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Замечание об инвариантной мере для нелинейного уравнения Шредингера

Строится инвариантная мера для нелинейного уравнения Шредингера со степенной нелинейностью. Доказательство опирается на более раннюю статью автора. Полученный результат позволяет применить теорему о возвращении Пуанкаре к нелинейному уравнению Шредингера.

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A Remark on the Invariant Measure for the Nonlinear Schrödinger Equation

The invariant measure for the nonlinear Schrödinger equation with the power nonlinearity is constructed. The proof is based on the earlier author's paper. Due to the result obtained one can apply the Poincare recurrence theorem to the nonlinear Schrödinger equation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1°. In this paper the invariant measure for the dynamical system defined by the nonlinear Schrödinger equation is constructed. Really, the polynomial nonlinearities are considered. In the previous author's paper [1] the analogical result was proved for stronger assumptions. There are a lot of papers on this matter for other equations of mathematical physics [2–4]. Unfortunately, in the paper [2] the proofs of the important properties are omitted. In the papers [3,4] the invariant measures for the Vlasov equation and the Euler equations are constructed, respectively.

Let us consider the following problem:

$$iu_t + u_{xx} + f(x,|u|^2)u = 0, \quad x \in (0,A),\ t \in R, \qquad \text{(I)}$$

$$u(x,0) = u_0(x) \qquad (2)$$

with two kinds of the boundary conditions:

(a) the vanishing boundary conditions

$$u(0,t) = u(A,t) = 0; \qquad (3)$$

(b) the periodic problem with periodic functions $f(x,s)$, $u_0(x)$ with respect to x with a period $A>0$; it is required in this case that the solution $u(x,t)$ is periodic with respect to x with the same period A.

In what follows let us consider the problem (I)–(3) because the periodic problem may be treated by analogy.

Let us formulate the basic assumption.

(A) Let the function f be such that for any $u_0 \in L_2(0,A)$ there exists a unique solution $u(x,t)$ of the problem (I)–(3) defined for all $t \in R$ of the class $C(R; L_2(0,A))$ Such that $|u|_{L_2} = \{\int_0^A |u(x,t)|^2 dx\}^{1/2}$ does not depend on t. Let the operator $S(t): u_0 \to u(x,t)$ from $L_2(0,A)$ to $L_2(0,A)$ be continuous for each fixed t. Let there exist a sequence of continuously differentiable functions $f_N(x,s)$ converging to $f(x,s)$ for any fixed x, s, such that for any fixed N the problem

(I)-(3) has a unique solution $u_N(x,t)\in C(R;L_2(0,A))$ converging to $u(x,t)$, when $N\to\infty$ for a fixed t in $L_2(0,A)$, and let there exist $C_N>0$ such that

$$|f_N(x,s)| + \left|\frac{\partial}{\partial x} f_N(x,s)\right| + \left|(1+s)\frac{\partial}{\partial s} f_N(x,s)\right| \le C_N$$

for all x, s.

Remark 1

For $f(x,|u|^2)u = \lambda |u|^{\kappa} u$, $\kappa \in (0,4)$, the assumption (A) was proved in the paper [5] for the Cauchy problem (with $x\in(-\infty,\infty)$ and the space $L_2(-\infty,\infty)$ in place of $L_2(0,A)$). For this function f the proof is valid for the problem (I)-(3).

Let us rewrite the problem (I)-(3) for $v=\mathrm{Re}\,u,\ w=\mathrm{Im}\,u$:

$$v_t + w_{xx} + f(x,v^2+w^2)w = 0, \tag{4}$$

$$w_t - v_{xx} - f(x,v^2+w^2)v = 0,\ x\in(0,A),\ t\in R, \tag{5}$$

$$v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \tag{6}$$

$$v(0,t) = v(A,t) = w(0,t) = w(A,t) = 0. \tag{7}$$

One can easily formulate the assumption (A) for the problem (4)-(7).

Let $F(x,s) = \frac{1}{2}\int_0^s f(x,p)\,dp$, $\Phi(u) = \int_0^A F(x,|u(x)|^2)\,dx$.

On the phase space $L = L_2(0,A)\times L_2(0,A)$ (here $L_2(0,A)$ is the space of real functions) consider the centered gaussian measure with the correlation operator $B = \left(-\frac{d^2}{dx^2}\right)^{-1}$ (the operator $-\frac{d^2}{dx^2}$ is taken with the vanishing boundary conditions on L). It is proved [1] that the Borel measure

$$\mu_N(\Omega) = \int_\Omega e^{\Phi_N(u)}\, w(du)$$

is invariant for the problem (4)-(7) with $f=f_N$ (here $\Phi_N(u) = \int_0^A F_N(x,|u(x)|^2)\,dx$, $F_N(x,s) = \frac{1}{2}\int_0^s f_N(x,p)\,dp$).

Let $S_N(t): u_0 \to u(x,t)$ be the operator of the evolution for the problem (4)-(7) with $f=f_N$ (u_N is the solution of the problem (4)-(7) corresponding to $f=f_N$).

The invariance of the measure means that $\mu_N(S_N(t)\Omega) = \mu_N(\Omega)$ for any $t\in R$ and Borel's set $\Omega\subset L$. Let us consider the measure μ on L:

$$\mu(\Omega) = \int_\Omega e^{\Phi(u)}\, w(du).$$

The basic result of the paper consists of

Theorem

Let the assumption $(A)_1$ be valid and let $C_1(1+|s|^d) \le f(x,s) \le C_2(1+|s|^{1-d_1})$ for some $C_1<0<C_2$, $d>0$, $0<d_1<1$ and for all x, s. Then, the measure μ is invariant for the problem (4)-(7).

Example

The assumptions of the theorem are valid for the equation

$$iu_t + u_{xx} - |u|^2 u = 0.$$

2°. Let us prove the theorem. Let $|u|_L = \left\{\int_0^A [v^2(x) + w^2(x)]\,dx\right\}^{1/2}$ be the norm in L and let $B_\rho = \{u\in L \mid |u|_L \le \rho\}$.

Lemma 1

$\lim_{N\to\infty} \mu_N(\Omega) = \mu(\Omega)$ for any $\rho>0$ and for any measurable $\Omega\subset B_\rho$.

Proof

Admitting the value $\Phi(u) = +\infty$ we shall get that for any $u\in L$ $\Phi(u) = \lim_{N\to\infty}\Phi_N(u)$; hence, $\Phi(u)$ is a measurable functional. After that the statement of the lemma is followed by the Lebesgue theorem.

Let $\Omega_N = S_N(t_2-t_1)\,\Omega(t_1)$, $A_K = \bigcap_{N\ge K}\Omega_N$, $A = \bigcup_{K\ge 1} A_K$.

It is clear that $A_1\subset A_2\subset\ldots\subset A_K\subset\ldots$.

Lemma 2

Let $\Omega(t_2)$ be open. Then, $\Omega(t_2)\subset A\subset\overline{\Omega}(t_2)$, where $\overline{\Omega}(t_2)$ is the closure of $\Omega(t_2)$.

Proof follows from the assumption (A).

Using lemmas 1,2 we get the sequence of inequalities

$$\mu_N(\Omega(t_1)) = \mu_N(\Omega_N) \geq \mu_N(A_k) \qquad (N \geq k).$$

Hence, by lemma 1 for any $k \geq 1$

$$\mu(\Omega(t_1)) \geq \mu(A_k)$$

and then the inequality

$$\mu(\Omega(t_1)) \geq \mu(A) \geq \mu(\Omega(t_2))$$

follows.

Due to the arbitrariness of t_1, t_2 it implies

$$\mu(\Omega(t_1)) = \mu(\Omega(t_2)) \tag{8}$$

for arbitrary open $\Omega \subset B_\rho$. Obviously, (8) is valid for closed $\Omega \subset B_\rho$, too.

For an arbitrary Borel set $\Omega \subset B_\rho$ we get the equality (8) by the approximation of the set Ω by closed sets from within and by open sets from outside.

Theorem is proved.

3°. Let us prove

Proposition

Under the assumptions of the theorem $0 < \mu(B_\rho) < \infty$ for any $\rho \in (0, \infty)$.

Proof

Let Δ be the closure of the operator $-\frac{d^2}{dx^2}$ with the vanishing boundary conditions on L. Then, Δ is a self-adjoint operator. Let us denote by H^α the space of elements $u \in L$ such that $\Delta^{\frac{\alpha}{2}} u \in L$, and let $(u,v)_\alpha = (u,v) + (\Delta^{\frac{\alpha}{2}} u, \Delta^{\frac{\alpha}{2}} v)$, $\|u\|_\alpha = (u,u)_\alpha$ be the scalar product and the norm in H^α, where $(u,v) = \int_0^A \{u_1(x)v_1(x) + u_2(x)v_2(x)\}dx$, $u = (u_1, u_2), v = (v_1, v_2)$.

As it is proved in [1] (see [6], too), for any $\alpha \in (0, \frac{1}{2})$ $w(H^\alpha) = 1 = w(L)$; hence, for any Borel $\Omega \subset L$

$$w(\Omega) = w(\Omega \cap H^\alpha).$$

Furthermore, we have

$$w(B_\rho) = w(B_\rho \cap H^\alpha) = \sum_{k=1}^{\infty} w(B_k) \tag{9}$$

where $B_k = \{u \in B_\rho \cap H^\alpha \mid \|u\|_\alpha \in [k-1, k)\}$. Then, by

the embedding theorem there exist $\alpha \in (0, \frac{1}{2})$, $C > 0$ such that

$$\int_0^A |u|^{2\alpha+2} dx \leq C \|u\|_\alpha^{2\alpha+2} \tag{10}$$

for each $u \in H^\alpha$. By (10) and by the assumptions on f for any k there exists $a_k > 0$ such that

$$e^{\Phi(u)} > a_k \tag{11}$$

for any $u \in B_k$. Using (9)-(11) and the inequality $w(B_\rho) > 0$ [1] we get $\mu(B_\rho) > 0$.

Furthermore, by the theorem's conditions on f and the embedding theorem we have

$$e^{\Phi(u)} \leq C_1 e^{C_2(\rho)\|u\|_{1/2-\tau}^{2-\gamma}}, \quad \gamma, \tau, C_1, C_2 > 0,$$

hence (see [6], ch.III, theorem 3.1)

$$\mu(B_\rho) < \infty$$

Proposition is proved.

It follows from the proved theorem and proposition that one can apply the Poincare's recurrence theorem for each ball B_ρ [7].

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