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GROWTH RATE OF NUMBER OF PERIODIC POINTS IN COUNTABLE CHAINS

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1.The purpose of this paper is to obtain asymptotic estimates of the number of periodic paths (cycles) in countable directed graphs.This problem is deeply connected with the theory of dynamical systems and statistical physics. Here the Bowen-Ruelle-Sinai thermodynamical formalism (BRS) serves as a connecting chain.

BRS theory deals with the so-called hyperbolic dynamical systems.The latter include one dimensional maps (on the interval or circle), the hyperbolic torus automorphisms, Anosov diffeomorphisms, Axiom A Smale's systems, the hyperbolic attractors which are very popular today, and finally, the hyperbolic billiards. The latter include classical model of hard balls,the Lorentz gas, the Rayleigh gas and others. A universal method for investigating all of these systems is the symbolic representation. One may construct the so-called Markov partition of the phase space.This partition allows one to code all (or almost all with respect to Liouville measure) trajectories by infinite sequences of letters from some formal alphabet. The motion in the phase space corresponds to the shift of each symbolic sequence. In this way we get a symbolic dynamical system of special kind called a topological Markov chain (or subshift of finite type). It has quite universal structure independently from specific features of initial model. However all essential and interesting dynamical properties are inherited in this symbolic representation.

2. Now we give exact definitions. Let $\Omega=\{\alpha_1,\alpha_2,...\}$ be a finite or countable set of letters (alphabet). Let $\Omega^{\mathbb{Z}}$ be a set of all doubly infinite sequences $\{..., \alpha_{i_{-n}},..., \alpha_{i_{-1}},\alpha_{i_0},...\alpha_{i_n},...\}$ with

entries from Ω.Let also **A** be a matrix of zeroes and ones.The number of rows and columns in **A** equals the number of letters in the alphabet Ω. .

We consider a subset $\Sigma_A \subset \Omega^Z$ of admissible sequences which are constructed by the rule $a(\alpha_{i_n}, \alpha_{i_{n+1}})=1\ \forall n,\ -\infty < n < \infty$. Here $a(\alpha,\beta)$ denotes an element of the matrix **A** corresponding to the pair $\alpha,\beta\in\Omega$. In other words the matrix **A** gives admissible transitions from one letter to another. All transitions between neighbouring letters in the sequences $\sigma=\{\alpha_{i_n}\}_{-\infty}^{\infty}$ belonging to the set Σ_A must be admissible. In the space Σ_A we introduce the shift transformation $\theta:\Sigma_A\to\Sigma_A$ defined by $\theta\sigma=\theta\{\alpha_n\}_{-\infty}^{\infty}=\{\beta_n\}$ where $\beta_{i_n}=\alpha_{i_{n+1}}$.The pair (Σ_A,θ) is called a topological Markov chain (TMC).

Any TMC has a simple graphic representation. Let us take a directed graph whose vertices correspond to the letters of alphabet Ω and directed arrows give admissible transitions.Then an admissible sequence $\sigma\in\Sigma_A$ defines an infinite path with one marked vertex on it (whish corresponds to zero element of σ).

At present BRS thermodynamical formalism has given very impressive results in the case of finite TMC (i.e. when the alphabet Ω has a finite number of letters). The notions of topological entropy and topological pressure have been defined; the Gibbs measures (equilibrium states) have been constructed; the variational principle,the exponential correlation decay,the central and local limit theorems have been proved /1/,/2/,/3/,/4/Finally, the question about periodic sequences $\sigma\in\Sigma_A$in TMC (or cycles in the related directed graphs) has been considered. The first result is that the number P_n of periodic sequences of period n in TMC (Σ_A,θ) (i.e. the number of solutions of the equation $\theta^n\sigma=\sigma$) grows exponentially fast

$$\lim_{n\to\infty} \frac{\ln P_n}{n} = h(\theta,\Sigma_A) \qquad (1)$$

Here $h(\theta,\Sigma_A)$ is the topological entropy of TMC.More subtle results are connected with the distribution of periodic sequences in the space Σ_A /5/ and various limit theorems for the distribution. The interest to these questions is also due to the fact that periodic points are used to describe the structure and the properties of Gibbs measures.

On the contrary there are very few results in the case of countable alphabet. The topological entropy /6/,/7/ and the topological pressure /8/ have been defined; the variational principle has been also proved /9/. But Gibbs measures have not been constructed in general case (moreover, sometimes they do not exist). Nothing is known about the asymptotics of periodic sequences $\sigma\in\Sigma_A$ (even the law (1) fails in general case). At the same time a number of models which are interesting from the physical point of view are reduced just to the countable TMC. Such are certain hyperbolic attractors and all billiards /10/. For example, the following relations for billiards have been obtained in /11/,/10/

$$K^n<P_n<L^n \qquad (2)$$

where $1<K<L<\infty$ It would be very desirable to prove the law (1) at least for this case. But, apparently one needs some additional properties of the countable TMC serving as a symbolic representation of billiard. No such suitable properties have been known so far.

Here we make the first step in studying asymptotics of periodic sequences in general countable TMC. We present a simple clear and easy to check condition under which the limit (1) exists and equals the topological entropy of TMC. It is remarkable that in a way our condition turns to be necessary as will be shown in theorem 2.

3. Now let (Σ_A,θ) be a countable TMC, **G** and **A** are related directed graph and transition matrix. We consider connected irreducible graphs. The first means that any two vertices can be connected by a path, and the second that one can find a cycle containing both of them. According to Gurevič /5/ the topological entropy $h(\theta,\Sigma_A)$ can be defined as

$$h(\theta,\Sigma_A)=\sup_{\hat{G}} h(\hat{G})=\sup_{\hat{G}} \log \lambda(\hat{G}),$$

where the sup is taken over all connected finite subgraphs $\hat{G}$ of **G**, $h(\hat{G})$ being the usual topological entropy of finite TMC determined by $\hat{G}$, the logarithm of maximal eigenvalue of the transition matrix.

It is known [12] that

$$P_n(A)=\mathrm{tr}\ A^{(n)}=\sum \lambda_i^n ,$$

where λ_i are all eigenvalues of $\mathbf{A}$ including multiplicities. Then

$$\underline{\lim}_{n\to\infty} \frac{\log P_n(A)}{n} \geq h(\theta,\Sigma_A) \qquad (3)$$

Hence the limit (1) may exist only if $h(\theta,\Sigma_A)< \infty$. But as the following example shows (fig.1) it isn't sufficient to provide the existence of the limit (1)

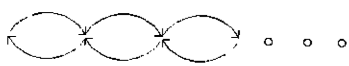

fig.1

The topological entropy of this graph equals log2, but there is an infinite number of periodic points of periods 2,4,6,... Thus the second necessary condition for fulfilment of law (1) is that the number of periodic sequences of period n must be finite for any n. We may try the condition (2) which does appear in symbolic representation of billiards [10] However, as was shown in [13] there was a connected countable irreducible graph with finite entropy satisfying (2), for which

$$\underline{\lim}_{n\to\infty} \frac{\log P_n}{n} \neq \overline{\lim}_{n\to\infty} \frac{\log P_n}{n} \qquad (*)$$

The idea of construction of this graph is that one takes a finite sequence of simple cycles (i.e. cycles whose vertices are distinct), of the same period , say n_1, then a sequence of simple cycles of sufficiently large period n_2 and so on. We may note that in this example infinitely large simple cycles are used. If we restrict the lenghs of simple cycles keeping (2) then as is easy to see, we come to the following situation.

Preposition 1. Let G be a connected graph, $h(G)<\infty$ Let the lengths $l_i(G)$ of all simple cycles are uniformly bounded.

$$l_i(G)< N .$$

Let the condition (2) be true. Then G is a graph with finite number of vertices.

4. Now we suggest another condition forbidding large number of cycles with the same period. We mark one vertex, say **W**. Then we define a distance $\rho(s)$ between vertex s and the initial vertex **W** as the number of arrows of the shortest path from **W** to s. Then a distance $\rho^*(n)$ between some cycle **p** of period n and **W** can be defined as

$$\rho^*(n)= \min_{s\in p} \rho(s) .$$

Let for any cycle of period **n** the following condition be true

$$\rho^*(n)<F(n) , \qquad (4)$$

where **F(n)** is some increasing function (the above example shows that **F(n)** is to be slower than exponent) Roughly speaking **(4)** forbids appearance of small cycles at large distance from **W**.

Theorem 1. Let G be an aperiodic irreducible countable graph, and $h(G)<\infty$. Let the number of arrows going out of any vertex of G be uniformly bounded. Let **(4)** be true $\forall\ n\geq 1$ where $\lim\limits_{n\to\infty} \frac{F(n)}{n} = 0$. Then

$$\lim_{n\to\infty} \frac{\log P_n}{n} = h(G) . \qquad (5)$$

Proof. First, from (3) we have

$$\varliminf_{n\to\infty} \frac{\log P_n}{n} \geq h(G) .$$

Now we want to estimate the upper limit $\overline{\lim} \frac{\log P_n}{n}$ Let the number of arrows going out of any vertex be less than say **k**(such chains are called iniformly bounded (forward)). Let $\{Y_n\}$ be a set of those points s for which

$$\forall\ s\in\{Y_n\} \qquad \rho(s)\le F(n)$$

(i.e. these are points, where cycles of period n may start) Then

$$\#\{Y_n\} \le k^{F(n)}.$$

Let A be the related transition matrix of infinite size. Then

$$P_n \le \sum_{i\in Y_n} A_{ii}{}^n .$$

According to /14/ in the case of countable irreducible aperiodic graphs we have

$$\{A_{ii}^n\}^{1/n} \to R, \qquad A_{ii}^n \le (1/R)^n ,$$

where R is the radius of convergence of the sum $\sum A_{ij}^n t^n$ (in fact it is independent of vertex i,j) Gurevič /5/ showed that if G is a connected graph with $h(g)<\infty$, then $h(g)=-\log R$. Hence

$$\frac{\log P_n}{n} \le \frac{\log((1/R)^n)\ k^{F(n)})}{n} \le \frac{\log n}{n} + \frac{F(n)\log k}{n} - \log R .$$

Thus

$$\overline{\lim_{n\to\infty}}\ \frac{\log P_n}{n} \le -\log R = h(G)$$

from which the assertion is obvious.

Thus we can provide the law (1) if the function $F(n)$ grows slower than n. What happens if this condition fails, i.e. if there is a sequence $n_1<n_2<n_3\ldots\ldots, n_i\to\infty$ and a constant C such that

$$F(n_i) \ge Cn_i \quad \forall\ i . \qquad (6)$$

The following theorem shows that then one can construct a counterexample, where the limit (5) does not exist.

Theorem 2. There is a graph with all above properties, $F(n)$ satisfying (6) for which (*) is true.

Proof. We construct the required graph. First, we make a loop at the initial vertex (in this way the constructed graph becomes aperiodic). Let our graph T be such that for each vertex there k are arrows going out of it. Then

$$h(T) \le \log k . \tag{7}$$

Let $\{n_i\}$ be a sequence, defined by (6). Further we shall give additional properties of $\{n_i\}$. We consider the set S_n of those points, for which

$$\rho(s) \le n_1 \qquad \forall\ s \in S_{n_1} .$$

Each of these points will be a "starting point" of cycles of period n_1 (i.e. for any $s \in S_{n_1}$ there is a path of length $n_1 - 1$ and the last arrow returns to the initial point.) Thus the whole number of cycles of period n_1 will be

$$P_{n_1} = k^{n_1} k^{n_1 - 1} ,$$

Then we stop making cycles, so that the cycles of periods $n_1+1, n_1+2, \ldots$ are formed due to the loop at the beginning. Thus $\frac{\log P_n}{n}$ decreases. Let n_2 be such that

$$\frac{\log P_{n_2}}{n_2} \le \log k + \varepsilon$$

for some small ε. We begin to construct all possible cycles of period n_2 according to the same scenario. All ponts for which

$$\rho(s) \le n_2$$

will be starting points of cycles of period n_2. Then

$$P_{n_2} > (k-1)^{n_2} (k-1)^{n_2 - 1}$$

and so on. Thus

$$\overline{\lim_{n\to\infty}} \frac{\log P_n}{n} > 2 \log (k-1), \text{ while } \underline{\lim} \frac{\log P_n}{n} \le \log k .$$

If k is sufficiently large then we are done.

Apparently, the assertion of theorems are independent of the choice of the initial vertex **W**.

Remark. It is interesting to compare our condition **(4)** with other known properties of countable graphs. The only classification for such graphs has been elaborated by Vere-Jones /12/(see also Salama /15/) It distinguishes three types of graphs: transient,null-recurrent and positive-recurrent. The latter ones are the most suitable for thermodynamical formalism, because only these graphs possess the measure of maximal entropy /16/.But it seems that the existence of the limit **(*)** does not relate to this classification. Really, the example in /13/(in which the limit **(1)** does not exist) is a positive-recurrent. On the other hand, it is easy to construct transient and null-recurrent graphs where our condition **(4)** holds and even $F(n)\equiv 0$ (i.e. all loops in the graph have a common vertex).

Example. Take a vertex **W** and **k** arrows going out of **W**.From each of their ends take another **k** arrows going out and so on. Then we add to this graph some arrows returning from far located vertices to the initial vertex **W** (no more than one returning arrow per vertex). Finally, we can remove some number of vertices(and arrows) to obtain an irreducible graph. It is clear that our graph is uniformly bounded (forward) and all the loops have a common vertex **W**. The number f_n of the simple loops of length n can be made arbitrary between **0** and k^{n-1}. Taking $f_n \sim n^{-2}k^{n-1}$ for all $n \geq n_0$ and $f_n \equiv 0$ for $n < n_0$ we obtain a transient or null-recurrent graph,depending on n_0(see definitions in /15/)

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