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дубна

E5-91-571

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SOME NEW INTEGRALS ORIGINATING FROM THE FRESNEL DIFFRACTION THEORY

1991

Studying the electron scattering on the toroidal solenoid[1] we have encountered the situation when the scattering amplitude on the circular aperture can be presented in two different forms. Their comparison permits us to obtain new integrals which are absent in mathematical literature. The list of the available references is given in two our previous publications[2].

Consider the scattering of a plane wave on the circular aperture S of the radius R lying in the z = 0 plane. Let the center of the aperture coincide with the origin and the wave-vector be parallel to the z axis. Then, in the framework of the scalar Kirchoff theory the Fresnel approximation gives the following scattering amplitude[3] at the observation point P

$$\Psi(P) = \frac{k}{2\pi i d}\exp(ikd)\cdot\iint dS_1 \exp\left[\frac{ik}{2d}((x-x_1)^2 + y_1^2)\right].$$

Here k is the wave-number of the incoming wave, d is the distance between the aperture and observation plane, x is the distance of P from the z axis. The integration is performed over the aperture S. It can be fulfilled either in the cartesian (x_1, y_1) or polar (ρ_1, ϕ_1) coordinates. In the first case, one obtains[4]

$$\Psi(P) = \frac{1}{i}\exp(ikd)\,\rho\int_{-1}^{1}\exp\left[\frac{i\pi\rho^2}{2}(r-r_1)^2\right][C(q) + iS(q)]\,dr_1.$$

Here $\rho = R\sqrt{\frac{k}{\pi d}}$, $r_1 = \frac{x_1}{R}$, $r = \frac{x}{R}$; $q^2 = \rho^2(1 - r_1^2)$; C(x) and S(x) are the Fresnel integrals:

$$C(x) = \int_0^x \cos\frac{\pi x^2}{2}dx, \quad S(x) = \int_0^x \sin\frac{\pi x^2}{2}dx.$$

In the second case, one obtains[5]

$$\Psi(P) = \frac{1}{i}\exp(ikd)\exp\left(ik\frac{x^2 + R^2}{2d}\right)(U_1 - iU_2).$$

Here $U_1 \equiv U_1(u,v)$ and $U_2 \equiv U_2(u,v)$ are the Lommel functions of two variables[6], $u = \pi\rho^2$, $v = \pi\rho^2 r$. The comparison of these amplitudes results in

$$\rho\int_{-1}^{1}\exp\left[\frac{iu}{2}(r-r_1)^2\right][C(q) + iS(q)]\,dr_1 = \exp\left(iu\frac{1+r^2}{2}\right)(U_1 - iU_2). \qquad (1)$$

It is more convenient to use the auxiliary functions f and g[7] instead of C and S

$$C + iS = \frac{1+i}{2} + [g(q) + if(q)]\exp\left(\frac{i\pi}{2}q^2\right).$$

Substituting this into (1) and separating real and imaginary parts, one obtains after a trivial change of the integration variable

$$2\int_0^{\rho}\cos(\pi\rho r\sqrt{\rho^2 - t^2})\frac{tdt}{\sqrt{\rho^2 - t^2}}g(t) = \sin\left(u\frac{1+r^2}{2}\right) - $$

$$- U_1(u,v) + \frac{1}{2}(\cos v - \sin v)[f(\rho(1-r)) - g(\rho(1+r))] + \qquad (2)$$

$$+ \frac{1}{2}(\cos v + \sin v)[f(\rho(1+r)) - g(\rho(1-r))],$$

$$2\int_0^{\rho}\cos(\pi\rho r\sqrt{\rho^2 - t^2})\frac{tdt}{\sqrt{\rho^2 - t^2}}f(t) = \cos\left(u\frac{1+r^2}{2}\right) + U_2(u,v) - $$

$$- \frac{1}{2}(\cos v - \sin v)[f(\rho(1+r)) + g(\rho(1-r))] - \qquad (3)$$

$$- \frac{1}{2}(\cos v + \sin v)[f(\rho(1-r)) + g(\rho(1+r))].$$

The differentiation of these Eqs. wrt τ (or integration by parts) permits one to obtain integrals of the type

$$\int_0^{\rho} tdt\,(\rho^2 - t^2)^n \sin(\pi\rho r\sqrt{\rho^2 - t^2})\,h(t),$$

and

$$\int_0^\rho t\,dt\,(\rho^2-t^2)^{n-1/2}\cos(\pi\rho r\sqrt{\rho^2-t^2})\,h(t)\,,$$

where h is either f or g and integer $n \geq 0$. For example, the single differentiation of (2) and (3) leads to

$$2\pi\rho\int_0^\rho \sin(\pi\rho r\sqrt{\rho^2-t^2})\,t\,dt\,f(t) = -\rho\sin v + uJ_1(v) + v\,\mathrm{RHS}(2)\,, \qquad (4)$$

$$2\pi\rho\int_0^\rho \sin(\pi\rho r\sqrt{\rho^2-t^2})\,t\,dt\,g(t) = \rho\sin v - v\,\mathrm{RHS}(3)\,.$$

($J_n(x)$ is the Bessel function).

Eqs.(2-4) are rather complicated. They are simplified in two different cases which will be considered separately.

The first case corresponds to the observation point P lying on the z axis. Then, $v = r = 0$ and $U_1(u,0) = \sin(u/2)$, $U_2(u,0) = 1 - \cos(u/2)$ [6]. Eqs.(2,3) are transformed into

$$2\int_0^\rho t\,dt\,g(t)(\rho^2-t^2)^{-1/2} = f(\rho) - g(\rho),$$

$$2\int_0^\rho t\,dt\,f(t)(\rho^2-t^2)^{-1/2} = 1 - f(\rho) - g(\rho). \qquad (5)$$

The integration by parts of Eqs.(5) leads to the following recurrence relations for $F_n = \int_0^\rho t\,dt(\rho^2-t^2)^{n-1/2}f(t)$ and $G_n = \int_0^\rho t\,dt\,(\rho^2-t^2)^{n-1/2}g(t)$:

$$\pi F_{n+1} = (2n+1)\,G_n + \frac{\pi}{2^{n+2}}\rho^{2n+2}\frac{(2n+1)!!}{(n+1)!} - \frac{1}{2}\rho^{2n+1}$$

$$\pi G_{n+1} = \frac{1}{2}\rho^{2n+1} - (2n+1)\,F_n\,.$$

The first terms of this succession correspond to n=0 and n=1:

$$2\pi F_1 = \frac{1}{2}\pi\rho^2 - \rho + f(\rho) - g(\rho),$$

$$2\pi G_1 = f(\rho) + g(\rho) + \rho - 1, \qquad (6)$$

$$\frac{2}{3}\pi^2 F_2 = \frac{1}{8}\pi^2\rho^4 - \frac{1}{3}\pi\rho^3 - 1 + f(\rho) + g(\rho) + \rho,$$

$$\frac{2}{3}\pi^2 G_2 = \frac{1}{3}\pi\rho^3 - \frac{1}{2}\pi\rho^2 + \rho - f(\rho) + g(\rho).$$

The second case corresponds to the observation point P lying at the boundary of the shadow. In this case $r = 1$ and [6] $U_1(u,u) = \frac{1}{2}\sin u$, $U_2(u,u) = \frac{1}{2}[J_0(u) - \cos u]$. Eqs.(2-4) take the form

$$2\int_0^\rho \cos(\pi\rho\sqrt{\rho^2-t^2})\frac{t\,dt}{\sqrt{\rho^2-t^2}}\,g(t) = \frac{1}{2}(\cos u + \sin u)\,f(2\rho) - \frac{1}{2}(\cos u - \sin u)\,g(2\rho),$$

$$2\int_0^\rho \cos(\pi\rho\sqrt{\rho^2-t^2})\frac{t\,dt}{\sqrt{\rho^2-t^2}}\,f(t) = \frac{1}{2}J_0(u) - \frac{1}{2}(\cos u - \sin u)\,f(2\rho) - \frac{1}{2}(\cos u + \sin u)\,g(2\rho),$$

$$2\pi\rho\int_0^\rho \sin(\pi\rho\sqrt{\rho^2-t^2})\,t\,dt\,g(t) = -\rho\sin u + uJ_1(u) - \frac{1}{2}u\,g(2\rho)(\cos u - \sin u) + \frac{1}{2}u\,f(2\rho)(\cos u + \sin u)\,, \qquad (7)$$

$$2\pi\rho\int_0^\rho \sin(\pi\rho\sqrt{\rho^2-t^2})\,t\,dt\,f(t) = \rho\sin u - \frac{u}{2}J_0(u) + \frac{1}{2}u(\cos u - \sin u)\,f(2\rho) + \frac{1}{2}u(\cos u + \sin u)\,g(2\rho).$$

Eqs.(2-7) are lacking in the mathematical handbooks, treatises and original publications. They may be useful for testing the numerical calculations of the scattering amplitude which is much easier obtained in terms of integrals in the LHS of Eq.(1) (see, e.g./4/). There is no doubt that the integrals obtained here will be derived later without reliance on the physical aspects.

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