91-534

сообщения
объединенного
института
ядерных
исследований
дубна

E5-91-534

M. Vinduška*

NON-EUCLIDEAN SPACE FOR LOCAL HIDDEN VARIABLES?

*Permanent address: Nuclear Physics Institute, Czechoslovak Academy of Sciences, 250 68 Řež, Czechoslovakia

1991

1. INTRODUCTION

In preceding papers /5-7/ we have demonstrated that the so-called "no-go" theorems about hidden variables in quantum mechanics of singlet systems (Bell inequalities /1/ , Braunstein and Caves inequalities /2/) express metric conditions* for certain vector spaces and/or rest upon existence of a continuous group of transformations in space of hidden variables (R.P.Feynman /3/).

We have also proposed the use of relative probability measure, which permits one to restore the quantum mechanical results for correlations of singlet systems and at the same time to overcome limitations of the mentioned no-go theorems.

Generalized inequalities, which must be satisfied by correlations on the base of relative measure are presented summarily in the preceding letter /12/.

In this work we demonstrate that the relative measure of probability can be understood as a manifestation of the non-metric properties of the space of hidden variables by itself. Actually, in such a case there arises a necessity of introducing a definite reference frame and geometrical or physical terms become frame-dependent. In statistical theories it means that the concept of absolute, independent measure of probability must be abandoned.

Introduction of the relative measure of probability (generally it must be connected with orientations of apparatuses) breaks down symmetry of the quantum systems and may express in such a way a peculiarity of the quantum measurements**.

In this connection there arises a problem of relation between different reference frames used - which is in fact the problem of covariance and also the problem of locality in relation to the special theory of relativity. We have treated these questions in our first two papers /5-6/, here we restrict

*We understand metric conditions or metricity in the usual sense: in a given space it is possible to define a distance, which fulfils triangular inequality.

**A similar idea was recently expressed by Y.J.Ng /4/ during the discussion of the physical content of the deformed quantum groups.

ourselves to the statement that the transformation procedures connecting the different reference frames can be formulated in such a way that all frames are equivalent (covariant description becomes possible) and also that each concrete event is invariant (i.e., it does not depend on the reference frame used). These characteristics do not permit signalization with superluminal speeds, they make the concept of superluminal connections redundant and, hence, fulfil the Einstein condition of locality. Because of their local character there is also no need for the concept of contextuality[8]. They allow one to use the concept of counterfactual definiteness[9], but do not allow exploiting it for deriving relations between experimentally measured mean values.

We shall demonstrate some aspects of the proposed theory analyzing a model of linear polarization of photons. We shall show that the use of the Euclidean geometry in space of hidden variables leads naturally to the classical inequalities for correlation functions (i.e., existence of metrics in space of hidden variables induces a metric character of inequalities). On the contrary, it will be seen that the use of Minkowski geometry leads naturally to the concept of relative probability measure and to the correct quantum mechanical correlations which fulfil generalized inequalities (i.e., absence of metricity in space of hidden variables induces corresponding non-metric character of inequalities).

2. DEFINITIONS

We shall consider correlations of linear polarized photons in the singlet systems, which are described quantum-mechanically as

$$\psi^{(+)}(1,2) = \frac{1}{\sqrt{2}}\{(1)_x(2)_x + (1)_y(2)_y\}, \tag{1a}$$

$$\psi^{(-)}(1,2) = \frac{1}{\sqrt{2}}\{(1)_x(2)_y - (1)_y(2)_x\}. \tag{1b}$$

Here the indices denote the projections of the polarizations along the corresponding axis, (+) and (-) correspond to the states with even and odd parity.

For the description of the correlations in the Bell scheme of hidden variables we use the usual notation

$$P(\vec{a},\vec{b}) = \int A(\vec{a},\lambda)\, B(\vec{b},\lambda)\, \rho(\lambda)\, d\lambda, \tag{2}$$

where $\rho(\lambda)$ is normalized probability measure of hidden variables λ and results of measurements $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ take values ± 1.

For the further process we need to express (2) in invariant geometrical terms. We must define more or less independently the expressions for $\rho(\lambda)d\lambda$. $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ and also determine the connection between both particles in accord with the quantum states (1a-1b).

3. GEOMETRICAL CONSIDERATIONS

We shall start with the analogy with the classical concept of linear polarization, namely, with its vector description in the polarization plane. We shall suppose that the same picture can be used for the space of hidden variables. As translations do not change relations between vectors we shall be interested in the rotations of the plane as a whole. There exist only two realizations of geometry in a plane with invariant quadratic bilinear forms: Euclidean geometry and geometry of Minkowski, which are not mutually isomorphic.

3a. Euclidean Geometry

The rotation of the plane as a whole (we write down a transformation of coordinates) has a form

$$x' = x\cos\theta + y\sin\theta,$$

$$y' = -x\sin\theta + y\cos\theta, \tag{3}$$

and $x^2 + y^2$ is to be invariant under such transformations. Consequently, the scalar product of two vectors has the usual form

$$(\vec{a},\vec{b})_e = a_x b_x + a_y b_y. \tag{4}$$

In this geometry, distances and angles between vectors are conserved under rotations and it is possible to introduce a metrics, which satisfies the triangular inequality. For the invariant description in terms of Euclidean geometry expressions with the scalar product (4) can be used.

3b. Minkowski Geometry

Instead of (3) we now have

$$x' = x\cosh\theta - y\sinh\theta,$$
$$y' = -x\sinh\theta + y\cosh\theta. \tag{5}$$

The invariant under hyperbolic rotation is $x^2 - y^2$ and a scalar product takes a form

$$(\vec{a},\vec{b})_h = a_x b_x - a_y b_y. \tag{6}$$

In this geometry a metric does not exist because the distances generally do not fulfil the triangular inequality. The only invariant which is at our disposal is a hyperbolic norm

$$\|\vec{a}\|_h^2 = a_x^2 - a_y^2, \tag{7}$$

whose value is fixed under rotations (5).

Besides, the Minkowski geometry is suitable for the description of properties of classical polarizations. Let us consider a passage of linearly polarized light ($\vec{p}$-direction of polarization, I-intensity) through an analyser oriented along the direction $\vec{a}$. We denote as $I_{a_\perp}$ and $I_{a_{||}}$ the intensity of light which passes with polarization parallel and perpendicular to $\vec{a}$. The nontrivial linear invariant form containing both $I_{a_{||}}$ and $I_{a_\perp}$ is equal to

$$\mathrm{Inv}(I_{a_{||}}, I_{a_\perp}) = \frac{I_{a_{||}} - I_{a_\perp}}{I}. \tag{8}$$

For expressing it in terms of the Minkowski geometry we must take into account that the value of Minkowski invariant $\|\vec{a}\|_h^2$ depends on the choice of orientation of the coordinate system in respect to $\vec{a}$: for any vector $\vec{a}$ the values of $\|\vec{a}\|_h^2$ can lie between $-a^2$ to $+a^2$. When the value of the norm (7) is fixed, only then it is an invariant under hyperbolic rotations (5).

Let us suppose that the polarization vector $\vec{p}$ has a unit Euclidean norm and put coordinate axis x parallel to $\vec{a}$. Then (8) becomes

$$\mathrm{Inv}(I_{a_{||}}, I_{a_\perp}) = \|\vec{p}\|_h^2, \tag{9}$$

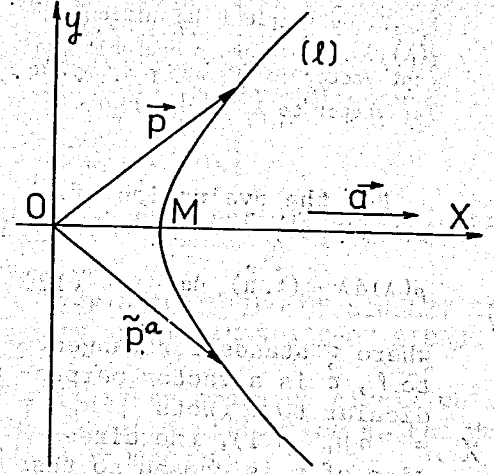

Fig.1. Illustration of the use of Minkowski geometry for the classical polarization. Here the curve is equal to $L: x^2 - y^2 = \text{const}$. For vector $\vec{p}$ it holds $\overline{OM} = \|p^a\|_h^2 = (\vec{p}, \tilde{p}^a)_e$ (see text).

where

$$\|\vec{p}\|_h^2 = p_{a_{||}}^2 - p_{a_\perp}^2 = \cos^2\phi_{pa} - \sin^2\phi_{pa} = \cos 2\phi_{pa}. \tag{10}$$

In the Euclidean geometry we must introduce a fictitious vector $\tilde{p}^a$ conjugated with $\vec{p}$ through $\vec{a}$ (see Fig.1) in order to obtain

$$\mathrm{Inv}(I_{a_{||}}, I_{a_\perp}) = (\vec{p}, \tilde{p}^a)_e. \tag{11}$$

We remind here that both expressions (9) and (11) can be understood as a generalized Malus law (M.l. follows from (9) and (11) when one-channel measurement is realized, i.e., for $I_{a_\perp} = 0$). This example explains in which way the frame-dependence appears when a pseudo-Euclidean geometry is used. Nevertheless, the expression (9) gives a true picture of the physical situation considered: both vectors $\vec{p}$ and $\vec{a}$ have a singular role here.

4. GEOMETRIZATION OF $\rho(\lambda)d\lambda$, $A(\vec{a},\lambda)$ AND $B(\vec{b},\lambda)$

Our imminent task is to express the mentioned functions in invariant geometric terms. For simplicity we shall use terms of the Euclidean geometry and the more concrete specification will be done later.

4a. Definition of $\rho(\lambda)d\lambda$

We suppose that λ is a vector lying in the polarization plane with the beginning at the centre of coordinate system. The function of $\rho(\lambda)d\lambda$ is determined as a distribution of the ends of $\vec{\lambda}$ on the curve ℓ lying also in the polarization plane.

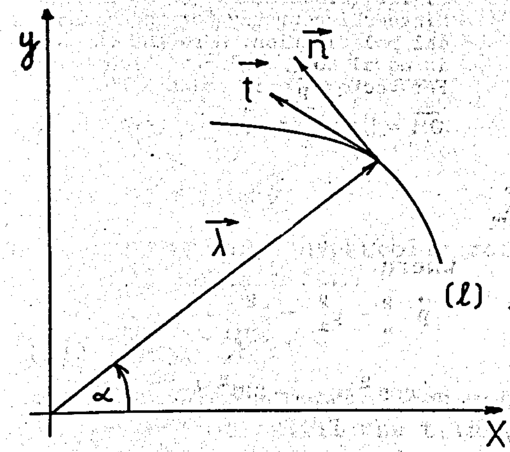

Fig.2. Geometrical definition of $\rho(\lambda)\,d\lambda$. The $\vec{t}$ and $\vec{n}$ are unit tangent vector to (ℓ) and perpendicular vector to $\vec{\lambda}$, respectively.

For the evaluation of $\rho(\lambda)\,d\lambda$ we postulate

$$\rho(\lambda)\,d\lambda = (\vec{t},\vec{n})_e\, d\alpha, \tag{12}$$

where $\vec{t}$ stands for tangent to ℓ, $\vec{n}$ is a vector perpendicular to $\vec{\lambda}$ (both $\|t\|_e = \|\vec{n}\|_e = 1$), the direction of $\vec{n}$ is chosen so that $(\vec{t},\vec{n})_e \geq 0$ and α is a central angle, fixing the direction of $\vec{\lambda}$ (see Fig.2).

The function $\rho(\lambda)$ is supposed to be normalized according to

$$\int \rho(\lambda)\,d\lambda = \oint_{\ell} (\vec{t},\vec{n})_e\, d\alpha = 1. \tag{13}$$

The choice of (12) is motivated by two reasons: in a simplified version it corresponds to the usual picture of distribution in an Euclidean space and it is also suited for normalization of $\rho(\lambda)$ in pseudo-Euclidean spaces when infinite curves (ℓ) appear.

4b. Definitions of $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$

We postulate for these functions the following relations

$$A(\vec{a},\lambda) = \text{sign}\{\cos^2\phi_{a\lambda} - \sin^2\phi_{a\lambda}\}, \tag{14a}$$

$$B(\vec{b},\lambda) = \pm\,\text{sign}\{\cos^2\phi_{b\lambda} - \sin^2\phi_{b\lambda}\}, \tag{14b}$$

where (+) sign holds in the case of system with (+) parity (1a) and (-) sign is used for the system with (-) parity (1b). The vectors $\vec{a}$ and $\vec{b}$ correspond to the orientation of apparatuses. Let us notice here that because of properties of linear polarizations and singlet photon systems and due to the formulation of problem the expressions (14a) and (14b) are the only ones which can be used.

4c. Rotational Invariance of the System

Rotational symmetry of (1a) and (1b) (here we mean an ordinary rotation in Euclidean space) leads to the condition

$$P(\vec{a},\vec{b}) = P(\phi_{ab}) \tag{15}$$

because no other direction is preferred in our treatment.

5. THE DESCRIPTION OF SINGLET SYSTEM OF PHOTONS IN EUCLIDEAN SPACE OF HIDDEN VARIABLES

It is an ordinary circle which describes the rotationally symmetric distribution in the Euclidean plane. In this case (13) reduces to

$$\rho(\lambda)\,d\lambda = \frac{1}{2\pi}\,d\alpha \tag{16}$$

because in the Euclidean geometry exist linear relations between the central angle and the elements of circular perimeter.

The relations (14a) and (14b) can be rewritten as

$$A(\vec{a},\lambda) = \text{sign}\{(\vec{\lambda},\tilde{\lambda}^{a})_e\}, \tag{17a}$$

$$B(\vec{b},\lambda) = \pm\,\text{sign}\{(\vec{\lambda},\tilde{\lambda}^{b})_e\}, \tag{17b}$$

where we have used the notation of conjugated vectors as in Fig.1. Here the vectors λ and $\tilde{\lambda}$ may not be the unit ones.

Using (16), (17) we can evaluate the correlation function

$$P^{\pm}(\vec{a},\vec{b}) = \pm 1 \mp \frac{4\phi_{ab}}{\pi}, \tag{18}$$

which is a linear function of ϕ_{ab}, $(0 \leq \phi \leq \pi/2)$.

The correlation function received fulfils all classical inequalities which are presented in the preceding letter [12]: metric Bell inequalities, metric inequalities of Braunstein and Caves and Feynman inequality. In the case of Bell inequalities it is not a surprise because of the linear dependence of $P(\vec{a},\vec{b})$ on ϕ_{ab}. This is also the reason, why the metric Braunstein and Caves inequalities are satisfied [11].

Moreover, as in this model there are fulfiled the conditions of extremality due to the prescription (14a) and (14b) used (for details see preceding letter [12]), all cited inequalities become equalities in a corresponding interval of ϕ_{ab}.

It is not difficult to indicate a classical physical situation which is described by the correlation function (18). It corresponds to measurement of intensities $I_{a_{||}}$, $I_{a_{\perp}}$ and $I_{b_{||}}$, $I_{b_{\perp}}$ of polarized source of light onto two directions $\vec{a}$ and $\vec{b}$ according to formulas

$$A(\vec{a},\lambda) = \mathrm{sign}\{I_{a_{||}} - I_{a_{\perp}}\},$$
$$B(\vec{b},\lambda) = \pm\,\mathrm{sign}\{I_{b_{||}} - I_{b_{\perp}}\}, \qquad (19)$$

when the orientation of source is equally distributed on the circle.

6. THE DESCRIPTION OF SINGLET SYSTEMS OF PHOTONS IN MINKOWSKI SPACE IN HIDDEN VARIABLES

Let us suppose, in analogue to the preceding consideration, that a distribution curve ℓ is a circle but in a Minkowski plane. Here the invariant $x^2 - y^2 = \pm$const describes four branches of hyperbolas as it is indicated in Fig.3 and logically a question arises of orientation of Minkowski coordinates in relation to $\vec{a}$ and $\vec{b}$. We put coordinate axis x parallel to $\vec{a}$. Then (12) becomes

$$\rho(\lambda)d\lambda = \rho_{\vec{a}}(\lambda)d\lambda = \frac{1}{4}\,\frac{||\lambda^a||_h^2}{||\lambda||_e^2}\,d\alpha, \qquad (20)$$

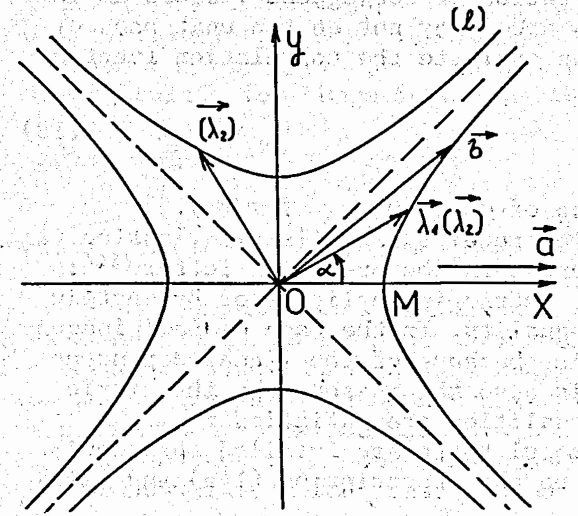

Fig.3. Description of hidden variables with the use of Minkowski circle. The orientation of x axis is chosen to be parallel to $\vec{a}$. For vector $\vec{b}$ it holds $||\vec{b}^a||_h^2 = \overline{OM}$.

where index a indicates which reference vector was used. The relation (14) can be rewritten accordingly

$$A(\vec{a},\lambda) = \mathrm{sign}\{||\lambda^a||_h^2\}, \qquad (21a)$$

$$B(\vec{b},\lambda) = \pm\,\mathrm{sign}\{||\lambda^b||_h^2\}. \qquad (21b)$$

After performing corresponding integration we obtain

$$P^{(\pm)}(\vec{a},b) = \pm||b^a||_h^2, \qquad (22a)$$

or after interchanging $a \to b$

$$P^{(\pm)}(\vec{a},\vec{b}) = \pm||b^a||_h^2 = \pm||a^b||_h^2 = \pm\cos 2\phi_{ab}. \qquad (22b)$$

In this last two expressions $\vec{a}$ and $\vec{b}$ are supposed to be normalized to unit.

We see that the use of Minkowski space for the description of local hidden variables led naturally to the concept of relative probability measure (20). This additional freedom of choice of an orientation for no-equally distributed measure allows a maximal strengthening of the correlations.

As we have shown in the preceding letter the use of relative measure of the probability leads to the generalized inequalities which have a nonmetric form [12]. It is possible to derive the relation for four vectors (see expression (11) of the cited paper)

$$D(\vec{a}_0,\vec{a}_1) + D(\vec{a}_1,\vec{a}_2) + D(\vec{a}_2,\vec{a}_3) - D(\vec{a}_3,\vec{a}_0) \geq +2 - 2\sqrt{2}, \qquad (23)$$

which, generally, for n vectors takes a form

$$D(\vec{a}_0,\vec{a}_1) + D(\vec{a}_1,\vec{a}_2) + \dots D(\vec{a}_{n-1},\vec{a}_n) - D(\vec{a}_n,\vec{a}_0) \geq n-1\,,$$
$$-(n-1)\cos\left(\frac{\pi}{n+1}\right), \qquad (24)$$

in accordance with quantum mechanical values.

These results could be expected, because (14a) and (14b) in the Minkowski space also fulfil the extremality conditions which lead to the equality of (24) in an interval when the minimum is evaluated.

We have also checked that for three and four vectors the generalized Braunstein and Caves inequalities are fulfiled, but we cannot prove it generally because of complexity of the problem (discussion about relation between Bell inequalities and these of Braunstein and Caves is contained in the preceding paper[7]).

The results of the preceding sections can be summarized as it is done in the Table.

Table. Comparison of description of local hidden variables for singlet photon states in different geometries

COMMON ASPECTS

$$P(\vec{a},\vec{b}) = \int A(\vec{a},\lambda) B(\vec{b},\lambda) \rho(\lambda) d\lambda$$

$$\rho(\lambda) d\lambda = (\vec{t},\vec{n})\, d\alpha \qquad A(\vec{a},\lambda) = \mathrm{sign}(\cos^2 \phi_{a\lambda} - \sin^2 \phi_{a\lambda})$$

$$\int \rho(\lambda) d\lambda = \oint (\vec{t},\vec{n})\, d\alpha = 1 \qquad B(\vec{b},\lambda) = \pm \mathrm{sign}(\cos^2 \phi_{b\lambda} - \sin^2 \phi_{b\lambda})$$

EUCLIDEAN GEOMETRY	MINKOWSKI GEOMETRY
The invariance under ordinary rotations $x^2 + y^2 = \mathrm{const}$ allows us to introduce a metrics, distances and angles are conserved, therefore a scalar product of two vectors can be used for invariant description.	The invariance under hyperbolic rotations $x^2 - y^2 = \mathrm{const}$ do not permit one to introduce a metrics. The distances and angles are not conserved, the only invariant is a hyperbolic norm which is frame-dependent.
ℓ: Euclidean circle	ℓ: Minkowski circle
$x^2 + y^2 = \mathrm{const}$	$x^2 - y^2 = \mathrm{const}$
$\rho(\lambda) d\lambda = \dfrac{d\alpha}{2\pi}$	$\rho(\lambda) d\lambda = \dfrac{1}{4} \dfrac{\lVert\lambda^a\rVert_h^2 d\alpha}{\lVert\lambda\rVert_e^2}$.
$A(\vec{a},\lambda) = \mathrm{sign}\{(\vec{\lambda},\vec{\lambda}^a)_e\}$	$A(\vec{a},\lambda) = \mathrm{sign}\{\lVert\lambda^a\rVert_h^2\}$
$B(\vec{b},\lambda) = \pm \mathrm{sign}\{(\vec{\lambda},\vec{\lambda}^b)_e\}$,	$B(\vec{b},\lambda) = \pm \mathrm{sign}\{\lVert\lambda^b\rVert_h^2\}$.

POSSIBILITY

to introduce general $\rho(\lambda)$ which guarantees the rotational invariance of $P(\vec{a},\vec{b})$.	to introduce a relative measure $\rho_{\vec{n}}(\lambda)$ only. The ordinary rotational invariance of $P(\vec{a},\vec{b})$ is guaranteed only for $\vec{n} = \vec{a}$ or $\vec{n} = \vec{b}$.

CORRELATION FUNCTION

$P(\vec{a},\vec{b}) = \pm 1 \mp \dfrac{4\phi_{ab}}{\pi}$.	$P(\vec{a},\vec{b}) = \pm \cos 2\phi_{ab}$
There is a linear relation between $d\alpha$, dl, $P(\vec{a},\vec{b})$ and $D(\vec{a},\vec{b})$.	the relation between $d\alpha$, dl, $P(\vec{a},\vec{b})$ and $D(\vec{a},\vec{b})$ are nonlinear.

INEQUALITIES SATISFIED[12]

Metric inequalities for $D(\vec{a},\vec{b})$ and $H(\vec{a}\mid\vec{b})$.	Generalized inequalities for $D(\vec{a},\vec{b})$ and $H(\vec{a}\mid\vec{b})$ which do not have a metric form.

DUE TO THE EXTREMAL PROPERTIES of evaluating $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ for coplanar vectors placed according to their increasing index, there are satisfied:

Metric equalities in certain interval of angles between vectors. For Feynman's value of $W(\phi = \pi/n,\ A,B, = +1)$ for symmetric system of photons it holds $W(\frac{\pi}{n}) = \dfrac{n-2}{n}$.	Generalized equalities in certain interval of angles between vectors. For symmetric system of photons it holds $W(\phi) \leq 1$.

TRANSFORMATION OF $\rho(\lambda)$[6]

Relation $\rho(\lambda) = \mathrm{const}$ guarantees a covariancy of description and the invariance of concret events, successive transformations of $\rho(\lambda)$ form trivially a continuous group.	The invariance of events and covariancy of the description must be independently postulated. Then the transformations of $\rho(\lambda)$ form a cyclic group. The number of independent elements is equal to the number of commuting quantum operators.

7. CONCLUSIONS

The proposed scheme of relative probability measure taken as a measure on pseudo-Euclidean space can be used for the description of other systems where discrete values with different signs are used. It is not restricted to bivalued variables. It is possible to show, that it also suites for description of spins with $s > 1/2$.

REFERENCES

1. Bell J.S. - Physics, 1964, 1, p.195.
2. Braunstein S.L., Caves C.M. - Ann. Phys., 1990, 202, p.22.
3. Feynman R.P. - Int. J.Theor.Phys., 1982, 21, p.467.
4. Ng Y.J. - J. Phys., 1990, A23, p.1023.
5. Tyapkin A.A., Vinduška M. - Found. Phys., 1991, 21, p.185.
6. Vinduška M. - JINR E5-91-485, Dubna, 1991.
7. Vinduška M. - "Information-Theoretic Bell Inequalities and the Relative Measure of Probability", to be published, Found. Phys., March 1992.
8. Santos E. - Found Phys., 1991, 21, p.221.
9. Stapp H.P. - Found Phys., 1991, 21, p.1.
10. Jefimov N.V., Rozendorn E.R.- "Linear Algebra and Many-Dimensional Geometry", Izd. Nauka, Moscow, 1970 (in Russian).
11. Jaglom A.M., Jaglom J.M. - "Probability and Information", Izd. Nauka, Moscow, 1973 (in Russian).
12. Vinduška M. - JINR E5-91-533, Dubna, 1991.

Received by Publishing Department
on December 3, 1991.