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STABILITY CONDITIONS FOR METHOD OF SELF-SIMILAR APPROXIMATIONS

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1. Introduction

Recently a new method[1] has been suggested for finding out an effective limit of divergent or poorly convergent sequences. This method, called the method of self - similar approximations, possesses the following main advantages. First, it is formulated as a general approach that may be applied to arbitrary sequences. Second, it needs to know only the first few terms of a sequence. The latter advantage is especially important for those complicated problems when one is not able to calculate many terms, and the usual resummation techniques fail. The standard difficulties is such a case are beautifully described by Stevenson[2, 3]. Third, the method of self - similar approximations[1] is constructed so that to force a sequence to converge as fast as possible, thus providing a maximal accuracy extracted from a minimal information. The convergence is to be governed by specially introduced governing functions.

Although the general formulation of the method includes the fast convergence, this can be disturbed in particular realizations, for example, by an inadequate choice of the governing functions. Therefore, it is necessary for practical purposes to define the conditions using which one could check the convergence in each concrete case. The conditions of this kind in the mapping theory, dynamical theory or renormalization - group theory are called the contraction conditions or stability conditions. The aim of the present paper is to define such conditions for the method of self - similar approximations[1] and to illustrate their use.

The paper is organized as follows. In Section 2 a generalized construction of the method is produced. The details that are thoroughly explained in Ref.1 are, of course, omitted, but the points generalizing the derivation are stressed. As a result of the generalization, a new parameter appears showing the number of steps needed to reach an effective limit of a sequence starting from its k -th term. The sought effective limit of a sequence plays the role of a fixed point for the self - similar mapping[1]; this is why the appearing new parameter showing the distance of the fixed point apart an approximate term can be called the fixed - point distance. In Section 3 the stability conditions controlling the convergence are formulated. The condition for the self - similar mapping to be contracting checks the choice of the governing functions, while the Lyapunov stability condition for the differential form of the self - similar relation regulates the option of a distribution function which is inversely proportional to the Gell - Mann - Low function. Section 4 considers two main variants of the fixed - point conditions defining the governing functions: the principle of minimal difference[4, 5] and the principle of minimal sensitivity[6, 7]. The former variant is analysed from the point of view of stability using as an illustration the anharmonic oscillator problem in Section 5. The accuracy of the method based on this choice of the governing function can be essentially improved by treating the fixed - point distance as a continuous parameter. The value of the latter may be found from the strong coupling limit. The fixed - point condition in the form of the principle of minimal sensitivity is examined in Section 6. It is demonstrated that this variant is preferable since it makes the method stable and accurate and does not need additional parameters, like the fixed - point distance. Finally, in Section 7, the optimal general scheme is described of the stable way which one should follow applying the method of self - similar approximations.

2. Self-Similar Approximation

Here the main steps of the derivation of the method[1] are adduced with an emphasis on the novelties generalizing this approach. Let us be interested in a function $f(g)$ of the variable $g \in I\!R$. Suppose the function $f(g)$ is a solution of a very complicated equation which cannot be solved

exactly. Invoking some iterative procedure or perturbation theory one can construct a sequence of approximations $\{f_k(g)|k = 0, 1, 2, \ldots\}$. If this sequence were convergent, $f(g)$ would be its limit. However, in many realistic cases such sequences diverge. To make a sequence convergent, we have to renormalize it by introducing an additional sequence of governing functions $\{z_k(g)|k = 0, 1, 2, \ldots\}$. According to their role, the governing functions should govern the convergence of a renormalized sequence formed by the functions

$$f_k(g) = f_k(g, z_k(g)). \tag{1}$$

Generally, it is possible to introduce a set of governing functions

$$z_k(g) = \{z_k^\alpha(g)|\alpha = 1, 2, 3, \ldots\}.$$

Define the coupling function $g(f)$ by the equation

$$f_0(g, z(g)) = f; \quad g = g(f), \tag{2}$$

where the notation

$$z(g) \equiv z_0(g) \equiv z_1(g), \tag{3}$$

is used. The substitution of the coupling function into (1) specifies the function

$$\bar{f}_k(f) \equiv f_k(g(f), z_k(g(f))), \tag{4}$$

for which (2) takes the form

$$\bar{f}_0(f) = f. \tag{5}$$

Function (4) is invented in order to be able to write[1] the fastest convergence condition, directly following from the Cauchy criterion, as the property of the functional self - similarity

$$\bar{f}_{k+p}(f) = \bar{f}_k(\bar{f}_p(f)). \tag{6}$$

It would be convenient to rewrite Eq.(6) with additive indices as a self - similar relation with multiplicative indices. To this end, we need to have a new variable $t_k = t(k)$ satisfying the property

$$t_{k+p} = t_k t_p; \qquad t_0 = 1; \qquad t_k > 1 \qquad (k > 0). \tag{7}$$

From (7) it follows that

$$t_k = a^k; \qquad a > 1. \tag{8}$$

Remind that in Ref.1 a particular form of (8) was taken, when $t_k = e^k$. Introducing the notation

$$z(t_k, g) \equiv z_k(g),$$

$$f(t_k, g, z(t_k, g)) \equiv f_k(g, z_k(g)),$$

$$\bar{f}\,(t_k, f) \equiv \bar{f}_k\,(f), \tag{9}$$

we rewrite (6) as the property of functional self - similarity with multiplicative indices

$$\bar{f}\,(t_k t_p, f) = \bar{f}\,(t_k, \bar{f}\,(t_p, f)). \tag{10}$$

Then from the discrete variable t_k we pass to the continuous variable t by the substitution

$$t_k \to t \in [1, \infty) \tag{11}$$

accompanied by the analytical continuation of all functions depending on t_k to functions depending on t so that when t crosses t_k, the continued functions of t cross the values coinciding with the corresponding initial functions of t_k. In this way, the analytical continuation of (4) is the function

$$\bar{f}\,(t, f) \equiv f(t, g(f), z(t, g(f))), \tag{12}$$

and the continuous analog of (5) is the equality

$$\bar{f}\,(1, f) = f. \tag{13}$$

The continuous representation of (10) is the self-similar relation

$$\bar{f}\,(\mu t, f) = \bar{f}\,(t, \bar{f}\,(\mu, f)) \tag{14}$$

with $\mu \geq 1$.

The functional equation (14) can also be written in the differential form

$$\frac{\partial\, \bar{f}\,(t, f)}{\partial \ln t} = \beta(\bar{f}\,(t, f)), \tag{15}$$

in which

$$\beta(f) \equiv \left[\frac{\partial\, \bar{f}\,(t, f)}{\partial \ln t}\right]_{t=1} \tag{16}$$

is the Gell - Mann - Low function.

The sought effective limit of sequence (1) is the self - similar approximation $f_*(g)$ playing the role of a fixed point of relation (14), which is reached at $t = t_*$ called the saturation point[1]. Integrating (14) over t from t_k up to t_*, when function (12) changes from the approximation $f_k(g)$ to the self - similar approximation $f_*(g)$, we have

$$\int_{f_k(g)}^{f_*(g)} \frac{df}{\beta(f)} = \ln \frac{t_*}{t_k}. \tag{17}$$

Returning to the discrete representation we have to replace the derivatives in (15) or (16) by the corresponding finite differences with respect to the variation of the discrete variable

$$\tau_k \equiv \ln t_k = k \ln a. \tag{18}$$

Then for the Gell - Mann - Low function (16) we can write the finite - difference representation

$$\beta_{sk}(f) = \Delta_{sk}(f)/(\tau_s - \tau_k), \tag{19}$$

in which

$$\Delta_{sk}(f) = f_s(g, z_k) - f_k(g, z_k) + (z_s - z_k)\frac{\partial}{\partial z_k} f_k(g, z_k), \qquad (20)$$

where

$$g = g(f), \qquad z_k = z_k(g(f)); \qquad k < s. \qquad (21)$$

As is seen, if the left - hand side of (15) is written in the discrete representation and function (19) is used for the right - hand side, then equation (15) becomes an identity.

For what follows it is useful to introduce the function

$$y_{sk}(f) \equiv \{\beta_{sk}(f)\ln(t_*/t_k)\}^{-1} \qquad (22)$$

and the distance

$$\delta_k \equiv k_* - k; \qquad k_* \equiv \ln t_* \qquad (23)$$

indicating the number of iterative steps needed for reaching the fixed point $f_*(g)$ starting from the k - th approximate term. I shall call (23) the fixed - point distance. With (19) - (21) and (23) function (22) reads

$$y_{sk}(f) = (s-k)/\delta_k\Delta_{sk}(f). \qquad (24)$$

Note that if in (23) we replace the saturation number k_* by s and, consequently, the fixed - point distance δ_k by $s-k$, then we return to the case of Ref.1. In general, the fixed - point distance can be treated as an additional fitting parameter, not necessarily being an integer.

Thus, from equation (17) we obtain the normalization condition

$$\int_{f_k(g)}^{f_*(g)} y_{sk}(f)df = 1 \qquad (25)$$

for function (24). The latter function can be called the distribution of approximations since it shows the distribution of approximate functions

between $f_k(g)$ and $f_*(g)$. Normalization (25) is the main equation defining the self - similar approximation $f_*(g)$.

Strictly speaking, as is clear from (25), each distribution of approximations $y_{sk}(f)$ defines the corresponding self - similar approximation $f_*^{sk}(g)$. When we consider a sole fixed distribution $y_{sk}(f)$, we may write the self - similar approximation simply as $f_*(g)$, without indices. However, if we take a set of distributions $y_{sk}(f)$ with different indices, we shall obtain a sequence of self - similar approximations. For example, we can get the sequence $f_*^{21}(g), f_*^{31}(g), f_*^{41}(g), \ldots$ or $f_*^{21}(g), f_*^{32}(g), f_*^{43}(g), \ldots$, or $f_*^{31}(g), f_*^{42}(g), f_*^{53}(g), \ldots$ or other sequences. It seems that the simplest way is to construct the sequence $\{f_*^{(k)}(g)\}$ with $f_*^{(k)}(g) \equiv f_*^{k+1k}(g)$, though a thorough investigation of these possibilities is a separate problem to be considered in another paper.

3. Stability Conditions

By construction of the method it is assumed that one can find the governing functions such that the sequence of functions (1) would fastly converge to an effective limit called the self - similar approximation because it is a fixed - point of the self - similar mapping (6). This means that the fixed point is to be stable.

However, in each concrete case we always deal with a particular choice of the governing functions. An inadequate choice of these functions can disturb the assumption of stability. Therefore, it is necessary to know the general stability conditions providing the convergence of the sequence $\{f_k(g, z_k(g))\}$ and making it possible to check the adequacy of a particular option of the governing functions for each concrete problem.

The self - similar approximation $f_*(g)$ in terms of notation (4) reads

$$f_*(g(f)) = \bar{f}_*(f), \qquad (26)$$

where the coupling function $g(f)$ is given by equation (2). The nomination of (26) as a fixed point of mapping (6) implies

$$\bar{f}_k(\bar{f}_*(f)) = \bar{f}_*(f). \tag{27}$$

Considering a variation near the fixed point,

$$\bar{f}_p(f) = \bar{f}_*(f) + \delta\bar{f}_p(f), \tag{28}$$

we have from (6)

$$\delta\bar{f}_{k+p}(f) = \left[\frac{d\bar{f}_k(\xi)}{d\xi}\right]_{\xi=\bar{f}_*(f)} \delta\bar{f}_p(f). \tag{29}$$

The self - similar mapping (6) defines a convergent sequence when this mapping is contracting[8], which needs

$$\lim_{\xi\to\bar{f}_*(f)} \left|\frac{d\bar{f}(\xi)}{d\xi}\right| < 1. \tag{30}$$

For the mapping multipliers

$$M_k(g) \equiv \lim_{f\to f_*(g)} \left|\frac{d}{df} f_k(g(f), z_k(g(f)))\right|. \tag{31}$$

condition (30) reads

$$M_k(g) < 1. \tag{32}$$

We could equally consider the continuous representation (12) for which the fixed point is defined by the equation

$$\bar{f}(t, \bar{f}_*(f)) = \bar{f}_*(f).$$

Then, substituting the variation

$$\bar{f}(t,f) = \bar{f}_*(f) + \delta\bar{f}(t,f)$$

into the continuous self -similar mapping (14), we have

$$\delta\bar{f}(\mu t, f) = \left[\frac{\partial\bar{f}(t,\xi)}{\partial\xi}\right]_{\xi=\bar{f}_*(f)} \delta\bar{f}(\mu, f).$$

Returning to the discrete representation we again obtain the contraction condition (32) for the mapping multipliers (31).

An additional condition follows from the analysis of the continuous representation if we treat the differential equation (15) as the law of motion for the function $\bar{f}(t,f)$ with respect to the variable $\tau \equiv \ln t$. Then we can use the Lyapunov theory of stability[9]. Linearizing (15) we get

$$\frac{\partial}{\partial\tau}\delta\bar{f}(t,f) = \left[\frac{d\beta(\xi)}{d\xi}\right]_{\xi=\bar{f}_*(f)} \delta\bar{f}(t,f). \tag{33}$$

The motion to be stable requires

$$\lim_{\xi\to\bar{f}_*(f)} \frac{d\beta(\xi)}{d\xi} < 0. \tag{34}$$

This condition, after returning to the discrete representation (19), introducing the Lyapunov exponent

$$\Lambda_{sk}(g) \equiv \lim_{f\to f_*(g)} \frac{d}{df}\Delta_{sk}(f) \tag{35}$$

and taking into account that $\tau_s > \tau_k$, yields

$$\Lambda_{sk}(g) < 0. \tag{36}$$

Inequalities (32) and (36) are sufficient conditions for the self - similar approximation $f_*(g)$ to be a stable fixed point of the self - similar mapping. In this case $f_*(g)$ is the sought effective limit of sequence (1). If conditions (32) and (36) are valid, we shall say for breavity that the method is stable. It is unstable when either (32) or (36) is not valid. When one of the equalities, $M_k(g) = 1$ or $\Lambda_{sk}(g) = 0$, is true, then we have a marginal situation and can say nothing about stability and convergence.

The accuracy of the self - similar approximation $f_*(g)$, as compared to the exact value $f(g)$, is defined by the error

$$\epsilon_*(g) \equiv \left| \frac{f_*(g)}{f(g)} - 1 \right|. \tag{37}$$

It is reasonable to define the accuracy of the method as a whole by the maximal error

$$\epsilon_* \equiv \sup_g \epsilon_*(g). \tag{38}$$

Analogously to (37) and (38), one can check the accuracy of the k-th approximation

$$\epsilon_k \equiv \sup_g \left| \frac{f_k(g)}{f(g)} - 1 \right|. \tag{39}$$

When the method is stable, then

$$\epsilon_1 > \epsilon_2 > \ldots > \epsilon_*. \tag{40}$$

For an unstable method Eq.(40) does not hold.

4. Fixed - Point Conditions

As is evident, it is very important to define the governing functions so as to provide the stability of the method. In accordance with the general approach[1], the governing functions are to be defined by a fixed - point condition. In the continuous representation, as follows from (15), the fixed point is given by zero of the Gell - Mann -Low function,

$$\beta(\bar{f}\,(t, f)) = 0. \tag{41}$$

In the discrete representation, with regard to (19) and (20), the fixed - point condition (41) can be written in two simple forms, either as the

principle of minimal difference[4, 5]

$$f_p(g, z_k) - f_k(g, z_k) = 0 \tag{42}$$

or as the principle of minimal sensitivity[6, 7]

$$\frac{\partial}{\partial z_k} f_k(g, z_k) = 0. \tag{43}$$

Both these conditions have been used in constructing the renormalized perturbation theory[4-7]. When (42) or (43) has no solution for $z_k = z_k(g)$, one can determine the latter by seeking for the minima of the corresponding left - hand sides.

Having two possibilities, (42) and (43), for defining governing functions, we should understand from the general point of view which of this possibilities is preferable. First of all we immediately see that in the case of several governing functions $z_k = \{z_k^\alpha(g) | \alpha = 1, 2, \ldots\}$ condition (43) yields the same number of equations, while (42) does not. We, of course, are able to find a way out of this trouble[4, 5] by considering several sequences $\{f_k^\alpha(g) | \alpha = 1, 2, \ldots\}$, but this would complicate the situation. Consequently, in the case of a set of governing functions it is easier to use the principle of minimal sensitivity (43).

What is more important, the principle of minimal difference (42), just because of its form, leads for some of the mapping multipliers (31) to the equality $M_k(g) = 1$. Therefore, condition (42), generally speaking,does not provide the stability of the method. Thus, the accuracy of the self - similar approximation with the governing functions obtained from (42) should be worse than that with the governing functions given by (43).

To make the above conclusions apparent and, in addition, to show how one could improve the accuracy of the method even working near an unstable fixed point, we will consider the anharmonic -oscillator problem with the Hamiltonian

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 + \lambda m^2 x^4, \tag{44}$$

in which $m, \omega, \lambda > 0$, and $x \in (-\infty, +\infty)$.

Take as a zero approximation the harmonic - oscillator Hamiltonian

$$H_0 = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega_0^2}{2}x^2. \tag{45}$$

Define the dimensionless coupling constant g and the trial parameter z,

$$g \equiv \frac{\lambda}{\omega^3}, \qquad z \equiv \frac{\omega_0}{\omega}. \tag{46}$$

Let us calculate with the Rayleigh - Schrödinger perturbation theory the dimensionless approximations $e_k(g)$ to the ground - state energy $e(g)$,

$$e_k(g) \equiv \frac{E^{(k)}}{\omega}, \qquad e(g) \equiv \frac{E}{\omega}. \tag{47}$$

Applying the method of self - similar approximations to this problem, we can compare the obtained results with numerical calculations[10] for $e(g)$ and with exact asymptotic expansions in the weak coupling limit

$$e(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2; \qquad g \to 0 \tag{48}$$

and in the strong coupling limit

$$e(g) \simeq 0.667986 g^{1/3} + 0.14367 g^{-1/3}; \qquad g \to \infty. \tag{49}$$

As representatives for the sequence of functions (1), we have now the zero approximation

$$e_0(g, z) = \frac{1}{2}z, \tag{50}$$

the first term

$$e_1(g, z) = e_0(g, z) + \frac{3g + z - z^3}{4z^2} \tag{51}$$

and the second one

$$e_2(g, z) = e_1(g, z) - \frac{6g^2 + (6g + z - z^3)^2}{16z^5}. \tag{52}$$

The coupling function, defined in (2), is given by the equation

$$e_0(g, z(g)) = f; \qquad g = g(f). \tag{53}$$

The fixed - point distance (23) entering into the distribution of approximations (24) will be determined in two ways: first, by putting, as in Ref.1,

$$\delta_1 = 1; \qquad k_* = 2, \tag{54}$$

and, second, by extracting its value from the condition

$$\lim_{g\to\infty} \frac{e_*(g)}{e(g)} = 1, \tag{55}$$

that is from the coincidence of the asymptotic forms for the self - similar approximation $e_*(g)$ and for the exact expansion (49).

5. Minimal Difference

Let us find the governing function from the principle of minimal difference (42) written as

$$e_1(g, z(g)) - e_0(g, z(g)) = 0, \tag{56}$$

which, together with (50) and (51), gives

$$z^3 - z - 3g = 0; \qquad z = z(g). \tag{57}$$

In what follows we shall assume for simplicity that $z_1 = z_2$ and use the notation

$$e_k(g) \equiv e_k(g, z(g)).$$

Equations (51) and (52), taking account of (53) and (57), become

$$e_1(g(f)) = f,$$

$$e_2(g(f)) = f - \frac{5(4f^2 - 1)^2}{384 f^3}. \tag{58}$$

For distribution (24) we get

$$y_{21}(f) = -\frac{384 f^3}{5\delta_1(4f^2-1)^2}. \tag{59}$$

Substituting (59) into normalization (25) we obtain the equation

$$\frac{4e_*^2(g)-1}{4e_1^2(g)-1} = \exp\left\{\frac{1}{4e_*^2(g)-1} - \frac{1}{4e_1^2(g)-1} - \frac{5}{12}\delta_1\right\} \tag{60}$$

for the self - similar approximation $e_*(g)$, where

$$e_1(g) = \frac{1}{2}z(g), \tag{61}$$

and the governing function $z(g)$ is defined by (57).

From (60), using the expansions

$$e_1(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{27}{16}g^2; \qquad g \to 0,$$

$$e_1(g) \simeq \frac{1}{2}(3g)^{1/3} + \frac{1}{6}(3g)^{-1/3}; \qquad g \to \infty, \tag{62}$$

we find for the self - similar approximation the weak coupling limit

$$e_*(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{3}{16}(9+5\delta_1)g^2; \qquad g \to 0 \tag{63}$$

and the strong coupling limit

$$e_*(g) \simeq A(3g)^{1/3} + B(3g)^{-1/3}; \qquad g \to \infty, \tag{64}$$

where

$$A \equiv \frac{1}{2}\exp(-\frac{5}{24}\delta_1), \qquad B \equiv \frac{3-8A^2}{12A}. \tag{65}$$

Now compare two ways, (54) and (55), of choosing the fixed - point distance (23).

i) Consider condition (54) which implies

$$\delta_1 = 1, \qquad A = 0.405968, \qquad B = 0.345166. \tag{66}$$

The weak coupling limit (63) becomes

$$e_*(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2; \qquad g \to 0, \tag{67}$$

and the strong coupling limit (64) is

$$e_*(g) \simeq 0.585507g^{1/3} + 0.239325g^{-1/3}; \qquad g \to \infty. \tag{68}$$

As is seen, the self - similar approximation given by (60) with condition (66) is very good at small coupling constants but worsens at high g. The maximal error (38) is $\varepsilon_* = 12\%$; which corresponds to $g \to \infty$. In this way, the accuracy of the method in this case is not so good.

ii) Define the fixed - point distance from condition (55), then

$$\delta_1 = 0.367416, \qquad A = 0.463156, \qquad B = 0.231005. \tag{69}$$

For the weak coupling limit (63) we get

$$e_*(g) \simeq \frac{1}{2} + \frac{3}{4}g - 2.031953g^2; \qquad g \to 0; \tag{70}$$

and for the strong coupling limit,

$$e_*(g) \simeq 0.667986g^{1/3} + 0.160170g^{-1/3}; \qquad g \to \infty. \tag{71}$$

The maximal error is $\varepsilon_* = 0.87\%$ at $g \approx 1$. As compared with the case (66) the accuracy of the method is improved by an order.

Check now the stability conditions. For the mapping multipliers (31) we find

$$M_1(g) = 1,$$

$$M_2(g) = 1 - \frac{5}{384e_*^4(g)}\left[4e_*^2(g)-1\right]\left[4e_*^2(g)+3\right]. \tag{72}$$

Although for $M_2(g)$ condition (32) is valid:

$$\frac{19}{24} < M_2(g) < 1; \qquad 0 < g < \infty, \tag{73}$$

but $M_1(g)$ does not satisfy (32). Substituting

$$\Delta_{21}(f) = e_2(g(f)) - e_1(g(f)) = -\frac{5(4f^2-1)^2}{384f^3}$$

into (35), we obtain the Lyapunov exponent

$$\Lambda_{21}(g) = -\frac{5}{384e_*^4(g)}\left[4e_*^2(g)-1\right]\left[4e_*^2(g)+3\right], \tag{74}$$

for which

$$-\frac{5}{24} < \Lambda_{21}(g) < 0; \quad 0 < g < \infty. \tag{75}$$

As far as $M_1(g) = 1$, we have here the marginal case, and to prove whether the method is stable or not, we can calculate the errors (39), for which we get $\epsilon_1 = 8\%$ and $\epsilon_2 = 15\%$. These values contradict condition (40). Hence the conclusion follows:

The principle of minimal difference does not provide the stability of the method. Although the accuracy of the latter can be sufficiently improved by a special choice of the fixed - point distance, the best accuracy is of the same order as that of the simple renormalized perturbation theory with the principle of minimal sensitivity[6].

6. Minimal Sensitivity

Turn now to the analysis of the method of self - similar approximations with the governing function given by the principle of minimal sensitivity (43). For the considered example of the anharmonic oscillator from the condition

$$\frac{\partial}{\partial z}e_1(g,z) = 0 \tag{76}$$

we have

$$z^3 - z - 6g = 0; \qquad z = z(g). \tag{77}$$

Eqs.(51) and (52), in agreement with (53) and(77), yield

$$e_1(g(f)) = \frac{12f^2+1}{16f},$$

$$e_2(g(f)) = \frac{12f^2+1}{16f} - \frac{(4f^2-1)^2}{768f^3}. \tag{78}$$

The distribution of approximations (24) is

$$y_{21}(f) = -\frac{768f^3}{\delta_1(4f^2-1)^2}. \tag{79}$$

From the normalization (25) we find the equation

$$\frac{4e_*^2(g)-1}{4e_1^2(g)-1} = \exp\left\{\frac{1}{4e_*^2(g)-1} - \frac{1}{4e_1^2(g)-1} - \frac{\delta_1}{24}\right\}, \tag{80}$$

in which

$$e_1(g) = \frac{3}{8}z(g) + \frac{1}{8z(g)}, \tag{81}$$

and the governing function is given by (77).

Using the expansions of (81)

$$e_1(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{9}{4}g^2; \qquad g \to 0,$$

$$e_1(g) \simeq \frac{3}{8}(6g)^{1/3} + \frac{1}{4}(6g)^{-1/3}; \qquad g \to \infty, \tag{82}$$

we obtain from (80) the weak coupling limit

$$e_*(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{3(24+\delta_1)}{32}g^2; \qquad g \to 0 \tag{83}$$

and the strong coupling limit

$$e_*(g) \simeq C(6g)^{1/3} + D(6g)^{-1/3}; \qquad g \to \infty \tag{84}$$

for the self - similar approximation $e_*(g)$, in which

$$C \equiv \frac{3}{8}\exp\left(-\frac{\delta_1}{48}\right), \qquad D \equiv \frac{9-40C^2}{36C}. \tag{85}$$

Consider two possibilities, (54) and (55), of setting the fixed - point distance (23).

i) Take condition (54) according to which

$$\delta_1 = 1, \qquad C = 0.367268, \qquad D = 0.272625. \tag{86}$$

Then the weak coupling limit (83) transforms to

$$e_*(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{75}{32}g^2; \qquad g \to 0, \tag{87}$$

and the strong coupling limit (84) reads

$$e_*(g) \simeq 0.667371 g^{1/3} + 0.150032 g^{-1/3}; \qquad g \to \infty. \tag{88}$$

For the maximal error (38) by a numerical solution of (80) we find $\epsilon_* = 0.28\%$ which is reached at $g \approx 1$.

ii) Accept condition (55) resulting in

$$\delta_1 = 0.955774, \qquad C = 0.367607, \qquad D = 0.271622. \tag{89}$$

The weak coupling limit (83) becomes

$$e_*(g) \simeq \frac{1}{2} + \frac{3}{4}g - 2.339604 g^2; \qquad g \to 0, \tag{90}$$

and the strong coupling limit (84) is

$$e_*(g) \simeq 0.667986 g^{1/3} + 0.149480 g^{-1/3}; \qquad g \to \infty. \tag{91}$$

By a numerical calculation the maximal error is found to be $\epsilon_* = 0.32\%$ at $g \approx 1$. As we see, the accuracy does not change practically when passing from (54) to (55). This, possibly, is connected with the stability of the method based on the principle of minimal sensitivity.

Really, the mapping multipliers (31) in this case are

$$M_1(g) = \frac{12e_*^2(g) - 1}{16e_*^2(g)},$$

$$M_2(g) = \frac{12e_*^2(g) - 1}{16e_*^2(g)} - \frac{[4e_*^2(g) - 1][4e_*^2(g) + 3]}{768 e_*^4(g)}, \tag{92}$$

from where

$$\frac{1}{2} < M_1(g) < \frac{3}{4}, \qquad \frac{1}{2} < M_2(g) < \frac{35}{48}. \tag{93}$$

Thus, the stability condition (32) is fulfilled. For the Lyapunov exponent (35) we get

$$\Lambda_{21}(g) = -\frac{[4e_*^2(g) - 1][4e_*^2(g) + 3]}{768 e_*^4(g)}, \tag{94}$$

which gives

$$-\frac{1}{48} < \Lambda_{21}(g) < 0; \qquad 0 < g < \infty. \tag{95}$$

The stability condition (36) is also true. The errors (39) are $\epsilon_1 = 2\%$ and $\epsilon_2 = 0.8\%$, which, together with $\epsilon_* = 0.3\%$, is in complete agreement with inequality (40). This analysis yields the following conclusion:

The method of self - similar approximations with the governing functions defined by the principle of minimal sensitivity is stable. It does not need additional fitting parameters, like the fixed - point distance. Its accuracy is an order higher than either the accuracy of the renormalized perturbation theory or the best possible accuracy of the method based on the principle of minimal difference.

7. Optimal Scheme

The results obtained above make it possible to formulate the general optimal scheme which one should follow applying the method of self - similar approximations. For the practical use it is convenient to divide the whole procedure into several main steps:

1) Construct a sequence of functions

$$f_k(g) = f_k(g, z); \qquad k = 0, 1, 2, \ldots, \tag{96}$$

containing a trial parameter z . The sequence can be obtained by an iterative procedure or perturbation theory.

2) Change the trial parameter by the governing functions given by the fixed - point condition in the form of the principle of minimal sensitivity

$$\frac{\partial}{\partial z} f_k(g,z) = 0, \qquad z = z_k(g). \tag{97}$$

When (97) has no solution for z_k but has for z_{k-1}, one can put $z_k = z_{k-1}$.

3) Define the coupling function $g(f)$ by the equation

$$f_0(g, z(g)) = f; \qquad g = g(f), \tag{98}$$

in which $z \equiv z_0 \equiv z_1$.

4) Introduce the distribution of approximations

$$y_{sk}(f) = \frac{s-k}{\delta_k \Delta_{sk}(f)}; \qquad \delta_k \equiv k_* - k, \tag{99}$$

where

$$\Delta_{sk}(f) = f_s(g(f), z_k(g(f))) - f_k(g(f), z_k(g(f))), \tag{100}$$

and the parameter δ_k is called the fixed - point distance.

5)Calculate the normalization integral

$$\int_{f_k(g)}^{f_*(g)} y_{sk}(f)df = 1 \tag{101}$$

yielding the equation for the self - similar approximation $f_*(g)$.

6) Check the stability of the method by finding the mapping multipliers

$$M_k(g) \equiv \lim_{f \to f_*(g)} \left| \frac{d}{df} f_k(g(f)) \right| \tag{102}$$

and the Lyapunov exponents

$$\Lambda_{sk}(g) \equiv \lim_{f \to f_*(g)} \frac{d}{df} \Delta_{sk}(f), \tag{103}$$

which have to satisfy the stability condition

$$M_k(g) < 1, \qquad \Lambda_{sk}(g) < 0. \tag{104}$$

As is evident, the method can be stable for some values of the coupling variable g but unstable for others. When (104) is true for all $g \in \mathbb{R}$, it can be called the condition of uniform stability.

7) Put the fixed - point distance

$$\delta_k = s - k \tag{105}$$

when the method is stable, or treat δ_k as a fitting parameter when the method is unstable. In the latter case δ_k can be obtained, for instance, from the strong coupling limit, if available.

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