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QUANTUM STATES OF THE PHOTON PHASE OPERATOR

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The first step towards the phase operator Φ for the quantum Bose-oscillator was done by Dirac in 1927 [1] who suggested the following "polar decomposition" of the creation a^+ and annihilation a operators

$$a^+ = \sqrt{N}\, e^{-i\Phi}, \quad a = e^{i\Phi}\sqrt{N}, \quad N = a^+ a, \quad [a, a^+] = 1. \qquad (\mathrm{I})$$

It was found afterwards that such a decomposition really cannot exist and a number of attempts to find some substitute of (I) was done. They may be found in the review [2]. The first mathematically correct construction of the operator Φ was proposed by Garrison and Wong in 1970 [3]. They considered the phase operator as a multiplication operator in the Hilbert space of analytical functions on the unit circle of complex plane and found the operator canonically conjugated to it. Unfortunately, the connection of their construction to the polar decomposition problem for the operators a, a^+ was not investigated; so this result proved to be helpless for physicists. The problem of polar decomposition (I) was solved by Popov and Yarunin in 1973 [4]. They considered the r-dimensional sub-space of the Hilbert space corresponding to the harmonic oscillator. The projections of a, a^+ on this subspace

$$a_r = P_r a P_r, \quad a_r^+ = P_r a^+ P_r, \quad P_r = \sum_{n=0}^{r-1} |n\rangle\langle n|, \quad N|n\rangle = n|n\rangle$$

have a single decomposition [x] (up to the unitary equivalence)

$$a_r = e^{i\Phi_r}\sqrt{N_r}\,, \quad a_r^+ = \sqrt{N_r}\,e^{-i\Phi_r}\,, \quad N_r = a_r^+ a_r \qquad (2)$$

$$\langle m|\Phi_r|n\rangle = \frac{2\pi}{r^2}\sum_{k=0}^{r-1} k \exp i\left[\frac{2\pi k}{r}(m-n)\right].$$

There is no mathematical problem to derive formulae (2). But there is a loss of physical meaning, in them, because in the subspace of projected variables Φ_r, N_r we have the second "vacuum state" $|r-1\rangle$ and commutation relation for the operators a_r, a_r^+ are identical to those for the angular momentum components [6]. So the problem of limit $r\to\infty$ in (2) is important. There are two various possibilities to solve this problem. One of them was realised in [4]. The second was proposed by Barnet and Pegg [5] and developed by other authors [7].

We start with the possibility [5,7].

It is based on the calculation of the average value of Φ_r in some physical state, characterised by the distribution operator ρ

$$\langle\Phi_r\rangle = \sum_{k=0}^{r-1}\langle f_k^r|\rho|f_k^r\rangle\,\varphi_k^r =$$

$$= \frac{1}{r}\sum_{k=0}^{r-1}\sum_{m,n=0}^{r-1}\langle m|\rho|n\rangle\, e^{i\frac{2\pi}{r}k(n-m)} \qquad (3)$$

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[x] The same formulae were derived by Barnett and Pegg in 1989 [5].

Quantum phase states $|f_k^r\rangle$ in the r-dimensional subspace are defined as follows:

$$\Phi_r|f_k^r\rangle = \varphi_k^r|f_k^r\rangle\,, \quad \varphi_k^r = \frac{2\pi}{r}k\,,$$

$$|f_k^r\rangle = \frac{1}{\sqrt{r}}\sum_{m=0}^{r-1}\exp\left(i\frac{2\pi}{r}km\right)|m\rangle\,,$$

$$k = 0, 1, \dots r-1\,.$$

It was noticed in [5,7] that the calculation of the limit $r\to\infty$ in (3) gives the phase distribution function $P(\varphi)$ in accordance with the usual formula

$$\lim_{r\to\infty}\langle\Phi_r\rangle = \int_0^{2\pi}\varphi\,P(\varphi)\,d\varphi \qquad (4)$$

By comparing (3) and (4) one can suppose, as it was done in [5,7], that the distribution function $P(\varphi)$ is expressed by formula

$$P(\varphi) = \frac{1}{2\pi}\sum_{m,n=0}^{\infty} e^{i(n-m)\varphi}\langle m|\rho|n\rangle\,. \qquad (5)$$

It should be noted that formula (5) looks like some quasi-classical approach in which the discrete number φ_k^r is changed by the variable $0\le\varphi\le 2\pi$.

Now we are going to consider the second possibility of taking the limit $\tau \to \infty$ in (2), and to show that really formula (5) is not exact.

This possibility was realised in [4]. In that work the quantum states of an operator Φ

$$\Phi = \lim_{\tau \to \infty} \Phi_\tau \,, \quad \langle m|\Phi|n\rangle = \begin{cases} \pi & n = m \\ \dfrac{i}{n-m} & n \neq m \end{cases}$$

were considered and their scalar products with the oscillator basic functions were found

$$\langle n|\Phi|f_\varphi\rangle = \varphi \langle n|f_\varphi\rangle \,, \quad 0 \leqslant \varphi \leqslant 2\pi$$
$$\langle n|f_\varphi\rangle = \frac{1}{i\,(2\pi)^{3/2}} \oint g(z,\varphi)\, z^{n-1}\, dz \,. \qquad (6)$$

Integration over the unit circle in (6) is made and the function $g(z,\varphi)$ is determined as follows:

$$g = \frac{1}{2\pi}\, \frac{z}{z - e^{i\varphi - \varepsilon}} \exp\left[\psi_-(z,\varphi) - \psi_-(e^{i\varphi}, \varphi)\right] ,$$

$$\psi_- = \frac{1}{2\pi i} \oint \frac{\psi(z')\,dz'}{z' - z\,e^{\varepsilon}} \,, \quad e^{\psi} = \frac{1 + i\,\frac{\ln z}{\varphi}}{1 - z\,e^{-i\varphi}} \,, \quad \varepsilon \to 0 \,.$$

Here the function ψ_- is an analytical function outside the unit circle of the complex plane z. It should be noticed that the operator Φ is a self-adjoint operator with a finite norm in the Hilbert space. The point of the oscillator phase problem is that the formulae [4]

$$\Phi = \pi + i\left[\ln(1 - V^+) - \ln(1 - V)\right] ,$$
$$a^+ = \sqrt{N}\, V^+ \,, \quad a = V\sqrt{N}$$

are fulfilled instead of (1) and that Φ and N is not a canonically conjugated pair of operators in an ordinary quantum mechanical sense. The mathematical equivalence of this construction to that proposed in [3] was proved in [8].

Taking (6) into account we find the exact formula for the average value of Φ

$$\langle\Phi\rangle = \sum_{m,n=0}^{\infty} \langle m|\rho|n\rangle\langle n|\Phi|m\rangle = \int_0^{2\pi} \varphi\, Q\, d\varphi \,,$$
$$Q = \sum_{m,n=0}^{\infty} \langle m|\rho|n\rangle\langle n|f_\varphi\rangle\langle f_\varphi|m\rangle \qquad (7)$$

It is shown in [4] that the asymptotics of the scalar product $\langle n|f_\varphi\rangle$ as $n \to \infty$ is

$$\langle n|f_\varphi\rangle \longrightarrow \frac{e^{in\varphi}}{\sqrt{2\pi}} \,. \qquad (8)$$

If we substitute (8) into Q , we can see that $Q(\varphi)$ becomes equal to $P(\varphi)$ in (5). It means that the function $P(\varphi)$ in (4,5) is an approximation which is derived from the exact formulae (7) if we change the matrix elements $\langle n | f_\varphi \rangle$ by their asymptotic values (8).

It is clear that the discrepancy in $\langle \Phi \rangle$, produced by this change, depends on the physical nature of the distribution ρ . Namely, if ρ describes a highly excited state of Bose-system, the terms in (7) with small m,n don't give much contribution to (7) and expression (5) is a good approximation. Such a situation is expected to be in most problems of coherent opticts. In the opposite case, when ρ describes the system near the ground state with small m,n, the mistake due to the change $Q(\varphi)$ by $P(\varphi)$ can be noticeable. Such a situation may happen in some low-temperature collective effects in oondensed matter physics.

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