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COMPUTER ALGEBRA TOOLS FOR HIGHER SYMMETRY ANALYSIS OF NONLINEAR EVOLUTION EQUATIONS

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1. INTRODUCTION

The symmetry analysis of differential equations is one of the central problems in modern applied mathematics and mathematical physics. Among numerous methods of analysis and integration of differential equations the most general and universal ones are based on their symmetry properties. S.Lie has introduced the concept of symmetry just for the purpose of creating solutions of differential equations. From the theoretical point of view the problems of symmetry analysis are investigated in sufficient detail. But in practice to find the symmetry group (or even some individual generators) of a given differential equation it is necessary to carry out extremely tedious algebraic manipulations. That is why computer algebra has continued to play an increasingly important part in the practical symmetry analysis [1].

Now there are several computer algebra packages for symmetry analysis of differential equations. Among them the big packages SODE for ordinary differential equations and SPDE for partial differential equations are the best developed [1]-[2] for determining so-called *classical* or *point* or *Lie* symmetries. They use the most general method of computation which is based on generating and solving of the determining system in the form of linear differential equations in functions which occur in the definition of a symmetry generator. Both Reduce and Scratchpad II versions of the packages SODE and SPDE have been designed according to basic concepts of software engineering. Moreover, data abstraction as one of the main attributes of the Scratchpad II system allowed one to gain very effective module organization of the package with the detailed investigation of its complexity [2]. The most difficult part of the whole computational process is simplification and integration of the determining equations. At this step a user has often to do a reasonable ansatz on the structure of symmetries. By this reason an interactive regime is always assumed.

In the searching of so-called *generalized* or *higher* (*Lie-Bäcklund*) symmetries, when functions which occur in the definition of a symmetry generator may depend not only on the point, i.e., the dependent and the independent, variables but also on the derivatives of the unknown functions, an appropriate ansatz plays even more important role. The point is that the existence of a higher symmetry imposes much more strong limitations on the equations under consideration than the existence of the classical Lie symmetries. Because of this, a universal computer algebra package for the construction of higher-order symmetries based on the most general scheme of computation (see, for example [3]) may not be usable for many nonlinear problems. Therefore special constructive and effective methods for finding the generalized symmetries in some sufficiently wide class of nonlinear differential equations are of interest for the design of the corresponding computer algebra packages.

In this paper a computer-aided approach to construction of higher symmetries is presented which can be applied to a wide class of multicomponent quasilinear partial differential equations of the evolution type. After necessary mathematical definitions and formulae (Sect.2), description of the computational procedure for higher symmetry analysis (Sect.3) and its implementation (Sect.4) in the form of package written in internal language (Rlisp) of the Reduce computer algebra system are given. The package consists of two functionally independent modules. One of them is destined for the symmetry analysis proper and the other for solving systems of nonlinear algebraic equations which arise in the presence of arbitrary numerical parameters. As an illustration, the computation of the third order Lie-Bãclund symmetries for eight-parametric family of coupled nonlinear Schrõdinger equations is considered (Sect.5).

2. MATHEMATICAL BACKGROUND

Among the partial differential equations of physical interest, of great importance is the class of polynomial-nonlinear evolution equations (NLEE) in one-spatial and one-temporal dimension of the following form

$$\begin{aligned} &u_t = \Phi(x,u,\ldots,u_N) = \Lambda u_N + F(x,u,\ldots,u_{N-1};\alpha_1,..\alpha_K), \quad N \geq 2 \\ &u = u(t,x) = (u^1,\ldots,u^M), \quad u_k = \partial^k u/\partial x^k, \quad F = (F^1,\ldots,F^M), \\ &\Lambda = diag(\lambda_1,\ldots,\lambda_M), \quad \lambda_i, \alpha_i \in \mathbf{C}, \quad \lambda_i \neq 0, \end{aligned} \tag{1}$$

where the vector function F is a polynomial in its arguments including numeric parameters α_i if any. F is said to be a differential function of $N-1$ order. The function Φ has the order N respectively.

The class (1) contains such well-known *integrable* NLEE as the Korteweg-de Vries equation, the Burgers equation, the nonlinear Schrõdinger equation and many other ones which are now under intensive investigation.

The concept of *integrability* is closely connected with the existence of higher symmetries [4]: NLEE is integrable if and only if it possesses infinitely many time-independent higher symmetries. But in practice the existence of M different higher symmetries is sufficient for integrability of M–component NLEE.

Definition. A vector function $H = (H^1,\ldots,H^M)$ of a finite number of differential variables $x,u,u_1,\ldots,u_n$ is a n-order (higher) *symmetry* of the system (1) if it leaves (1) invariant under the transformation $t' = t$, $x' = x$, $u' = u + \tau H(x,u,u_1,\ldots,u_n)$ within order τ. This means that H corresponds to the canonical Lie-Bãcklund operator [5]

$$X = \sum_{i=1}^{M} H^i \frac{\partial}{\partial u^i} + \cdots \tag{2}$$

and satisfies the differential equation

$$\frac{dH}{dt} = \Phi_*(H),$$

which is equivalent to the operator relation

$$\frac{dH_*}{dt} - [H_*, \Phi_*] = \frac{d\Phi_*}{d\tau}. \tag{3}$$

Here Φ_* and H_* are matrix differential operators

$$\Phi_* = \sum_{i=0}^{N} \Phi_i D^i, \quad [\Phi_i]_{kj} = \partial\Phi^k/\partial u^j_i, \quad H_* = \sum_{i=0}^{n} H_i D^i, \quad [H_i]_{kj} = \partial H^k/\partial u^j_i \tag{4}$$

and

$$D = \frac{d}{dx} = \frac{\partial}{\partial x} + \sum_{i=1}^{M}\sum_{j=0}^{\infty} u^i_{j+1}\frac{\partial}{\partial u^i_j}, \tag{5}$$

$$\frac{d}{dt} = \sum_{i=1}^{M}\sum_{j=0}^{\infty} D^j(\Phi^i)\frac{\partial}{\partial u^i_j}, \qquad \frac{d}{d\tau} = \sum_{i=1}^{M}\sum_{j=0}^{\infty} D^j(H^i)\frac{\partial}{\partial u^i_j}$$

are the total differentiation operators with respect to x, t and τ respectively.

3. CONSTRUCTION OF HIGHER SYMMETRIES

To compute higher-order ($n > N$) symmetries for a given NLEE of the form (1) the effective algorithms have been developed [6, 7, 8] which take into account the basic methods being used by experts [9] in their pencil and paper work.

The basic idea is to construct step by step the coefficients A_i, $i = n, n-1, \ldots, 0$ of the matrix differential operator

$$L = A_0 + A_1 D + \cdots + A_n D^n \tag{6}$$

as a solution of the operator equation

$$\frac{dL_*}{dt} - [L_*, \Phi_*] = \frac{d\Phi_*}{d\tau} = \frac{d\Phi_{N-1}}{d\tau} D^{N-1} + \cdots. \tag{7}$$

which corresponds to relation (3) with the constant diagonal matrix $\Phi_N = \Lambda$ as defined in (1).

The isolation of the coefficients of D^i in the operator equality (7) gives the following chain of equations in A_i

$$\begin{array}{ll} D^{N+n}: & [\Lambda, A_n] = 0, \\ D^{N+n-1}: & N\cdot\Lambda\cdot D(A_n) + [\Lambda, A_{n-1}] + [\Phi_{N-1}, A_n] = 0, \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ D^{N+n-i}: & N\cdot\Lambda\cdot D(A_{n-i+1}) + [\Lambda, A_{n-i}] + [\Phi_{N-1}, A_{n-i+1}] + B_i = 0, \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ D^N: & N\cdot\Lambda\cdot D(A_1) + [\Lambda, A_0] + [\Phi_{N-1}, A_1] + B_n = 0, \end{array} \tag{8}$$

where B_i is expressed in terms of A_j, $j > n - i + 1$.

The structure of the matrix Λ in (1) and the form of the i-th equation of the chain (8) make possible finding the diagonal parts of A_{n-i+1} and non-diagonal parts of A_{n-i}. For example, in the case of different eigenvalues λ_i, from the first two equations of (8) it follows that A_n is arbitrary diagonal number matrix $A_n = diag(\mu_1, \mu_2, \dots, \mu_M)$, $\mu_i \in \mathbb{C}$. The general recurrent formulae for A_i as solutions of (8) are given in [6, 7, 8]. Because of this, equations (8) allow one to compute sequentially matrices $A_n, A_{n-1}, \dots, A_1$ and the non-diagonal part of A_0.

To provide the existence of a *local* higher symmetry $H(x, u, u_1, \dots, u_n)$, the chain (8) must admit *local*, i.e. depending on a finite number of dynamic variables taken from an infinite set $x, u, u_1, \dots$, solutions A_i as well. From Eqs.(8) it follows that to find the diagonal part of A_i it is necessary to solve an equation of the form

$$D(Q) = S, \tag{9}$$

where the operator D is defined by expression (4). For a given local S, Eq.(9) admits a local solution $Q = D^{-1}(S)$ only if S satisfies a number of restrictions [6]. The reverse operator D^{-1} is none other than an integration operator with respect to x. Hence at each step of the chain (8) a number of arbitrary constants is generated. These constants may be important for the analysis of the next steps.

After the construction of the n-th order operator (6) by means of Eqs.(8) one can compute the n-th order symmetry using the operator relation

$$H_* - diag(H_0) = \tilde{L} \equiv L - diag(A_0), \tag{10}$$

which follows from Eqs.(3) and (7). Operating by both sides of (10) on $u_1 \equiv u_x$ we obtain

$$\tilde{D}(H) = \tilde{L}(u_1), \quad \tilde{L}_j = D - \partial/\partial x - {u_1}^j \cdot \partial/\partial u^j \tag{11}$$

Eq.(11) defines the components H^j of the symmetry H within arbitrary functions $h^j(u^j)$

$$H^j = \tilde{D}^{-1}(\tilde{L}u_1)^j + h^j(u^j). \tag{12}$$

The algorithms of D and $\tilde{D}$ reversion are described in [6]. They allow verifying the conditions of solvability of Eqs.(9) and (11)

$$S \in Im(D), \quad (\tilde{L}u_1)^j \in Im(\tilde{D}_j).$$

The notation $\rho \in ImD$ means that $\rho = D\sigma$ where σ is some local function. It is just solvability of (9) in terms of the corresponding local functions of the chain (8) that leads to the existence of higher symmetries for Eq.(1).

Since a higher symmetry of some fixed order may not exist for a given NLEE of the form (1), the best computational strategy is the following one.

Step 1. Verification of the necessary conditions for the existence of higher symmetries. Those necessary conditions follow from solvability of Eq.(7) in terms of the series (6) and have the form of the local conservation laws [6]-[7]

$$\frac{d}{dt}R(i,j) \in ImD, \quad i = 0, 1, \dots \ , j = 1, 2, \dots, M \ . \tag{13}$$

The densities $R(i,j)$ in (13) are computed in terms of the r.h.s. of (1) [6]-[7]. For example, $R(0,j) = \partial F^j / \partial u^j_{N-1}$.

In the presence of the arbitrary parameters α_i in (1) the necessary conditions (13) for a higher-order symmetry are equivalent to some system of nonlinear algebraic equations in those parameters. As an illustration, let us consider the two-component case $u = (v, w)$ and the following local expression $\rho = a * v_2 * w + b * v * w_2 + c * v_1 * w_1$. The condition $\rho = D\sigma$ is solvable in terms of the local function σ if and only if $c = a + b$. In that case $\sigma = a * v * w_1 + b * v_1 * w$.

In what follows we have to verify whether the obtained algebraic system has a solution. It is remarkable that the Gröbner basis technique [10], being the well-known tool of computer algebra, gives the most elegant and effective method for solving that problem.

Step 2. The previous step gives very important information on the existence of a higher symmetry. Now it is possible to try to construct the explicit form of the latter for some fixed order using the above algorithm. At this step we may obtain new restrictions on the r.h.s. of (1) in the form of algebraic equations in its parameters.

Step 3. Solving of the resulting system of the nonlinear algebraic equations obtained at steps 1,2. Here the Gröbner basis technique again provides a means for simplifying the problem drastically. Moreover, in many cases, in particular, in problems of classification of integrable NLEE [12], it allows one to find all (even infinitely many) the solutions in explicit algebraic form.

4. IMPLEMENTATION IN REDUCE

We have implemented the above computational scheme for the polynomial-nonlinear evolution equations (1) in the Reduce computer algebra system [13]. Our package consists of the two functionally different modules written in the language Rlisp of the Reduce symbolic mode.

The first module HSYM, which abbreviates Higher Symmetry, provides the procedures for the sequential verifying of the necessary conditions (13) in the case when there are no arbitrary parameters in the initial NLEE (1). If they are the HSYM generates an equivalent system of nonlinear algebraic equations. The solvability of the latter guarantees the existence of the higher-order conservation laws (13). Their densities $R(i,j)$ are computed in explicit form. The HSYM has also a special procedure realizing the method of Sect.3 for finding the explicit form of the higher symmetry of the order specified by a user.

The restriction imposed in the HSYM that F is a polynomial in its arguments, being very important from the viewpoint of applications, has made possible establishing the efficient algorithms for the realization of all the necessary algebraic manipulations. They are based on the built-in recursive representation for polynomials in "standard form" and effectively use the corresponding built-in procedures acting at "standard forms" and "standard quotients" of the Reduce internal data.

The second module ASYS, which abbreviates Algebraic System, provides verifying the consistency of the systems of algebraic equations which arise at step 1 of Sect.3 as the necessary conditions for the existence of higher symmetries. For this purpose it is sufficient to compute [10] a Gröbner basis G for an ideal generated by a set of the polynomials under consideration. The system is unsolvable if $\{1\} \in G$. The ASYS contains the procedures for a Gröbner basis computation realizing the well-known Buchberger algorithm [10].

Solving the systems of algebraic equations at step 3 of Sect.3 is accomplished in the ASYS as follows. A lexicographic Gröbner basis is constructed. Then the ASYS computes the dimension and independent sets of variables for the ideal according to the method described in [11]. If our algebraic equations have infinitely many solutions the ideal has a positive dimension and the variables of each independent set can be considered as free parameters. In this case the obtained Gröbner basis is recomputed for each set of parameters leaving the order of the other variables unchanged. As a result a set of Gröbner bases is obtained with a simple structure and with "separated" variables (G is "triangularized") [10]. In this way the problem of solving a (often very complicated) system of nonlinear algebraic equations is always reduced to solving an equation in one variable.

In the general case only this last stage of computation may not be done automatically by our package. But our experience shows that the solutions can often be found with the help of the Reduce polynomial factorization facilities [13]. In the case of integrable NLEE their higher symmetry analysis leads to algebraic equations which can certainly be solved in completely algebraic way by using the ASYS [12].

5. EXAMPLE

As an example of application of our package let us consider the following eight-parametric system of two coupled nonlinear Schrödinger equations

$$\begin{cases} i(\Psi_1)_t = \alpha_1(\Psi_1)_{xx} + \beta_1|\Psi_1|^2\Psi_1 + \gamma_1|\Psi_2|^2\Psi_1 + \delta_1\Psi_2^2\Psi_1^*, \\ i(\Psi_2)_t = \alpha_2(\Psi_2)_{xx} + \beta_2|\Psi_2|^2\Psi_2 + \gamma_2|\Psi_1|^2\Psi_2 + \delta_2\Psi_1^2\Psi_2^*. \end{cases} \tag{14}$$

Here Ψ_i are complex functions and α_i, β_i, γ_i, δ_i $(i = 1, 2)$ are real parameters. This family of nonlinear evolution equations includes, for example, the systems describing

the interaction of electromagnetic waves with different polarizations in nonlinear optics [14] and the resonant interaction of long acoustic and short waves [15]. The complete integrability of (14) at $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$ have been studied by another method in [16].

In order to be integrable (14) must have the higher symmetries of the order $n \geq 3$ of the form

$$H_i(\Psi_j, (\Psi_j)_x, ..., (\Psi_j)_{x...x(n-times)}), \;\; i, j = 1, 2; \;\; n \geq 3,$$

which correspond to the canonical Lie-Bäcklund operators (2).

Introducing the notations $u = \Psi_1$, $v = \Psi_1^*$, $p = \Psi_2$, $q = \Psi_2^*$, $\tau = it$ we can rewrite (14) in the form (1)

$$\begin{aligned} u_\tau &= \alpha_1 u_{xx} + \beta_1 u^2 v + \gamma_1 upq + \delta_1 vp^2 \,, \\ v_\tau &= -\alpha_1 v_{xx} - \beta_1 uv^2 - \gamma_1 vpq - \delta_1 uq^2 \,, \\ p_\tau &= \alpha_2 p_{xx} + \beta_2 p^2 q + \gamma_2 uvp + \delta_2 u^2 q \,, \\ q_\tau &= -\alpha_2 q_{xx} - \beta_2 pq^2 - \gamma_2 uvq - \delta_2 v^2 p \,. \end{aligned}$$

As a result of the first two necessary conditions, the module HSYM generates the three set of algebraic equations in dependence on the relation between α_1 and α_2 and under assumption that $\alpha_1\alpha_2 \neq 0$ in accordance with (1):

1) $\alpha_1 \neq \pm\alpha_2$,

$$\begin{aligned} &\alpha_1\gamma_2\delta_1 - \alpha_2\gamma_2\delta_1 = \beta_1\gamma_1\delta_2 - \gamma_2^2\delta_1/2 = \beta_1\beta_2\gamma_2 - \gamma_1\gamma_2^2/4 = \beta_1\delta_1 - \gamma_1\delta_2/2 = 0, \\ &\gamma_1^2\delta_2 - 2\beta_2\gamma_2\delta_1 = \gamma_1\gamma_2\delta_1 - 2\beta_2\gamma_2\delta_1 = \gamma_1\gamma_2\delta_2 - \gamma_2^2\delta_1 = \beta_2\delta_2 - \gamma_2\delta_1/2 = 0, \\ &\alpha_1\beta_1\gamma_1 - \alpha_2\gamma_1\gamma_2/2 = \alpha_1\gamma_1\gamma_2 - 2\alpha_2\beta_2\gamma_2 = \alpha_1\gamma_1\delta_2 - \alpha_2\gamma_1\delta_2 = 0, \end{aligned} \tag{15}$$

2) $\alpha_1 = \alpha_2$,

$$\begin{aligned} &\beta_2^2\delta_1 - \delta_1^3 = \beta_2\gamma_2\delta_1 - 2\delta_1^2\delta_2 = \beta_2\delta_2 - \gamma_2\delta_1/2 = \gamma_2^2\delta_1 - 4\delta_1\delta_2^2 = \beta_1\delta_1 - \gamma_2\delta_1/2 = 0, \\ &\gamma_2^2\delta_2 - 4\delta_2^3 = \gamma_1^2 - \gamma_1\beta_2 - 2\delta_1^2 = \gamma_1\gamma_2 - \beta_2\gamma_2 - 2\delta_1\delta_2 = \beta_1\delta_2 - \gamma_2\delta_2/2 = 0, \\ &\gamma_1\delta_1 - 2\beta_2\delta_1 = \gamma_1\delta_2 - \gamma_2\delta_1 = \beta_1\gamma_1 - \beta_2\gamma_2 = \beta_1\gamma_2 - \gamma_2^2 + 2\delta_2^2 = 0, \end{aligned} \tag{16}$$

3) $\alpha_1 = -\alpha_2$,

$$\begin{aligned} &\beta_1\gamma_1 - \beta_2\gamma_2 = \beta_1\gamma_2 + \gamma_2^2 = \beta_1\delta_1 = \gamma_1^2 + \gamma_1\beta_2 = \gamma_1\gamma_2 + \beta_2\gamma_2 = 0, \\ &\gamma_1\delta_1 = \gamma_1\delta_2 = \beta_2\delta_2 = \gamma_2\delta_1 = \gamma_2\delta_2 = 0. \end{aligned} \tag{17}$$

The module ASYS allows one readily to obtain all the solutions of (15)-(17). But the construction of a symmetry according to the algorithms of Sect.3 which are implemented in the module HSYM, may lead to new restrictions on the initial evolution equations in addition to those which follow from the necessary integrability conditions.

In the case of polynomial-nonlinear evolution equations with arbitrary parameters the HSYM allows one to produce an extra set of algebraic equations for a given order of a higher symmetry (see Sect.2,3). We omit here those extra equations because of their awkwardness.

Table 1 gives all the solutions of (15)-(17) such that (14) possesses the Lie-Băclund symmetries of the order $n \geq 3$. The corresponding third order symmetries are listed in Table 2.

Table 1
Subset of solutions of (15)-(17) which provides the existence of Lie-Băcklund symmetries

Free variables	Solutions
1) $\alpha_1, \alpha_2, \beta_1, \beta_2$	$\gamma_1 = 0, \gamma_2 = 0, \delta_1 = 0, \delta_2 = 0.$
2) $\alpha_1, \beta_1, \beta_2$	$\alpha_2 = \pm\alpha_1, \gamma_1 = \pm\beta_2, \gamma_2 = \pm\beta_1, \delta_1 = 0, \delta_2 = 0.$
3) $\alpha_1, \delta_1, \delta_2$	$\alpha_2 = \alpha_1, \beta_1 = \pm\delta_2, \gamma_1 = \pm 2\delta_1, \beta_2 = \pm\delta_1, \gamma_2 = \pm 2\delta_2.$

Table 2
Lie-Băcklund symmetries of the third order for the solutions of Table 1

Free variables	Symmetries
1) $\alpha_1, \alpha_2, \beta_1, \beta_2$	$H_1 = \alpha_1(\Psi_1)_{xxx} + 3\beta_1(\Psi_1)_x \mid \Psi_1 \mid^2$ $H_2 = \alpha_2(\Psi_2)_{xxx} + 3\beta_2(\Psi_2)_x \mid \Psi_2 \mid^2$
2) $\alpha_1, \beta_1, \beta_2$	$H_1 = \alpha_1(\Psi_1)_{xxx} \pm 3/2\ \beta_2(\Psi_1\Psi_2)_x\Psi_2^* + 3\beta_1(\Psi_1)_x \mid \Psi_1 \mid^2$ $H_2 = \pm\alpha_1(\Psi_2)_{xxx} \pm 3/2\ \beta_1(\Psi_1\Psi_2)_x\Psi_1^* + 3\beta_2(\Psi_2)_x \mid \Psi_2 \mid^2$
3) $\alpha_1, \delta_1, \delta_2$	$H_1 = \alpha_1(\Psi_1)_{xxx} \pm 3(\Psi_1)_x(\delta_1 \mid \Psi_2 \mid^2 + \delta_2 \mid \Psi_1 \mid^2) + 3\delta_1(\Psi_2)_x(\Psi_2\Psi_1^* \pm \Psi_1\Psi_2^*)$ $H_2 = \alpha_1(\Psi_2)_{xxx} \pm 3(\Psi_2)_x(\delta_1 \mid \Psi_2 \mid^2 + \delta_2 \mid \Psi_1 \mid^2) + 3\delta_2(\Psi_1)_x(\Psi_1\Psi_2^* \pm \Psi_2\Psi_1^*)$

We conclude that all the systems of the form (1) possessing the canonical Lie-Băcklund symmetries of the above structure are exhausted by Table 1. This conclusion is consistent with the results of Ref.[16]. The complete list of the third order symmetries is given in Table 2. The computation of the symmetries $1) - 3)$ with our Reduce package requires about 20, 40 and 50 seconds on an IBM PC AT-386 (25 Mhz) respectively. Other canonical Lie-Băcklund symmetries of the order $n \geq 4$ can be found in a completely automatic way as well.

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