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SELF-SIMILAR APPROXIMATIONS FOR EIGENVALUE PROBLEM

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1. INTRODUCTION

The eigenvalue problem is one of the most frequent problems of quantum mechanics, statistical mechanics and various branches of mathematical physics. In the standard realistic situation the problem cannot be solved exactly. Then, one resorts to a certain perturbation theory which usually results in a divergent series. When a number of perturbative terms are known, say, about ten of them, one can restore an effective sum of an asymptotic series by means of one of resummation techniques like the Borel summation or Pade approximation. However, if we know only a few first terms of perturbation theory, these resummation techniques are meaningless. A worth discussion of these difficulties has been done by Stevenson[1].

To overcome this trouble, an approach was constructed[2] to restore the sum of a divergent series on the basis of a minimal number of perturbative terms. This approach was called the method of self-similar approximations[2] since its foundation lies on a close connection between the criterion of convergence and the property of functional self-similarity written for specially introduced functionals. The method was applied for finding out the ground-state energies for several statistical and quantum-mechanical models[2-4].

The aim of the present paper is to show that the method of self-similar approximations is applicable not solely for finding the lowest energy level of a corresponding Hamiltonian but for calculating all its eigenvalues with an accuracy not worse than that obtained for the ground state.

In Section 2 we formulate the principle of the method whose foundation with all the details is given in Refs.[2,3]. In Section 3 we apply the method for calculating the eigenvalues of the anharmonic-oscillator Hamiltonian using only two terms of perturbation theory. We find the maximal error of the method for all eigenvalues and all anharmonicity parameters to be of the order of 10^{-3}. We show that our method is more simple and accurate than the quasiclassical approximation (Section 4) and renormalized perturbation theory (Section 5).

2. METHOD OF SELF-SIMILAR APPROXIMATIONS

In Ref.[2] a general formulation for the method of self-similar approximations was given, being applicable to sequence of arbitrary nature. Here, without repeating the formulation of the method expounded in Ref.[2] we adapt its scheme to suit the calculation of eigenvalues.

Suppose we need to find the eigenvalues of an operator $\hat{F}(x, g)$ depending on a multiparameter g and a space variable $x = \{x^1, x^2, \ldots x^d\}$, where d is a space dimensionality. The eigenvalue problem written in the standard form

$$\hat{F}(x, g)\Psi_n(x, g) = f(n, g)\Psi_n(x, g)$$

defines the sought eigenvalues $f(n, g)$ and eigenfunctions $\Psi_n(x, g)$ in which $n = \{n^1, n^2, \ldots n^d\}$ is a multiindex of the order of the space dimensionality. If the problem does not allow an exact solution, one resorts to one or another kind of perturbation theory obtaining a sequence of approximate terms $\{f_k(n, g)\}$, where the index $k = 0, 1, 2, \ldots$ enumerates the approximations.

To make the sequence convergent, we renormalize it introducing into the initial approximation $f_0(n, g, z)$ a trial multiparameter z, as a result of which all farther approximations

$$f_k(n, g) = f_k(n, g, z); \; k = 0, 1, 2\ldots \qquad (1)$$

become dependent on this parameter too. The latter is to be transformed into the sequence $\{z_k(n, g)\}$ of the governing functions $z_k(n, g)$ whose role is to govern the fastest convergence of the sequence $\{f_k(n, g, z_k(n, g))\}$. The governing functions are to be defined by a fixed point condition whose different variants have been analyzed in Refs.[2-4]. First fixed point conditions for defining the governing functions were proposed[5,6] in the form of the principle of minimal difference

$$f_i(n, g, z) - f_j(n, g, z) = 0.$$

Later Stevenson[7] argued that the principle of minimal sensitivity

$$\frac{\partial}{\partial z} f_k(n, g, z) = 0; \; z = z_k(n, g) \qquad (2)$$

suits better for improving the convergence of asymptotic series. In what follows we shall use condition (2) with the notation

$$z_0(n, g) \equiv z_1(n, g) \equiv z(n, g).$$

Persuing the scheme of the method of self-similar approximations[2] we define the coupling function $g(n, \xi)$ by the equation

$$f_0(n, g, z(n, g)) = \xi; \; g = g(n, \xi). \qquad (3)$$

Then, we introduce the distribution of approximations

$$y_{ks}(n, \xi) = [f_s(n, g,(n, \xi), z_k(n, g(n,\xi))) - f_k(n, g(n,\xi), z_k(n, g(n,\xi)))]^{-1} \qquad (4)$$

satisfying the normalization

$$\int_{f_k(n,g)}^{f_*(n,g)} y_{ks}(n,\xi)d\xi = 1, \qquad (5)$$

in which $f_*(n,g)$ is just the self-similar approximation for the sought function $f(n, g)$.

3. ONE-DIMENSIONAL ANHARMONIC OSCILLATOR

Now the scheme of Section 2 will be applied for calculating the eigenvalues of the Hamiltonian

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 + \lambda m^2 x^4 \qquad (6)$$

describing the one-dimensional anharmonic oscillator, in which m, ω, λ are positive parameters and $x \in (-\infty, +\infty)$. For an illustration we choose the anharmonic oscillator for the following reasons. The mathematical structure of Hamiltonian (6) is common for a great number of various physical problems[2]. The strandard perturbation theory in powers of the anharmonicity constant yields strongly divergent series for all finite values of this constant[8]. Exact numerical solutions for the eigenvalues are avariable[9] making it possible to check the accuracy of our method.

For the eigenvalues $E_n (n = 0, 1, 2, \ldots)$ of operator (6) it is convenient to use the dimensionless quantities

$$e(n,g) \equiv \frac{E_n}{\omega}, \quad g \equiv \frac{\lambda}{\omega^3}. \tag{7}$$

By the Rayleigh - Schrödinger perturbation theory, starting with the Hamiltonian

$$H_0 = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega_0^2}{2}x^2, \tag{8}$$

we can find the approximate eigenvalues

$$e_k(n,g,z) \equiv \frac{E_n^{(k)}}{\omega}; \quad z \equiv \frac{\omega_0}{\omega} \tag{9}$$

depending on a trial parameter z. In this notation, beginning with the zero approximation

$$e_0(n,g,z) = (n + \tfrac{1}{2})z, \tag{10}$$

we have in the first order

$$e_1(n,g,z) = e_0(n,g,z) - (n + \tfrac{1}{2})\frac{z^2 - 1}{2z} + (n^2 + n + \tfrac{1}{2})\frac{3g}{2z^2}, \tag{11}$$

and in the second order

$$e_2(n,g,z) = e_1(n,g,z) - (n + \tfrac{1}{2})\frac{(z^2 - 1)^2}{8z^3} + \tag{12}$$

$$+ (n^2 + n + \tfrac{1}{2})\frac{3g(z^2 - 1)}{2z^4} - (n + \tfrac{1}{2})(n^2 + n + \tfrac{21}{17})\frac{17g^2}{4z^5}.$$

Define the governing function $z(n,g)$ by the fixed-point condition (2),

$$\frac{\partial}{\partial z} e_1(n,g,z) = 0; \quad z = z(n,g). \tag{13}$$

Substituting (11) into (13) we get the equation

$$z^3 - z - 6\gamma_n g = 0, \tag{14}$$

in which

$$\gamma_n \equiv (n^2 + n + \tfrac{1}{2})/(n + \tfrac{1}{2}). \tag{15}$$

The positive solution to equation (14) reads

$$z(n,g) = \begin{cases} (2/\sqrt{3})\cos(\alpha_n/3); & g \leq g_n \\ A_n^+ + A_n^-; & g \geq g_n \end{cases}, \tag{16}$$

where

$$\alpha_n = \arccos(g/g_n),$$

$$A_n^{\pm} = (3g)^{1/3}[1 \pm \sqrt{1 - (g/g_n)^2}]^{1/3},$$

$$g_n = (9\sqrt{3}\gamma_n)^{-1} = 0.064150/\gamma_n.$$

The weak anharmonicity region corresponds to $g << g_n$. From (16) it follows that this region becomes more and more narrow as the eigenvalue number increases. According to (15),

$$\gamma_0 = 1; \quad \gamma_n \simeq n \ (n \to \infty).$$

Therefore, for higher eigenvalues the weak anharmonicity region practically disappears, since

$$g_n \to 0; \quad n \to \infty.$$

This explains why the techniques based on the standard perturbation theory seize to provide a good accuracy for sufficiently high eigenvalues.

Taking account of (14) we may rewrite (11) and (12) in the form

$$e_1(n,g,z) = (n + \tfrac{1}{2})\frac{3z^2 + 1}{4z},$$

$$e_2(n,g,z) = (n + \tfrac{1}{2})\left[\frac{3z^2 + 1}{4z} - \frac{(z^2 - 1)^2}{48z^3}a_n\right], \tag{17}$$

where

$$a_n \equiv (n + \tfrac{1}{2})\frac{17}{3\gamma_n} + \frac{25}{6\gamma_n^2} - 6. \tag{18}$$

Eq.(18) has the property

$$a_0 = 1, \quad \lim_{n\to\infty} a_n = -\frac{1}{3}.$$

To abbreviate the subsequent expressions we shall use the notation

$$e_k(n,g) \equiv e_k(n,g,z(n,g)). \tag{19}$$

Equation (3) for the coupling function now reads

$$e_0(n,g) = \xi; \quad g = g(n,\xi). \tag{20}$$

For the distribution of approximations (4) we have

$$y_{12}(n,\xi) = [e_2(n,g(n,\xi)) - e_1(n,g(n,\xi))]^{-1}, \tag{21}$$

which, together with (17) and (20), gives

$$y_{12}(n,\xi) = -\frac{48\xi^3/(n+\frac{1}{2})^4}{a_n[\xi^2/(n+\frac{1}{2})^2 - 1]^2}. \tag{22}$$

In place of normalization (5) we get

$$\int_{e_1(n,g)}^{e_*(n,g)} y_{12}(n,\xi)d\xi = 1. \tag{23}$$

Integrating (23), we obtain the equation

$$\frac{e_*^2(n,g)/(n+\frac{1}{2})^2 - 1}{e_1^2(n,g)/(n+\frac{1}{2})^2 - 1} = \exp\left\{\frac{1}{e_*^2(n,g)/(n+\frac{1}{2})^2 - 1} - \frac{1}{e_1^2(n,g)/(n+\frac{1}{2})^2 - 1} - \frac{a_n}{24}\right\} \tag{24}$$

for the self-similar approximation of the eigenvalues, $e_*(n,g)$.

In the weak anharmonicity region, Eq.(24) yields

$$e_*(n,g) \simeq (n+\frac{1}{2})(1+\frac{3}{2}\gamma_n g); \quad g \to 0, \tag{25}$$

which coincides with the exact asymptotic axpansion[9] in powers of g. In the strong anharmonicity limit Eq.(24) leads to

$$e_*(n,g) \simeq \frac{3}{4}(n+\frac{1}{2})\exp\left(-\frac{a_n}{48}\right)(6\gamma_n g)^{1/3}; \quad g \to \infty. \tag{26}$$

For higher eigenvalues, (26) transforms to

$$e_*(n,g) \simeq \frac{\exp(1/144)}{2^{5/3}}(3n)^{4/3} g^{1/3}; \quad g,n \to \infty. \tag{27}$$

The latter expression can be compared with the corresponsing exact asymptotic expansion[9]

$$e(n,g) \simeq \frac{\pi^2}{\Gamma^{8/3}(1/4)}(3n)^{4/3} g^{1/3}; \quad g,n \to \infty. \tag{28}$$

For an easier comparison, (27) and (28) can be written as

$$e_*(n,g) \simeq 1.372\,338\, n^{4/3} g^{1/3}; \quad g,n \to \infty, \tag{29}$$

and

$$e(n,g) \simeq 1.376\,507\, n^{4/3} g^{1/3}; \quad g,n \to \infty, \tag{30}$$

respectively. The accuracy of (29) with respect to (30) is 0.3%.

It would not be, of course, honest to check an accuracy of a method by comparing only some asymptotic expansions, as well as by considering only several eigenvalues, for instance, solely that of the ground state. The true accuracy of a method must be characterized by the maximal error defined by comparing the calculated quantity with the corresponding exact results in the whole region of variation for all parameters given. In the present case, we have to compare the solution to equation (24) with the known numerical values[9] for all eigenvalue numbers $n = 0,1,2,\ldots\infty$ and all anharmonicity parameters $g \in (0,\infty)$. In the same way, if a fixed point condition would yield several possible branches for the governing function $z(n,g)$, then, for each given set of parameters, we should take the maximal error among all the branches. This is especially

important if one intends to estimate the accuracy of a method for its farther application to complicated problems whose exact solutions are not available. Thus by the maximal error we mean

$$\varepsilon_* = \sup_{g,n} \sup_{z(n,g)} \left| \frac{e_*(n,g)}{e(n,g)} - 1 \right| .$$

Fortunately, in our case we have the unique positive solution for the governing function (16). However, in Section 5 we shall present an example when there appear two positive branches of the governing function.

Calculating the maximal error of the self-similar approximation we conclude that it is of the order of 10^{-3} for all eigenvalues and all anharmonicity parameters.

4. COMPARISON WITH QUASICLASSICAL APPROXIMATION

Let us compare the accuracy of our method with that of other approximate approaches. For instance, for the anharmonic oscillator the quasiclassical (WKB) approximation can be used[9]. In this approximation the eigenvalues of the Hamiltonian (6) read

$$e_{WKB}(n,g) = \frac{(1 + 12gJ_n)^{4/3} - 1}{16g}, \qquad (31)$$

where $J_n = J_n(g)$ is to be obtained from the Bohr - Sommerfeld quantization condition which in the treated case is a transcendental equation

$$\frac{2}{\pi} K(p_n) J_n = n + \frac{1}{2} \qquad (32)$$

with $K(p_n)$ being a complete elliptic integral of the first kind,

$$K(p_n) = \int_0^1 \frac{dt}{[(1 - t^2)(1 - p_n^2 t^2)]^{1/2}} ,$$

and with the variable p_n given by the expression

$$p_n^2 = \frac{1}{2} [1 - (1 + 12gJ_n)^{-2/3}]. \qquad (33)$$

In this way, to obtain the eigenvalues in the WKB approximation one needs to solve the transcendental system of equations (31)-(33).

In the weak anharmonicity limit the WKB approximation leads to the expansion

$$e_{WKB}(n,g) \simeq (n + \tfrac{1}{2})[1 + 2(n + \tfrac{1}{2})g]; \; g \to 0, \qquad (34)$$

which is wrong as compared with the exact one[9] coinciding with the self-similar form (25). The WKB approximation has the correct asymptotic behaviour only in the strong anharmonicity and, simultaneously, high level limits $g \to \infty$, $n \to \infty$, when it becomes equivalent to the corresponding exact expansion (30),

$$e_{WKB}(n,g) \simeq e(n,g); \; g,n \to \infty. \qquad (35)$$

The accuracy of the WKB approximation is good only in this asymptotic region of $g \gg 1$, $n \gg 1$, and worsens for the low lying levels. For example, for the ground state $n = 0$ and $g \to \infty$ the error is 22%, and it becomes even higher for intermediate $g \sim 1$. The maximal error, as defined in the previous section, for the WKB approximation is too high to consider this approximation as a satisfactory one for the whole range of parameters g and n. Moreover, it is expressed through much more complicated formulae than the self-similar approximation.

5. RENORMALIZED PERTURBATION THEORY

Compare now our method of self-similar approximations with the renormalized perturbation theory[5-7,10]. As far as Stevenson[7] has shown that the minimal difference criterion is much poorer than the minimal sensitivity criterion, we shall use here the optimal variant defining z(n,g) by Eq.(13).

The first-order renormalized perturbation theory corresponds to $e_1(n,g,z)$ from (17) with z(n,g) defined by equation (14). In the weak anharmonicity limit one has the same expansion as the self-similar one given by (25). In the strong anharmonicity limit we get

$$e_1(n,g) \simeq \frac{3}{4} (n + \tfrac{1}{2})(6\gamma_n g)^{1/3}; \; g \to \infty, \qquad (36)$$

which differs from the self-similar form (26) so that the relation

$$e_*(n,g) \simeq e_1(n,g) \exp(-a_n/48) \qquad (37)$$

holds. For the high level limit, (36) transforms into

$$e_1(n,g) \simeq \frac{3}{4} n^{4/3} (6g)^{1/3}; \quad g,n \to \infty, \qquad (38)$$

which, for a better comparison with the exact expansion (30), can be written as

$$e_1(n,g) \simeq 1.362\,840\, n^{4/3} g^{1/3}; \quad g,n \to \infty. \qquad (39)$$

The accuracy of (39) is 1%.

The maximal error of the first-order approximation $e_1(n,g)$ is of the order of 10^{-2}, which is sufficiently higher than that of the self-similar approximation $e_*(n,g)$, being of the order of 10^{-3}.

The second-order renormalized approximation is equivalent in our case to $e_2(n,g,z)$ from (12) with $z(n,g)$ defined by the equation

$$\frac{\partial}{\partial z} e_2(n,g,z) = 0, \qquad (40)$$

which yields

$$z^6 - 2z^4 - 16\gamma_n g z^3 + z^2 + 16\gamma_n g z + \frac{170}{3}(n^2 + n + \frac{21}{17}) g^2 = 0. \qquad (41)$$

By using the substitution

$$u = z(z^2 - 1), \qquad (42)$$

eq.(41) can be changed to

$$u^2 - 16\gamma_n g u + \frac{170}{3}(n^2 + n + \frac{21}{17}) g^2 = 0. \qquad (43)$$

The latter equation has no real solution for $n = 0,1$ and has two real solutions

$$u_\pm = 8\gamma_n g\left(1 \pm \frac{1}{4}\sqrt{1 - \frac{5}{2} a_n}\right) \qquad (44)$$

for $n \geq 2$. Here γ_n is the same as (15) and a_n is given by (18). Therefore, the function $z(n,g)$ also has two real branches defined by the equation

$$z_\pm^3 - z_\pm - 6\gamma_n^\pm g = 0; \quad n \geq 2, \qquad (45)$$

in which

$$\gamma_n^\pm \equiv \frac{4}{3}\left(1 \pm \frac{1}{4}\sqrt{1 - \frac{5}{2} a_n}\right)\gamma_n. \qquad (46)$$

The high level limit of (46) is

$$\gamma_n^\pm \simeq \frac{4}{3}\left(1 \pm \frac{1}{4}\sqrt{\frac{11}{6}}\right) n; \quad n \to \infty,$$

hence

$$\gamma_n^+ \simeq 1.784669\, n, \quad \gamma_n^- \simeq 0.881998\, n. \qquad (47)$$

The weak anharmonicity limit coincides with the self-similar expression (25). In the strong anharmonicity limit, Eq.(45) gives

$$z_\pm(n,g) \simeq (6\gamma_n^\pm g)^{1/3}; \quad g \to \infty. \qquad (48)$$

Consequently, approximations (11) and (12) become

$$e_1^\pm(n,g) \simeq \frac{1}{2}(n + \frac{1}{2})\left(1 + \frac{\gamma_n}{2\gamma_n^\pm}\right)(6\gamma_n^\pm g)^{1/3}, \qquad (49)$$

and, respectively,

$$e_2^\pm(n,g) \simeq \frac{1}{2}(n + \frac{1}{2})\left(1 + \frac{\gamma_n}{2\gamma_n^\pm} - \frac{a_n^\pm}{24}\right)(6\gamma_n^\pm g)^{1/3}, \qquad (50)$$

where

$$a_n^\pm \equiv (n + \frac{1}{2})\frac{17\gamma_n}{3(\gamma_n^\pm)^2} + \frac{25}{6(\gamma_n^\pm)^2} + 6 - 12\frac{\gamma_n}{\gamma_n^\pm}. \qquad (51)$$

Eq.(51) is similar to (18) but defines two different branches which in the high level limit are

$$\lim_{n\to\infty} a_n^+ = 1.055\,213, \quad \lim_{n\to\infty} a_n^- = -0.321095. \qquad (52)$$

Approximation (11) in the strong anharmonicity and high level limit reads

$$e_1^+(n,g) \simeq 1.410\ 821\ n^{4/3}\ g^{1/3},$$
$$e_1^-(n,g) \simeq 1.365\ 262\ n^{4/3}\ g^{1/3}. \qquad g,n \to \infty \qquad (53)$$

The corresponding accuracies are 3% and 0.8%, respectively, as compared with the exact expansion (30).

Approximation (12) has the asymptotic form

$$e_2^+(n,g) \simeq 1.362\ 366\ n^{4/3}\ g^{1/3},$$
$$e_2^-(n,g) \simeq 1.376\ 920\ n^{4/3}\ g^{1/3}. \qquad g,\ n \to \infty \qquad (54)$$

These two branches have the asymptotic accuracies of 1% and 0.03%, respectively. In the whole range of the parameters g and $n \geq 2$ the approximation $e_2^+(n,g)$ corresponds to an error of 1%; and $e_2^-(n,g)$, to an error of 0.3%.

Thus, the second-order renormalized perturbation theory is applicable for calculating the eigenvalues of the anharmonic oscillator only for $n \geq 2$, where its maximal error, as defined in Section 3, is of the order of 10^{-2}.

Again, we see that the method of self-similar approximations is simpler and more accurate than the renormalized perturbation theory.

A natural question can arise whether it is possible to improve the second-order renormalized perturbation theory by an additional self-similar renormalization. To chek this, let us build the distribution of approximations (21). More exactly, we have now two distributions corresponding to $e_k^{\pm}(n,g)$ which are

$$y_{12}^{\pm}(\xi) = -\frac{48\xi^2/(n+\frac{1}{2})^4}{b_n^{\pm}\,[\xi^2/(n+\frac{1}{2})^2 - 1]^2}; \qquad (55)$$

here

$$b_n^{\pm} \equiv \frac{2}{3}\left(\frac{\gamma_n}{\gamma_n^{\pm}}\right)^2\left(6\,\frac{\gamma_n^{\pm}}{\gamma_n} - a_n - 6\right).$$

From equation (46) we can express

$$a_n = \frac{6\gamma_n^{\pm}}{5\gamma_n}\left(8 - 3\,\frac{\gamma_n^{\pm}}{\gamma_n}\right) - 6,$$

because of which

$$b_n^{\pm} = \frac{12}{5}\left(1 - \frac{\gamma_n}{\gamma_n^{\pm}}\right). \qquad (56)$$

The high level limit for (56) gives

$$\lim_{n\to\infty} b_n^+ = 1.055\ 213; \quad \lim_{n\to\infty} b_n^- = -0.321\ 095. \qquad (57)$$

The normalization (23) yields two branches of the self-similar approximation satisfying the equation

$$\frac{[e_*^{\pm}(n,g)/(n+\frac{1}{2})]^2 - 1}{[e_1^{\pm}(n,g)/(n+\frac{1}{2})]^2 - 1} = \exp\frac{1}{[e_*^{\pm}(n,g)/(n+\frac{1}{2})]^2 - 1} - \frac{1}{[e_1^{\pm}(n,g)/(n+\frac{1}{2})]^2 - 1} - \frac{b_n^{\pm}}{24}; \quad n \geq 2. \qquad (58)$$

In the strong anharmonicity limit, (58) leads to

$$e_*^{\pm}(n,g) \simeq e_1^{\pm}(n,g)/\exp(-b_n^{\pm}/48); \quad g \to \infty. \qquad (59)$$

Passing to the high level limit for these two branches we obtain

$$e_*^+(n,g) \simeq 1.380\ 144\ n^{4/3}\ g^{1/3},$$
$$e_*^-(n,g) \simeq 1.374\ 426\ n^{4/3}\ g^{1/3}. \qquad g,n \to \infty \qquad (60)$$

Comparing (60) with the exact expansion (30) we see that the asymptotic accuracy of $e_*^+(n,g)$ is 0.3%; and of $e_*^-(n,g)$, 0.2%. The numerical calculation shows that in the whole range of g and $n \geq 2$ the error of $e_*^+(n,g)$ is not more than 0.7% and that of $e_*^-(n,g)$ is less or of the order of 0.3%. Thus, an additional self-similar renormalization smoothes the difference between the branches of second-order renormalized perturbation theory and slightly improves it by making the maximal error 0.7% for $n \geq 2$.

However, no principal improvement is achieved, since, as before, there are no solutions for $n = 0,1$ and the maximal error is of the order of 10^{-2}. Therefore, it is more reasonable to use directly the method of self-similar approximations applied in Section 3, as it works for all $n = 0,1,2,\ldots\infty$ and $g \in (0, \infty)$ having the maximal error of the order of 10^{-3}.

The main difference between the renormalized perturbation theory and the method of self-similar approximations is that the former yields an additive-type renormalization while the latter leads to an exponential-type renormalization, which can be illustrated in the following way: Let us substitute an approximate form

$$f_s(n,g(n,\xi), z_k(n, g(n,\xi))) \approx f_k(n, g(n,\xi), z_k(n,g(n,\xi))) + R_{ks}(n)\xi$$

into the distribution of approximations (4), which gives

$$y_{ks}(n,\xi) \approx [R_{ks}(n)\,\xi]^{-1}.$$

Then, from normalization (5) we find the self-similar approximation

$$f_*(n, g, z(n,g)) \approx f_k(n, g, z(n,g))\,\exp[R_{ks}(n)]$$

with an exponential-type renormalization. A particular example of such a renormalization is given by formula (26).

Thus, we have shown that the method of self-similar approximations can be successfully used for the eigenvalue problem. The eigenvalues of the anharmonic oscillator can be calculated with a maximal error of the order of 10^{-3} for all anharmonicity parameters. The method is more simple and accurate than the quasiclassical approximation and renormalized perturbation theory.

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