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GENERALIZED INEQUALITIES FOR QUANTUM CORRELATIONS WITH HIDDEN VARIABLES

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1. INTRODUCTION

There exists a set of inequalities for quantum correlations of singlet systems which must be satisfied by local hidden variables theories[1-3]. It can be shown[4-7] that they rest upon existence of a global metrics in a certain space and, consequently upon existence of a continuous group of transformations of the probability measure.

An essential feature of these approaches is an assumption of existence of absolute, independent measure of probability which determines a future behaviour of systems through a hidden parameter λ.

Our aim is to generalize the existing inequalities for correlations with variables in such a way, that they could also be used for realization of the hidden variables theory in non-Eucliden spaces, where only a relative probability measure can be defined (we mean here special cases of non-Euclidean spaces in which the distance does not fulfill metric conditions, i.e., triangular inequality).

Our immediate goal will be a generalization of Bell inequalities[1], that of Braunstein and Caves[2], and of Feynman[3]. The main results were obtained in our preceding works[5-7]*, here we restrict ourselves to a brief summary and commentary.

2. DEFINITIONS

We consider, as usual , the correlations of polarizations of photons or of particles with spin s = 1/2 also in singlet systems

$$\psi^{(+)}(1,2) = \frac{1}{\sqrt{2}} \{\psi_x(1)\psi_x(2) + \psi_y(1)\psi_y(2)\}, \qquad (1a)$$

*Metric Bell inequalities were independently introduced by E.Santos [4] and similar considerations are contained in a paper of Fivel [9] which has appeared recently. Metric inequalities for conditional information entropy in a slightly different version were commented by Braunstein and Caves [2] in connection with paper of Zurek [8].

$$\psi^{(-)}(1,2) = \frac{1}{\sqrt{2}} \{\psi_x(1)\psi_y(2) - \psi_y(1)\psi_x(2)\}, \qquad (1b)$$

$$\psi(1,2) = \frac{1}{\sqrt{2}} \{\psi_{+z}(1)\psi_{-z}(2) - \psi_{-z}(1)\psi_{+z}(2)\}. \qquad (1c)$$

Here the indices denote the projections of the polarizations along the corresponding axis. Results of measuring of the polarizations onto different $\vec{a}$ and $\vec{b}$ are denoted as $A, B = \pm 1$.

In the scheme of hidden variables the correlations are equal to

$$P(\vec{a}, \vec{b}) = \int A(\vec{a},\lambda)\, B(\vec{b},\lambda)\rho(\lambda)d\lambda, \qquad (2)$$

here $\rho(\lambda)$ is a normalized measure of probability of hidden variables λ.

3. METRIC FORM OF INEQUALITIES

Metric form of the mentioned inequalities can be demonstrated under consideration of a set of correlation functions on a closed polygon, where one common measure of probability is used (see the Figure).

3a. Bell Inequalities in Metric Form [5]

We put*

$$D(\vec{a}, \vec{b}) = \frac{1}{P(\vec{a},\vec{a})} \{P(\vec{a},\vec{a}) - P(\vec{a},\vec{b})\}, \qquad (3)$$

Then the function $D(\vec{a},\vec{b})$ has the following properties $D(\vec{a},\vec{b}) \geq 0$, $D(\vec{a},\vec{b}) = D(\vec{b},\vec{a})$ and $D(\vec{a},\vec{a}) = 0$, i.e., it can be taken as a distance.
Moreover, it holds

$$D(\vec{a}_0,\vec{a}_1) + D(\vec{a}_1,\vec{a}_2) + \ldots D(\vec{a}_{n-1}, \vec{a}_n) - D(\vec{a}_n, \vec{a}_0) \geq 0. \qquad (4)$$

The last inequality can be easily proven. It is sufficient to consider all possible values of $A(\vec{a}_i, \lambda)$ and $B(\vec{a}_i, \lambda)$ and to take into account that $p(\lambda) > 0$.

*We omit here a constant κ, which we have used in preceding work, it fixes the scale which is not important in these considerations.

3b. Braunstein and Caves Inequalities in Metric Form

Braunstein and Caves have derived inequalities for conditional information entropy of the considered correlations [2]. Using relation

$$H(\vec{a}|\vec{b}) = -\frac{1}{2}\{P(\vec{a}, \vec{b}) + 1\} \log \frac{1}{2}\{P(\vec{a}, \vec{b}) + 1\} - \frac{1}{2}\{1 - P(\vec{a},\vec{b})\} \log \frac{1}{2}\{1 - P(\vec{a}, \vec{b})\}$$

which holds due to the symmetry of singlet systems, it is possible to generalize inequalities of Braunstein and Caves as [6]

$$H(\vec{a}_0|\vec{a}_1) + H(\vec{a}_1|\vec{a}_2) + \ldots H(\vec{a}_{n-1}|\vec{a}_n) - H(\vec{a}_n|\vec{a}_0) \geq 0 \qquad (5)$$

and also to show, that $H(\vec{a}|\vec{b})$ obeys the properties of distance $H(\vec{a}|\vec{b}) \geq 0$, $H(\vec{a}|\vec{b}) = H(\vec{b}|\vec{a})$ and $H(\vec{a}|\vec{a}) = 0$. Here $H(\vec{a}|\vec{b})$ denotes the conditional information entropy of the treated correlations

$$H(\vec{a}|\vec{b}) = -\sum_{\alpha\,\beta} p(\alpha,\beta) \log p(\alpha|\beta),$$

and $p(\alpha,\beta)$ is the joint probability; $p(\alpha|\beta)$, the conditional probability for $A(\vec{a}, \lambda) = \alpha$ and $B(\vec{b}, \lambda) = \beta$ $(\alpha, \beta = \pm 1)$, and the base 2 was used for logarithm. For details see the cited papers [2,6].

4. FEYNMAN'S INEQUALITY

R.P.Feynman in his lecture [3] has introduced an inequality which must be satisfied by the probability of getting the same values of $A(\vec{a}, \lambda)$ and $B(\vec{b}, \lambda)$ (i.e. both + 1 or both - 1) for certain choice of the angle between vectors $\vec{a}$ and $\vec{b}$ (the systems of photons with (+) parity are considered here):

His result for such a system can be generalized in the following way [6]

$$W(\phi_{ab} = \frac{\pi}{n};\ A(\vec{a}, \lambda)\cdot B(\vec{b}, \lambda) = +1) \leq \frac{n-2}{n} \qquad (6)$$

$n = 4, 6, 8, \ldots$

In our preceding work we have shown, that the derivation of inequality (6) rests upon the assumption of existence of a continuous group of transformation of $\rho(\lambda)$ (the details can be found in the cited paper[6]).

Concluding this summary part let us remind that it is well known that quantum correlations do not satisfy introduced inequalities (4), (5) and (6).

5. GENERALIZED INEQUALITIES

5a. Destroyed Independence of $\rho(\lambda)$

We can generalize the considered inequalities when we shall suppose that general independence of $\rho(\lambda)$ of coordinate system is destroyed in such a way that

$$\rho(\lambda) \equiv \rho_{\vec{n}}(\lambda), \tag{7}$$

where $\vec{n}$ is certain vector.

We shall not especially discuss the physical meaning of this procedure here, but as it follows from the subsequent contribution[10] the need for a relative measure of the probability is essential for description of local hidden variables in a Pseudo-Euclidean space.

We shall use the notation

$$P_{\vec{n}}(\vec{a}, \vec{b}) = \int A(\vec{a}, \lambda)\, B(\vec{b}, \lambda)\, \rho_{\vec{n}}(\lambda) d\lambda \tag{8}$$

and a similar meaning will have symbols of $D_{\vec{n}}(\vec{a},\vec{b})$ and $H_{\vec{n}}(\vec{a},\vec{b})$.

5b. Restored Rotational Invariance of $P(\vec{a}, \vec{b})$

The rotational invariance of the considered systems leads to the condition

$$P(\vec{a}, \vec{b}) = P(\phi_{ab}), \tag{9}$$

which must be fulfilled by the quantum correlations (a similar relation holds for $D(\vec{a}, \vec{b})$ and $H(\vec{a}|\vec{b})$).

It is evident that, generally, the condition of rotational invariance will be satisfied, if we put $\vec{n} = \vec{a}$ or $\vec{n} = \vec{b}$.

Hence, we shall consider as "quantum-mechanical correlation with hidden variables" only such functions and we shall use the index "QM" for them

$$P_{QM}(\vec{a}, \vec{b}) = \int A(\vec{a}, \lambda)\, B(\vec{b}, \lambda)\, \rho_{\vec{n}}(\lambda) d\lambda \tag{10}$$

We shall use the same notation for other functions as $D_{QM}(\vec{a},\vec{b})$ and $H_{QM}(\vec{a}|\vec{b})$.

Now we are prepared to derive the generalized inequalities. Again, we shall consider a closed polygon, when different ρ_i must be used as a consequence of (10).

Thus the inequalities for four vectors $(\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3)$ can be described with $\rho_{\vec{a}_0}$ and $\rho_{\vec{a}_2}$ or with $\rho_{\vec{a}_1}$ and $\rho_{\vec{a}_3}$ as it is shown in the Figure. As the metric inequalities (4) hold for any $\rho(\lambda) > 0$, they hold also for $\rho_{\vec{a}_1}$ and $\rho_{\vec{a}_3}$:

$$D_{\vec{a}_1}(\vec{a}_0, \vec{a}_1) + D_{\vec{a}_1}(\vec{a}_1, \vec{a}_2) + D_{\vec{a}_1}(\vec{a}_2, \vec{a}_3) - D_{\vec{a}_1}(\vec{a}_3, \vec{a}_0) \geq 0$$

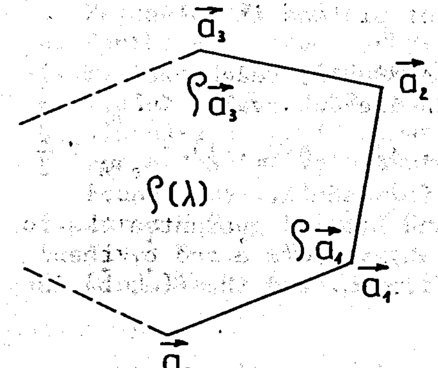
$$D_{\vec{a}_3}(\vec{a}_0, \vec{a}_1) + D_{\vec{a}_3}(\vec{a}_1, \vec{a}_2) + D_{\vec{a}_3}(\vec{a}_2, \vec{a}_3) - D_{\vec{a}_3}(\vec{a}_3, \vec{a}_0) \geq 0.$$

After summation and separation of terms according to (10) we get generalized Bell inequalities (GBI) for four vectors

$$D_{QM}(\vec{a}_0, \vec{a}_1) + D_{QM}(\vec{a}_1, \vec{a}_2) + D_{QM}(\vec{a}_2, \vec{a}_3) - D_{QM}(\vec{a}_3, \vec{a}_0) \geq$$
$$\geq D_{\vec{a}_1}(\vec{a}_3, \vec{a}_0) - D_{\vec{a}_1}(\vec{a}_2, \vec{a}_3) - D_{\vec{a}_3}(\vec{a}_0, \vec{a}_1) - D_{\vec{a}_3}(\vec{a}_1, \vec{a}_2). \tag{11}$$

By the same way we can get the generalized Braunstein-Caves inequalities (GBCI)

$$H_{QM}(\vec{a}_0|\vec{a}_1) + H_{QM}(\vec{a}_1|\vec{a}_2) + H_{QM}(\vec{a}_2|\vec{a}_3) - H_{QM}(\vec{a}_3|\vec{a}_0) \geq$$
$$\geq H_{\vec{a}_1}(\vec{a}_3|\vec{a}_0) - H_{\vec{a}_1}(\vec{a}_2|\vec{a}_3) - H_{\vec{a}_3}(\vec{a}_0|\vec{a}_1) - H_{\vec{a}_3}(\vec{a}_1|\vec{a}_2). \tag{12}$$



Let us briefly comment on received results. On the left-hand sides of both inequalities (11) and (12) stand the quantum-mechanical values (due to the definition (10)) which, generally, need not satisfy the condition of metricity, because the right-hand sides can take negative values. Unfortunately, we cannot interpret these right-hand sides as measurable quantities (generally it is impossible to express them using quantum-mechanical formalism, because they are model-dependent).

6. GENERALIZED FEYNMAN INEQUALITY (GFE)

In our preceding work[6] we have shown that the use of relative probability measure has as a consequence that its transformations do not form a continuous group and that the inequalities of Feynman can be rewritten as

$$W(\phi_{arbitrars};\ A(\vec{a},\ \lambda)\cdot B(\vec{b},\ \lambda) = +1) \leq 1 \quad (13)$$

which is fulfilled trivially.

7. AN EXAMPLE OF MODEL-DEPENDENT INEQUALITIES

In our preceding works[5,6] we have introduced two theorems, which can be used for the formulation of model-dependent relations.

Theorem 1.

The relation

$$D(\vec{a}_0,\ \vec{a}_1) + D(\vec{a}_1,\ \vec{a}_2) + \ldots D(\vec{a}_{n-1},\ \vec{a}_n) - D(\vec{a}_n,\ \vec{a}_0) = 0 \quad (14)$$

holds for each $\rho(\lambda) > 0$. $\int \rho(\lambda)d\lambda = 1$ if and only if, for each λ, the sequence $A(\vec{a}_0,\ \lambda)$, $A(\vec{a}_1,\ \lambda)\ \ldots\ A(\vec{a}_n,\ \lambda)$ changes its sign no more than once.

For the symmetric singlet state of protons it holds

Theorem 2.

The functions $P(\phi_{ab})$ and $W(\phi_{ab};\ A,B = +1)$ reach their maxima for arbitrary $\rho(\lambda) > 0$ and ϕ_{ab} in the interval $0 \leq \phi_{ab} \leq \frac{\pi}{2}$ only if the sequence $A(\vec{a}_0,\ \lambda)$, $A(\vec{a}_1,\ \lambda)\ldots\ A(\vec{a}_n,\ \lambda)$, $\phi_{a_0 a_n} \leq \frac{\pi}{2}$ changes its sign no more than once for each λ.

If the functions $\rho(\lambda)$, $A(\vec{a},\ \lambda)$ and $B(\vec{b},\ \lambda)$ guarantee the rotational invariance of $P(\vec{a},\ \vec{b})$ for any vectors $\vec{a}$ and $\vec{b}$, then the preceding condition is also sufficient and the $P(\vec{a},\ \vec{b})$ is equal to

$$P(\vec{a},\ \vec{b}) = 1 - \frac{4\phi_{ab}}{\pi} \quad (15)$$

for ϕ_{ab} in the interval $0 \leq \phi_{ab} \leq \pi/2$. The proofs of mentioned theorems can be found in the cited papers[5,6].

Let us suppose, that our scheme of hidden variables fulfils the conditions of both theorems. Then instead of the Bell metric inequalities (4) we get equalities in some interval of ϕ_{ab} and the Feynman inequality (6) also takes a form of an equality.

Using $\rho_{\vec{n}}(\lambda)$ permits to exploit the Theorem 1 only. In such a case (11) turns into generalized Bell equalities on some interval of $\phi_{a_0 a_n}$. The mentioned theorems do not touch the inequalities of Braunstein and Caves. As we have shown[7], the boundaries given by information entropy are wider than those given by any linear functions of $P(\vec{a},\ \vec{b})$.

8. CONCLUSION

The derived generalized inequalities (11), (12) and (13) which were obtained with relative measure of the probability are wider than the usual ones, (4), (5) and (6) and, therefore they need not be in contradiction with quantum mechanics. As we demonstrate in the subsequent contribution, they can be understood as the inequalities for local hidden variables theory in non-Euclidean spaces.

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