M)=(2\pi)^{-\frac{\ell}{2}}\prod_{k=1}^{n}\lambda_{j_k}^{1/2}\int_F e^{-\frac{1}{2}\sum_{k=1}^{\ell}\lambda_{j_k}x_k^2}\,dx=\rho(M),$$

where $x=(x_1,\dots,x_\ell)$, for sufficiently large n. Since there exists a unique continuation of the measure from an algebra to the minimal σ-algebra, $\rho_n\to\rho$ weakly.

Lemma 1 is proved.

Lemma 2

$$\lim_{n\to\infty}\inf\mu_n(\Omega)\ge\mu(\Omega)\qquad \text{if } \Omega\subset X \text{ is open.}$$

$$\lim_{n\to\infty}\sup\mu_n(K)\le\mu(K)\qquad \text{if } K\subset X \text{ is closed.}$$

For the proof see /13/.

Let $S(t):X\to X$ be an operator mapping $(u(\cdot,\tau),u_t'(\cdot,\tau))$ into $(u(\cdot,t+\tau),u_t'(\cdot,t+\tau))$, where $u(\cdot,t)$ is an arbitrary solution of (1)-(3). By analogy, let $S_n(t):(X_n\times X_n)\to(X_n\times X_n)$ be an operator mapping any $(u^n(\cdot,\tau),\frac{d}{dt}u^n(\cdot,\tau))$ into $(u^n(\cdot,t+\tau),\frac{d}{dt}u^n(\cdot,t+\tau))$ and let $T_n(t)=S_n(t)P_n$. By theorem 1 the operators $S(t),\ S_n(t),\ T_n(t)$ are continuous for any t.

Since $A\bigcap[X_n\times X_n]$ is open, if $A\subset X$ is open, this is correct.

Lemma 1

Let the hypothesis (f) be valid. Then, the sequence $\{\rho_n\}$ weakly coverges to ρ .

Proof

First, let us prove the weak compactness of $\{\rho_n\}$.

Let $s_1\in(0,\frac{1}{2})$, $B_R=\{(g,h)\in H^{s_1}\times H^{s_1-1}\,|\,\|g\|_{s_1}\le R, \|h\|_{s_1-1}\le R\}$ and let $\overline{B}_R$ be the closure of B_R in X . Then, $\overline{B}_R$ is a compact. By lemma II.1.1 from /12/ one has

$$\rho_n(X\setminus\overline{B}_R)\le\left[\frac{\operatorname{Tr}\Delta^{-1+s_1}}{R^2}\right]^2.$$

Hence, by the Prokhorov's theorem the sequence ρ_n is weakly compact.

Later, let $M=\{(g,h)\in X\,|\,[(g,e_{j_1}),\dots,(g,e_{j_m}),(h,\lambda^{1/2}_{j_{m+1}}e_{j_{m+1}}),\dots,(h,\lambda^{1/2}_{j_\ell}e_{j_\ell})]\in F\}$ be the cylindrical set in X where F is a Borel's set in R^ℓ and $j_k\neq j_i$ if $k,i\le m$ or $k,i>m$ and $k\neq i$. By the definition (10)

$$\rho_n(M)=(2\pi)^{-\frac{\ell}{2}}\prod_{k=1}^{n}\lambda_{j_k}^{1/2}\int_F e^{-\frac{1}{2}\sum_{k=1}^{\ell}\lambda_{j_k}x_k^2}\,dx=\rho(M),$$

where $x=(x_1,\dots,x_\ell)$, for sufficiently large n. Since there exists a unique continuation of the measure from an algebra to the minimal σ-algebra, $\rho_n\to\rho$ weakly.

Lemma 1 is proved.

Lemma 2

$$\lim_{n\to\infty}\inf\mu_n(\Omega)\ge\mu(\Omega)\qquad\text{if } \Omega\subset X \text{ is open.}$$

$$\lim_{n\to\infty}\sup\mu_n(K)\le\mu(K)\qquad\text{if } K\subset X \text{ is closed.}$$

For the proof see /13/.

Let $S(t):X\to X$ be an operator mapping $(u(\cdot,\tau),u'_t(\cdot,\tau))$ into $(u(\cdot,t+\tau),u'_t(\cdot,t+\tau))$, where $u(\cdot,t)$ is an arbitrary solution of (1)-(3). By analogy, let $S_n(t):(X_n\times X_n)\to(X_n\times X_n)$ be an operator mapping any $(u^n(\cdot,\tau),\frac{d}{dt}u^n(\cdot,\tau))$ into $(u^n(\cdot,t+\tau),\frac{d}{dt}u^n(\cdot,t+\tau))$ and let $T_n(t)=S_n(t)P_n$. By theorem 1 the operators $S(t)$, $S_n(t)$, $T_n(t)$ are continuous for any t .

Proof of theorem 2

Let $\Omega(t_1)$ be open, $\Omega(t_2)=S(t_2-t_1)\Omega(t_1)$, $\mu(\Omega(t_1))<\infty$. By theorem 1 $\Omega(t_2)$ is open, too. Let us fix $\varepsilon>0$. There exists a compact $K_1\subset\Omega(t_1)$ such that $\mu(\Omega(t_1)\setminus K_1)<\varepsilon$. Let $K_2=S(t_2-t_1)K_1$. Then, $K_2\subset\Omega(t_2)$ is a compact. Let

$$\alpha=\min\{\operatorname{dist}(K_1,\partial\Omega(t_1))\ ;\ \operatorname{dist}(K_2,\partial\Omega(t_2))\}.$$

Then $\alpha>0$. By theorem 1 for any $g\in K_1$ there exists a ball $B(g)\subset\Omega(t_1)$ such that $\operatorname{dist}(T_n(t_2-t_1)g,\ T_n(t_2-t_1)h)<\frac{\alpha}{3}$ for all $h\in B(g)$ and for all h . Let $\Omega_\beta(t_2)=\{g\in\Omega(t_2)\,|\,\operatorname{dist}(g,\partial\Omega(t_2))\ge\beta\}$ and let $B(g_1),\dots,B(g_\ell)$ be a finite covering of K_1 by the balls, $D=\bigcup_{i=1}^{\ell}B(g_i)$. By construction $T_n(t_2-t_1)D\subset\Omega_{\frac{\alpha}{4}}(t_2)$ for all sufficiently large n . Then, by lemma 2

$$\mu(\Omega(t_1))\le\mu(D)+\varepsilon\le\lim_{n\to\infty}\inf\mu_n(D)+\varepsilon=$$

$$=\lim_{n\to\infty}\inf\mu_n(S_n(t_2-t_1)[D\bigcap(X_n\times X_n)])+\varepsilon\le\mu(\Omega(t_2))+\varepsilon$$

Due to the arbitrariness of t_1,t_2 and $\varepsilon>0$ one gets the equality

$$\mu(\Omega(t_1))=\mu(\Omega(t_2)).\tag{11}$$

For any Borel's set $\Omega(t_1)\subset X$ we get the equality (11) approximating $\Omega(t_i)$ $(i=1,2)$ by open sets from outside and by closed sets from inside.

Thus, theorem 2 is proved.

5. Generalizations and applications

For the proof of theorem 2 the strong hypothesis (f) was assumed. In fact, this assumption was only used to prove theorem 1. Let us formulate the condition.

(C) Let a continuously differentiable function $f(x,u)$ be such that theorem 1 (a) is valid and let there exists a sequence $f_N(x,u)$ converging to $f(x,u)$ for any x,u and satisfying (f) with the following property: for any $u_0\in L_2^2$, $u_1\in H^{-1}$ and $T>0$ the sequence $u_N(x,t)$ converges to $u(x,t)$ in

Proof of theorem 2

Let $\Omega(t_1)$ be open, $\Omega(t_2)=S(t_2-t_1)\Omega(t_1)$, $\mu(\Omega(t_1))<\infty$. By theorem 1 $\Omega(t_2)$ is open, too. Let us fix $\varepsilon>0$. There exists a compact $K_1\subset\Omega(t_1)$ such that $\mu(\Omega(t_1)\setminus K_1)<\varepsilon$. Let $K_2=S(t_2-t_1)K_1$. Then, $K_2\subset\Omega(t_2)$ is a compact. Let

$$\alpha=\min\{dist(K_1,\partial\Omega(t_1));\ dist(K_2,\partial\Omega(t_2))\}.$$

Then $\alpha>0$. By theorem 1 for any $g\in K_1$ there exists a ball $B(g)\subset\Omega(t_1)$ such that $dist(T_n(t_2-t_1)g,\ T_n(t_2-t_1)h)<\frac{\alpha}{3}$ for all $h\in B(g)$ and for all n. Let $\Omega_\beta(t_2)=\{g\in\Omega(t_2)\mid dist(g,\partial\Omega(t_2))\ge\beta\}$ and let $B(g_1),\dots,B(g_l)$ be a finite covering of K_1 by the balls, $D=\bigcup_{i=1}^{l}B(g_i)$. By construction $T_n(t_2-t_1)D\subset\Omega_{\frac{\alpha}{4}}(t_2)$ for all sufficiently large n. Then, by lemma 2

$$\mu(\Omega(t_1))\le\mu(D)+\varepsilon\le\liminf_{n\to\infty}\mu_n(D)+\varepsilon=$$

$$=\liminf_{n\to\infty}\mu_n(S_n(t_2-t_1)[D\cap(X_n\times X_n)])+\varepsilon\le\mu(\Omega(t_2))+\varepsilon$$

Due to the arbitrariness of t_1,t_2 and $\varepsilon>0$ one gets the equality

$$\mu(\Omega(t_1))=\mu(\Omega(t_2)). \qquad (11)$$

For any Borel's set $\Omega(t_1)\subset X$ we get the equality (11) approximating $\Omega(t_i)$ $(i=1,2)$ by open sets from outside and by closed sets from inside.

Thus, theorem 2 is proved.

5. Generalizations and applications

For the proof of theorem 2 the strong hypothesis (f) was assumed. In fact, this assumption was only used to prove theorem 1. Let us formulate the condition.

(C) Let a continuously differentiable function $f(x,u)$ be such that theorem 1 (a) is valid and let there exists a sequence $f_N(x,u)$ converging to $f(x,u)$ for any x,u and satisfying (f) with the following property: for any $u_0\in L^2$, $u_1\in H^{-1}$ and $T>0$ the sequence $u_N(x,t)$ converges to $u(x,t)$ in

$C([t_0-T,t_0+T];L^2)\cap C^1([t_0-T,t_0+T];H^{-1})$, where u_N is a solution of (1)-(3) corresponding to $f=f_N$. Under the assumption (C) one can construct an invariant measure for (1)-(3) so as in the paper [5] for the nonlinear Schrödinger equation. For the application one can use the Poincare recurrence theorem.

Theorem 3 [14]

Let f be such that $\mu(X)<\infty$. Then, almost all points of X are stable in the Poisson sense.

This is an important result for the theory of "soliton" equations. There exists an old observation by Fermi, Past and Ulam. These authors considered a chain of balls with a nonlinear interaction between them. They discovered the phenomenon when an arbitrary solution of the Cauchy problem from time to time returns back to its initial data with any accuracy. Later, in the soliton theory this return was called the Fermi - Past - Ulam phenomenon. By computer simulation it was observed for many "soliton" equations (see [15]).

And finally, theorem 2 is valid for two physical nonlinearities $f(x,u)=au-\frac{u^2}{1+u^2}u$ and $f(x,u)=au-ue^{-\alpha u^2}$, where a,α are positive constants [16]. We remark that the methods of this paper are applicable to the nonlinear Schrödinger equation.

References

1. Arsen'ev A.A. On the invariant measures for the classical dynamical systems with infinite-dimensional phase space. Matem. Sbornik. v.121, 297 (1983) (in Russian).
2. Casati G., Guarneri I., Valz-Griz F. Preliminaries to the ergodic theory of the infinite-dimensional systems: A model of radiant cavity. J.Statist.Phys. v.30, No.1, 195 (1983).
3. Friedlander L. An invariant measure for the equation $u_{tt}-u_{xx}+u^3=0$. Commun.Math.Phys. v.98, 1 (1985).
4. Zhidkov P.E. On the invariant measure for the nonlinear Schrödinger equation. Doklady Akad.Nauk SSSR. v. 317, No.3, 543 (1991). Preprint JINR, E5-92-174, Dubna (1991).
5. Zhidkov P.E. A remark on the invariant measure for the nonlinear Schrödinger equation. Prepr.JINR, E5-92-23, Dubna (1992). To appear in "Different.Uravn."
6. Chueshov I.D. Equilibrium statistical solutions for dynamical systems with an infinite number of degrees of freedom. Math.USSR Sbornik, v.58, No.2, p.397 (1987).

7. Chueshov I.D. On the structure of equilibrium states for a class of dynamical systems connected with Lee – Poisson brackets. Teor.Matem.Fiz., v.75, No.3, 445. (1988) (in Russian).
8. Peskov N.V. On the Kubo – Martin – Schwinger state of the sin-Gordon system. Teor.Matem.Fiz. v.64, No.1, 32 (1985) (in Russian).
9. Lebowitz J.L., Rose R.A., Speer E.R. Statistical mechanics of the nonlinear Schrödinger equation. J.Statist.Phys. v.50, 657 (1988).
10. Reed M., Simon B. Methods of Modern mathematical physics, v.2, Acad.Press, New York (1975).
11. Ginibre J., Velo G. The global Cauchy problem for the non-linear Klein – Gordon equation. Math.Z. v.189, 487 (1985).
12. Duletskii Yu.L., Fomin S.V. Measures and differential equations in infinite-dimensional spaces. Moscow, Nauka (1983) (in Russian).
13. Prokhorov Yu.V., Rosanov Yu.A. Probability theory. Moscow, Nauka (1987) (in Russian).
14. Halmos P.R. Lectures on ergodic theory. Tokyo (1956).
15. Dodd R.K., Eilbeck J.C., Gibbon J.D., Morris H.C. Solitons and nonlinear wave equations. London Acad.Press (1984).
16. Michalache D., Nazmitdinov R.G., Fedyanin V.K. Nonlinear optical waves in layered structures. Prepr. JINR, E17-88-66, Dubna (1988).

Received by Publishing Department
on July 15, 1992.

Жидков П.Е. E5-92-305

Инвариантная мера для нелинейного волнового уравнения

Приведены достаточные условия корректности смешанной задачи

$u_{tt} - u_{xx} + f(x, u) = 0$, $x \in (0,A)$, $t \in R$,

$u(0,t) = u(A,t) = 0$,

$u(x,t_0) = u(x)$, $u^1_t(x,t_0) = u_1(x)$.

Построена инвариантная борелевская мера для динамической системы (с бесконечномерным фазовым пространством), определяемой этим уравнением. Важным приложением этого результата является теорема о возвращении Пуанкаре. Работа является продолжением нескольких публикаций автора на эту тему.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1992

Zhidkov P.E. E5-92-305

An Invariant Measure for a Nonlinear Wave Equation

Sufficient conditions for the correctness of the initial-boundary value problem $u_{tt} - u_{xx} + f(x,u) = 0$, $x \in (0,A)$, $t \in R$, $u(0,t) = u(A,t) = 0$,

$u(x,t_0) = u_0(x)$, $u^1_t(x,t_0) = u_1(x)$

are formulated. An invariant Borel measure is constructed for the dynamical system (with the infinite-dimensional phase space) defined by this equation. As an important application, the Poincare recurrence theorem follows from this result. The investigation is a continuation of several author´s papers on this matter.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1992