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ON THE INVARIANT MEASURE FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Introduction

In the paper the invariant measure for the nonlinear Schrödinger equation (NSE)

$$iu_t + u_{xx} + f(x,|u|^2)u = 0 \quad , t \in R \tag{I}$$

is constructed. Two problems are considered:
the periodic problem when

$$u(x,t_0) = u_0(x), \tag{2}$$

functions $u_0(x)$ and $f(x,s)$ are periodic with respect to x with period $A>0$, and it is required that the solution $u(x,t)$ is periodic with period A with respect to x . The second problem is the first boundary problem:

$$u(0,t) = u(A,t) = 0 \quad , t \in R, \tag{3}$$

$$u(x,t_0) = u_0(x) \quad , \quad x \in [0,A]. \tag{4}$$

Many important properties of the dynamical system can be proved by using the invariant measure /2,7,13/ . For the dynamical system with the compact phase space the existence of the invariant measure is proved by N.N.Bogolubov and N.M.Krylov /2/ (see also [13]).

Equation (I) describes many physical phenomena such as the waves on the deep water and in the plasma, behaviour of the non-ideal Bose--gas, the propagation of heat impulse in the solid states /I0/ . At the last time, in connection with various applioations in teohnology the problem of propagation of light impulse in a layered medium, which may be described also by equation (I), has been actual (see /11/). As it is proved in /16/, this equation may be introduoed by some general considerations and has a universal character, too.

Interest in equation (I) is oonneoted to a ground extent with the possibility of the application of the inverse problem method to it if $f(x,s) = \pm s$. But now in many physical articles the functions f of another kind are oonsidered, for example, whioh depend on x (see reviews /I0,11/).

Let $S(t)$ be the operator of the evolution for problems oonsidered. Then, the invariant measure for the problem (I)-(2) or (I),(3),(4)) is the Borel measure on the phase space L (which will be defined later) such that for any Borel set there follows $\mu(S(t)\Omega) = \mu(\Omega)$. Recently, the construoted measure was introduoed in the paper /6/ but ~~without sufficient mathematical arguments.~~
the invariance of the measure was not proved in this article

In 1° some results on the uniqueness and existence of solutions for the problems (I), (3), (4) and (I), (2) are established. Only sketches of the proofs are established because there were few articles on this matter (/4,6,12/ , for example).

In the 2° σ-additive Borel measure w is constructed on the phase space L where L is the space of periodic functions on R which belongs to L_2 locally for the problem (I)-(2) and $L = L_2(0,A)$ for the problem (I), (3), (4). In fact, w is a Gaussian measure.

In 3° the invariance of some measure μ for the problem is proved. The measure μ depends on w as follows

$$\mu(\Omega) = \int_{\Omega} e^{\Phi(u)}\, w(du),$$

where $\Omega \subset L$ is a Borel set and Φ is a continuous bounded functional on L.

The applications are contained in 4°. The important corollary is the Poincare recurrence theorem. By this theorem almost all points of L are stable in the Poisson sense. There is the old problem of Fermi-Past-Ulam (FPU). These authors considered the chain of balls with the nonlinear interaction. They studied the problem by computer and found out that the solution of the corresponding system of equations with any initial condition after some time is returned to the initial value with any accuracy. Later, in the soliton theory the property of solutions to return back was called the FPU phenomenon for other soliton equations, too. The results of the paper explain the FPU property.

The next functions are the examples of physically important functions for which the results of the paper are valid:

$$f(x,s) = 1 - e^{-\alpha s} \quad (\alpha > 0), \qquad f(x,s) = \frac{s}{1+s}.$$

At the time when this paper was written the author learned about the paper /1/ in which the invariant measure for two-dimensional Navie - Stooks equations was constructed. The methods in these ~~two~~ papers are completely different.

1°. Let us assu...

(f) functions $f(x,s)$, $f'_x(x,s)$, $(1+s)\, f'_s(x,s)$ be real, continuous and bounded on $[0,A] \times [0,\infty)$:

$$|f(x,s)| + |f'_x(x,s)| + |(1+s) f'_s(x,s)| \le C,$$

where $C = const > 0$.

$m_0 > 0$

Let $\frac{1}{2} m_0 > \sup_{x,s} |f(x,s)|$. Then, H is the space of functions g defined on R which are periodic with period A for the problem (I)-(2) belonging to ~~complex~~ $H^1(0,A)$ and H is the space $H^1_0(0,A)$ for the problem (I), (3), (4) but in both cases the space H is considered under the field of real numbers with the scalar product

$$\langle f,g \rangle = \int_0^A \{ Re\, f'(x) \cdot Re\, g'(x) + Im\, f'(x) \cdot Im\, g'(x) + m_0 [Re\, f(x) \cdot Re\, g(x) + Im\, f(x) \cdot Im\, g(x)] \}\, dx$$

and the norm $\|f\| = \langle f,f \rangle$. Similarly, let L be equal to the space of complex functions $L_2(0,A)$ under the field of real numbers with the scalar product

$$(f,g) = m_0 \int_0^A \{ Re\, f(x) \cdot Re\, g(x) + Im\, f(x) \cdot Im\, g(x) \}\, dx$$

and the norm $|f|_L = (f,f)$.

We denote by Δ the closure of the operator $\frac{d^2}{dx^2}$ with boundary conditions (3) or periodic on L. It is well-known that Δ is a self-adjoint operator. Let H^α be the subset of L consisting of such functions $u \in L$ that $(-\Delta)^{\frac{\alpha}{2}} u \in L$ and let $(u,v)_\alpha = (u,v) + m_0^{-1}((-\Delta)^{\frac{\alpha}{2}} u, (-\Delta)^{\frac{\alpha}{2}} v)$. Then L, H^α, H are the Hilbert spaces. It is clear also that $H = H^1$ and $\langle u,v \rangle = (u,v)_1$ for any $u,v \in H$.

In what follows we shall consider the problem (I),(3),(4) because the problem (I)-(2) may be considered similarly.

Let $N = 2^n$ for every integer n,

$$0 = x_0 < x_1 < \ldots < x_N = A,$$

where $x_i = ih$, $i = \overline{0,N}$, $h = AN^{-1}$.

Let us consider the system of ordinary differential equations

$$i r_{it} + \frac{r_{i-1} - 2r_i + r_{i+1}}{h^2} + f(x_i, |r_i|^2) r_i = 0, \quad i = \overline{1,N-1}, \quad (5)$$

$$r_0(t) = r_N(t) = 0, \quad t \in R, \quad (6)$$

$$\tau_i(t_0) = \tau_i^{(0)}, \quad i = \overline{0,N}, \quad \tau_0^{(0)} = \tau_N^{(0)} = 0. \tag{7}$$

Let

$$\tau_0^N = (\tau_0^{(0)}, \tau_1^{(0)}, \dots, \tau_N^{(0)}), \quad u_0^N = \operatorname{Re} \tau_0^N, \quad v_0^N = \operatorname{Im} \tau_0^N,$$

$$\tau^N(t) = (\tau_0(t), \tau_1(t), \dots, \tau_N(t)), \quad u^N(t) = \operatorname{Re} \tau^N(t), \quad v^N(t) = \operatorname{Im} \tau^N(t)$$

By (f) the system (5)-(7) has a unique solution for all $t \in R$ and every τ_0^N.

We consider the generalized solution of the problem (I), (3),(4) which satisfies the integral equation

$$u(x,t) = e^{i(t-t_0)\Delta} u_0 + i \int_{t_0}^{t} e^{i(t-s)\Delta} [f(x, |u(x,s)|^2) u(x,s)] \, ds. \tag{8}$$

where $e^{it\Delta}$ is the operator from L to L and

$$e^{it\Delta} u = ((\cos t\Delta) \operatorname{Re} u - (\sin t\Delta) \operatorname{Im} u, (\sin t\Delta) \operatorname{Re} u + (\cos t\Delta) \operatorname{Im} u).$$

It is clear that the operator $e^{it\Delta}$ is defined on the dense set in L and is bounded. Thus, by continuity it can be extended to all L and also

$$|e^{it\Delta}|_L \le C' < \infty$$

(see /14/).

Let X_n be the linear space which consists of every $\bar{\tau}_N = (\bar{u}_N, \bar{v}_N)$: where $\bar{u}_N, \bar{v}_N$ are the functions defined on $[0,A]$, the graphs of which are the broken lines connecting the points $\{x_i, \bar{u}_N(x_i)\}_{i=\overline{0,N}}$, $\{x_i, \bar{v}_N(x_i)\}_{i=\overline{0,N}}$ and $\bar{\tau}_N(0) = \bar{\tau}_N(A) = 0$. There is a natural one-to-one correspondence between the pairs of functions on the network $\{x_i\}_{i=\overline{0,N}}$ and elements of the space X_n. Then, we can compare the function $\bar{\tau}_N(t) \in X_n$ with the solution of (5)-(7) and the element $\bar{\tau}_N^{(0)}$ with τ_0^N.

Let us define some bounded operator $\bar{\Delta}_n$ on X_n. For every $\bar{\tau}_N \in X_n$ we compare $\tau^N = (\tau_0^N, \tau_1^N, \dots, \tau_N^N)$, where $\tau_0^N = \tau_N^N = 0$,

$$\tau_i^N = \Delta_n \bar{\tau}_N = \frac{\bar{\tau}_N(x_{i-1}) - 2\bar{\tau}_N(x_i) + \bar{\tau}_N(x_{i+1})}{h^2} \quad \text{for } i = \overline{1, N-1}$$

and let $\bar{\Delta}_n \bar{\tau}_N$ be the element of X_n which corresponds to τ^N. On the subspace $X_n^{\perp}$ of L orthogonal to X_n we define the operator $\bar{\Delta}_n$ as identical and extended it on all L by linearity. Then, $\bar{\Delta}_n$ is a bounded and self-adjoint operator on L.

For any $u \in L$ let

$$e^{it\bar{\Delta}_n} u = ((\cos t\bar{\Delta}_n) \operatorname{Re} u - (\sin t\bar{\Delta}_n) \operatorname{Im} u, (\sin t\bar{\Delta}_n) \operatorname{Re} u + (\cos t\bar{\Delta}_n) \operatorname{Im} u).$$

It is clear that there exists the constant $C'' > 0$ such that

$$|e^{it\bar{\Delta}_n}|_L \le C''$$

for all $n, t \in R$.

We compare the function $g_n(x,t)$ to the function $f(x, |\bar{\tau}_N|^2) \bar{\tau}_N$ such that $g_n(x,t) \in X_n$ for every fixed t and $g_n(x_i,t) = f(x_i, |\bar{\tau}_N(x_i,t)|^2) \bar{\tau}_N(x_i,t)$. It is clear that

$$\bar{\tau}_N(x,t) = e^{i(t-t_0)\bar{\Delta}_n} \bar{\tau}_N(x,t_0) + i \int_{t_0}^{t} e^{i(t-s)\bar{\Delta}_n} g_n(x,s) \, ds, \tag{9}$$

where $\tau_N(x,t)$ corresponds to the solution of (5)-(7).

We consider also the equation

$$u_n(x,t) = e^{i(t-t_0)\bar{\Delta}_n} u_{0n} + i \int_{t_0}^{t} e^{i(t-s)\bar{\Delta}_n} [f(x, |u_n(x,s)|^2) u_n(x,s)] \, ds. \tag{IO}$$

<u>Theorem I</u>

Let (f) be valid, T > 0. Then,

(a) for any $u_0, u_{0n} \in L$ **there exist** unique solutions of the problems (8) and (IO) which belong to $C([t_0-T, t_0+T]; L)$;

(b) $|u(\cdot,t)|_L$, $|u_n(\cdot,t)|_L$ do not depend on t;

(c) let $B \subset L$ **be bounded. Then, for every** $\varepsilon > 0$ **there exist** $\delta > 0$ **and the number** $M > 0$ **such that**

$$\max_{t \in [t_0-T, t_0+T]} |\bar{\tau}_N(\cdot,t) - u_n(\cdot,t)|_L < \varepsilon$$

for every n, u_{0n}, $\bar{\tau}_N(\cdot,t_0) \in B$ **such that** $n > M$ **and**

$$|\bar{\tau}_N(\cdot,t_0) - u_{0n}(\cdot)|_L < \delta;$$

(d) for every $u_0 \in L$

$$\lim_{n \to \infty} \max_{t \in [t_0-T, t_0+T]} |u(\cdot,t) - u_n(\cdot,t)|_L = 0$$

if $\lim\limits_{n\to\infty} u_{0n} \overset{L}{=} u_0$;

(e) for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\max_{t\in[t_0-T,t_0+T]} |u_1(\cdot,t)-u_2(\cdot,t)|_{L_1} < \varepsilon$$

if $|u_1(\cdot,t_0)-u_2(\cdot,t_0)|_{L} < \delta$.

where u_1, u_2 are the solutions of (8);

(h) the problem (I),(3),(4) defines the dynamical system on the phase space L.

Proof

Statements (a) and (b) can be proved by the standard methods of the papers /4,6,12/.

Lemma 1

For any $u\in L$ $\lim\limits_{n\to\infty} e^{it\bar\Delta_n}u = e^{it\Delta}u$ uniformly to any segment $t\in[a,b]$.

Proof

It is sufficient to prove that $\lim\limits_{n\to\infty}\sin(t\bar\Delta_n)u=\sin(t\Delta)u$ and $\lim\limits_{n\to\infty}\cos(t\bar\Delta_n)u=\cos(t\Delta)u$ uniformly to $t\in[a,b]$.

We will prove the first equality only.

Let $u\in L$, $\varepsilon>0$. The operator Δ has eigenfunctions and eigenvalues

$$u_k(x)=\sqrt{\frac{2}{A}}\sin\frac{\pi k x}{A}, \quad \lambda_k=-\left(\frac{\pi k}{A}\right)^2, \quad k=1,2,3,\dots,$$

and

$$\tau_{kn}=\sqrt{\frac{2}{A}}\sin\frac{\pi j k}{N} \quad (j=\overline{0,N}),\ \lambda_{kn}=-\frac{4}{h^2}\sin^2\frac{\pi k}{2N}, \quad k=\overline{1,N-1},$$

are the first (N-1) eigenfunctions and eigenvalues of the operator Δ_n on the network $\{x_i\}_{i=\overline{0,N}}$ (see /15/). Then, the operator $\bar\Delta_n$ has $2N-2$ eigenfunctions and eigenvalues $\bar\tau_{kn}\in X_n$ ($k=1,2N-2$), λ_{kn} where $\bar\tau_{2k-1,n}=(\bar u_{kn},0)$, $\lambda_{2k-1,n}=\lambda_{kn}$ and $\bar\tau_{2k,n}=(0,\bar u_{kn})$, $\lambda_{2k,n}=\lambda_{kn}$ and the graphs of $\bar u_{kn}$ are broken lines connecting the points $(x_j,\sqrt{\frac{2}{A}}\sin\frac{\pi j k}{N})$ ($j=\overline{0,N}$). Another part of the spectrum of $\bar\Delta_n$ is 1 and for the corresponding eigenfunctions one can take the orthogonal normed basis of $X_n^{\perp}$. It may be proved that $\bar u_{kn}\overset{L}{\to}u_k$, $\lambda_{kn}\to\lambda_k$ when $n\to\infty$.

Let $u=\sum\limits_{k=1}^{\infty}a_k u_k$. There exists $K>0$ such that $\left\{\sum\limits_{k=K+1}^{\infty}|a_k|^2\right\}^{1/2} < \frac{\varepsilon}{4(C'+C'')}$. Then

$$\left|\left(e^{it\Delta}-e^{it\bar\Delta_n}\right)u\right|_L \le \frac{\varepsilon}{2}+\left|\left(e^{it\Delta}-e^{it\bar\Delta_n}\right)\sum_{k=1}^{K}a_k u_k\right|_L$$

and hence

$$\left|\left(e^{it\Delta}-e^{it\bar\Delta_n}\right)u\right|_L<\varepsilon$$

for all $t\in[a,b]$ and sufficiently large n.

Lemma 1 is proved.

Lemma 2

Let $B\subset L$ be bounded. Then, there exist $C_1,C_2>0$ such that for any $u\in B$, $\bar\tau_N\in B\cap X_n$ $(n=1,2,3,\dots)$ the inequality

$$|g_n(x)-f(x,|u|^2)u|_L\le C_1|\bar\tau_N-u|_L+C_2 h$$

is valid.

(Here $g_n(x)\in X_n$ corresponds to $f(x_i,|\bar\tau_N(x_i)|^2)\bar\tau_N(x_i)$.

Proof is trivial.

Lemma 3

(c) of theorem 1 is valid.

Proof

By (I0), (11) and lemma 2 we have

$$|\bar\tau_N(\cdot,t)-u_n(\cdot,t)|_L\le|u_{0n}-\bar\tau_N(\cdot,t_0)|_L+\int_{t_0}^{t}|g_n(\cdot,s)-f(x)|u|^2)u|_L ds$$

$$\le|u_{0n}-\bar\tau_N(\cdot,t_0)|_L+C_2'h+C_1\int_{t_0}^{t}|\bar\tau_N(\cdot,s)-u_n(\cdot,s)|_L ds,$$

where C_2' = const >0, and lemma 3 is proved.

Lemma 4

(d) of theorem 1 is valid.

Proof

Using (8), (I0) and lemma 1 and 2 we have

$$|u_n(\cdot,t)-u(\cdot,t)|_4 \le |(e^{it\Delta}-e^{it\bar\Delta_n})u_0|_4 + C_3|u_0-u_{0n}|_4 + \int_{t_0}^{t} |(e^{it\Delta}-e^{it\bar\Delta_n})[f(x,|u|^2)u]|_4 ds +$$
$$C_4\int_{t_0}^{t} |f(x,|u_n(\cdot,s)|^2)u_n(\cdot,s)-f(x,|u(\cdot,s)|^2)u(\cdot,s)|_4 ds \to 0.$$

when $n\to\infty$ (here $C_3, C_4 = \text{const} > 0$).

Lemma 4 is proved.

(e) of theorem 1 may be proved as lemma 4 and (h) is obvious.

Theorem 1 is proved.

2°. ~~Here~~ We shall construct the σ-additive Gaussian measure W on L^{1}.

Let $H_1 \subset H$ be a subspace, $\dim H_1 = m < \infty$, P be the orthogonal projector on H_1 in H, and $e_1,\dots,e_m$ be the orthogonal normed basis in H_1. Sets of the kind

$$M_H = \{u\in H \mid [\langle u,e_1\rangle,\dots,\langle u,e_m\rangle]\in F\},$$

where $F\subset R^m$ is a Borel set, are called the cylindrical subsets of H. Let

$$\nu_H(M_H) = \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int_F e^{-\frac{|x|^2}{2}}\,dx,$$

where dx is the m-dimensional Lebesque measure. Then, $\nu_H(M_H)$ does not depend on the basis $e_1,\dots,e_m$ because the transitional matrix from one orthogonal normed basis to another is orthogonal. Let R_H be the set of cylindrical subsets of H. Then R_H is an algebra and the function ν_H is additive on R_H. But ν_H is not σ-additive on R_H [8].

Let $\alpha\in(0,\frac{1}{2})$ and $e_1,\dots,e_m \in H^\alpha$ be orthogonal normed vectors. For any m and Borel $F\subset R^m$ the sets of the kind

$$M_\alpha = \{u\in H^\alpha \mid [(u,e_1)_\alpha,\dots,(u,e_m)_\alpha]\in F\} \tag{11}$$

will be called cylindrical sets in H^α. We denote by R^α the set of such sets. Then, R^α is an algebra.

Let $M_H = M_\alpha \cap H$ for any $M_\alpha\in R^\alpha$.

Then, $M_H\in R_H$ because functionals $(u,e_i)_\alpha$ from (11) are continuous in H. Let

$$\nu_\alpha(M_\alpha) = \nu_H(M_H).$$

Then, ν_α is a Gaussian measure on H^α. Its characteristic functional is

$$\chi_\alpha(y) = e^{-\frac{1}{2}(B_\alpha y,y)}$$

for any $y\in H^\alpha$ where

$$B_\alpha = (-\Delta+m_0)^{-1}((-\Delta)^\alpha+m_0).$$

<u>Lemma 5</u>

Measure ν_α is σ-additive on R^α.

<u>Proof</u> follows from the well-known theorem (theorem II.2.1 from /3/) because B_α is a nuclear operator. By the well-known method /5/ the measure ν_α may be extended to the minimal σ-algebra containing R^α. Let W_α and $\mathfrak{M}_\alpha$ be this σ-additive measure and minimal σ-algebra.

<u>Lemma 6</u>

$\mathfrak{M}_\alpha$ is a Borel σ-algebra.

<u>Proof</u>

Let $B_\rho(a)=\{u\in H^\alpha \mid |u-a|_\alpha\le\rho\}$ $(a\in H^\alpha, \rho>0)$.

It is sufficient to prove that $B_\rho(a)\in\mathfrak{M}_\alpha$.

Let $\{z_n\}_{n=1,2,3,\dots}$ be a dense set on the unit sphere in H^α. Then,

$$B_\rho(a) = \bigcap_{n=1}^{\infty}\{u\in H^\alpha \mid |(u-a,z_n)_\alpha|\le\rho\}.$$

Lemma 6 is proved.

<u>Lemma 7</u>

$W_\alpha(B_\rho(a))>0$ for any $\rho>0$, $a\in H^\alpha$.

<u>Proof</u> see in /3/, ch.II, §4.

Let us consider the system (5)-(7). It is obvious that it is a Hamiltonian system with the Hamiltonian

$$H_h = \sum_{i=1}^{N}\left\{\frac{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}{2h^2} - F(x_i,u_i^2+v_i^2)\right\}.$$

where $F(x,s) = \frac{1}{2}\int_0^s f(x,p)\,dp$. As it is known, the Hamiltonian H_n is independent of t on the solutions $\tau^N(t)$ of (5)-(7).

Let E_n be the phase space of the system (5)-(7) and let for the Borel set $\Omega \subset E_n$

$$\mu_n^1(\Omega) = \int_\Omega e^{-hH_n(u_0,\dots,u_N,v_0,\dots,v_N)}\,du_1\dots du_{N-1}\,dv_1\dots dv_{N-1},$$

$$\nu_n^1(\Omega) = \int_\Omega e^{-\sum_{i=1}^{N}\left\{\frac{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}{2h}+\frac{1}{2}hm_0(u_i^2+v_i^2)\right\}}\,du_1\dots du_{N-1}\,dv_1\dots dv_{N-1}.$$

It is known /13/ that μ_n^1 is an invariant measure for the system (5)-(7), i.e. if $\Omega \subset E_n$ is a Borel set, then $\mu_n^1(\Omega_t) = \mu_n^1(\Omega)$, where

$$\Omega_t = \{\tau^N(t) \mid \tau^N(t_0) \in \Omega\}.$$

For the Borel set $\Omega \subset X_n$ let

$$\nu_n(\Omega) = \left(\frac{1}{2\pi}\right)^{N-1}\int_F e^{-\frac{|x|^2}{2}}\,dx$$

where dx is the Lebesque measure in R^{2N-2} and F is the set coordinates of $u \in \Omega$ in some orthogonal normed basis in X_n with the scalar product $\langle\cdot,\cdot\rangle$. Then, let

$$\mu_n(\Omega) = \int_\Omega e^{h\sum_{i=1}^{N}\{F(x_i,u_i^2+v_i^2)+\frac{1}{2}m_0(u_i^2+v_i^2)\}}\,\nu_n(du\,dv).$$

Lemma 8

The measure μ_n is invariant on X_n, i.e. for any Borel set $\Omega \subset X_n$ and any t $\mu_n(\Omega_t) = \mu_n(\Omega)$, where $\Omega_t = \{\bar\tau_N(t) \mid \bar\tau_N(t_0) \in \Omega\}$ and $\bar\tau_N(t)$ corresponds to the solution of (5)-(7) for which $\tau^N(t_0)$ corresponds to $\bar\tau_N(\cdot,t_0)$.

Proof

Let $e_1,\dots,e_{2N-2}$ be an orthogonal normed basis in X_n with the product $\langle\cdot,\cdot\rangle$, $\bar\tau_N = (\bar u_N,\bar v_N) \in X_n$, $\bar u_N(x_i) = u_i$, $\bar v_N(x_i) = v_i$. Let $\bar\tau_N = \sum_{i=1}^{2N-2} y_i e_i(x)$, then $\bar\tau_N(x_j) = \sum_{i=1}^{2N-2} y_i e_i(x_j)$, $j=\overline{0,N}$. For any Borel set $\Omega^1 \subset E_n$

$$\nu_n^1(\Omega^1) = \int_{\Omega^1} e^{-\sum_{i=1}^{N}\left\{\frac{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}{2h}+\frac{1}{2}m_0h(u_i^2+v_i^2)\right\}}\,du_1\dots du_{N-1}\,dv_1\dots dv_{N-1}$$

We introduce new coordinates $\bar y = (y_1,\dots,y_{2N-2})$, then

$$\nu_n^1(\Omega^1) = |\det Q|\int_F e^{-\sum_{i=1}^{2N-2}\frac{y_i^2}{2}}\,d\bar y =$$

$$(2\pi)^{N-1}\,|\det Q|^{-1}\,\nu_n(\Omega).$$

where Ω is the Borel set of coordinates $\bar y$ for which $\tau^N \in \Omega^1$.

Since Q is a constant matrix

Lemma 8 is proved.

Let us define the measure w on L using w_α by the role: for any $\Omega \subset L$

$$w(\Omega) = w_\alpha(\Omega \cap H^\alpha)$$

if the right-hand side is defined. It is clear that any set $M \subset R_L$ satisfies this property, where R_L is the algebra of a cylindrical set in L which is constructed as R^α and R_H. Hence w is defined on the Borel σ-algebra in L.

By analogy we can define the family of Borel measures for any $\alpha^1 : 0 \le \alpha^1 \le \alpha$: for any $\Omega \subset H^{\alpha^1}$

$$w_{\alpha^1}(\Omega) = w_\alpha(\Omega \cap H^\alpha).$$

Then, we can define measures $w_n^{\alpha^1}$ on H^{α^1} by the rule

$$w_n^{\alpha^1}(\Omega) = w_n(\Omega \cap X_n)$$

for any Borel set $\Omega \subset H^{\alpha^1}$, $0 \le \alpha^1 \le \alpha$. In fact, in this way we can define the measure $w_n^{\alpha^1}$ on any closed cylindrical set of H^{α^1} and thus on the Borel σ-algebra as in lemma 6.

We denote by w_n the measure w_n^0. It is important to prove that the sequence w_n converges weakly to the measure w ($w = w^0$). First, we shall prove some auxiliary results.

Lemma 9

Let $F \subset R^m$ be a Borel set, $e_1, \dots, e_m \in H$ are orthogonal normed vectors and $M_H = \{ u \in H \mid [\langle u, e_1 \rangle, \dots, \langle u, e_m \rangle] \in F \}$ is a cylindrical set, $M_n = M_H \cap X_n$. Then,

$$\lim_{n \to \infty} \nu_n (M_n) = \nu_H (M_H).$$

Proof

It is clear that M_n is the Borel set. Let $e_i^{(n)} = Pr_{X_n} e_i$ $(i = \overline{1,m})$ by the product of H. Then,

$$\lim_{n \to \infty} e_i^{(n)} \overset{H}{=} e_i \qquad (i = \overline{1,m}). \tag{11}$$

Then, the vectors $e_i^{(n)}$ $(i = \overline{1,m})$ are linearly independent for all large n. By the standard method of orthogonalization used for $e_i^{(n)}$ $(i = \overline{1,m})$, we get the vectors $\overline{e}_1^{(n)}, \dots, \overline{e}_m^{(n)}$ for which

$$\overline{e}_i^{(n)} = e_i + d_i^{(n)}, \tag{12}$$

where $\lim_{n \to \infty} d_i^{(n)} \overset{H}{=} 0$. Then, since

$$M_n = \{ u \in X_n \mid [\langle u, \overline{e}_1^{(n)} \rangle, \dots, \langle u, \overline{e}_m^{(n)} \rangle] \in F \},$$

we have

$$\nu_n (M_n) = \left(\frac{1}{2\pi} \right)^{\frac{m}{2}} \int_{F'} e^{-\frac{|x|^2}{2}} dx.$$

where F' is the set of coordinates of the vectors from M_n in the basis $e_i^{(n)}$ $(i = \overline{1,m})$. Then

$$\nu_n (M_n) = \left(\frac{1}{2\pi} \right)^{\frac{m}{2}} |\det Q_n|^{-1} \int_F e^{-\frac{|Q_n y|^2}{2}} dy,$$

where Q_n is the transformation matrix from the basis $\{ e_i^{(n)} \}_{i = \overline{1,m}}$ to the basis $\{ \overline{e}_i^{(n)} \}_{i = \overline{1,m}}$. By (11)-(12)

$$Q_n = E + Q_1^{(n)}.$$

where E is the unit matrix and $\lim_{n \to \infty} Q_1^{(n)} = 0$. Now

$$\lim_{n \to \infty} \nu_n (M_n) = \left(\frac{1}{2\pi} \right)^{\frac{m}{2}} \int_F e^{-\frac{|x|^2}{2}} dx = \nu_H (M_H)$$

and lemma 9 is proved.

Lemma 10

Let $M \in R^{\alpha_1}$, $0 \le \alpha_1 \le \alpha$. Then

$$\lim_{n \to \infty} w_n^{\alpha_1} (M) = w_{\alpha_1} (M).$$

Proof

By definition $w_{\alpha_1} (M) = \nu_H (M_H)$. where

$$M_H = M \cap H, \quad w_n^{\alpha_1} (M) = \nu_n (M \cap X_n) = \nu_n (M_H \cap X_n)$$

and lemma 10 is proved.

The characteristic functional $\chi_n^\alpha (y)$ of the measure w_n^α is equal to $e^{-\frac{1}{2}(B_n' y, B_n' y)_\alpha}$. where

$$B_n' = (m_0 - \Delta)^{-1/2} (m_0 + (-\Delta)^\alpha)^{1/2} P_n (m_0 - \Delta)^{-1} (m_0 + (-\Delta)^\alpha) P_\alpha.$$

where P_α and P_n are the orthogonal projectors on X_n in H^α and H, respectively. Hence $(B_n' y, B_n' y)_\alpha$ is a bounded quadratic form and hence there exists the bounded operator B_n defined on H^α such that

$$(B_n' y, B_n' y)_\alpha = (B_n y, y)_\alpha.$$

It is clear $0 \le B_n \le \mathfrak{D}_n = P_\alpha (m_0 - \Delta)^{-1} (m_0 + (-\Delta)^\alpha) P_\alpha$ because for any $y \in H^\alpha$

$$(B_n y, y)_\alpha = (B_n' y, B_n' y)_\alpha = \| P_n (m_0 - \Delta)^{-1} (m_0 + (-\Delta)^\alpha) P_\alpha y \|^2 \le$$

$$\| (m_0 - \Delta)^{-1} (m_0 + (-\Delta)^\alpha) P_\alpha y \|^2 = (\mathfrak{D}_n y, y)_\alpha.$$

Hence, B_n is a nuclear operator.

Lemma 11

The sequence $\{w_n\}_{n=1,2,3,\dots}$ is weakly compact in L.

Proof

By Prokhorov's theorem it is sufficient to prove that for any $\varepsilon>0$ there exists the compact $K_\varepsilon\subset L$ such that $w_n(K_\varepsilon)>1-\varepsilon$ for all n. It is clear (see ch.II, /3/) that for any $\varepsilon>0$ there exists the ball $B\subset H^\alpha$ ($\alpha\in(0,\frac{1}{2})$) for which

$$w_n^\alpha(B)>1-\varepsilon \qquad (n=1,2,3,\dots).$$

Then, we take the closure K_ε in L of the ball B and then K_ε is compact and $w_n(K_\varepsilon)>1-\varepsilon$.

Lemma 11 is proved.

Lemma 12

The sequence $\{w_n\}_{n=1,2,3,\dots}$ converges to w weakly in L.

Proof

For any $M\in R_L$ $\quad w_n(M)\to w(M)$ by lemma I0. Then, $\{w_n\}$ is weakly compact and lemma follows by the unity of the measure defined on algebra on minimal σ-algebra[5]

Later two important results on the measures w and μ are established (see lemmas 13 and 15) also we are not needed in them.

Lemma 13

Let $S=\{u\in L \mid |u|_L=1\}$, G be the opened set on S and let for every $v\in G$ there exists a unique $\tau=\tau(v)>0$ such that $\tau(v)\in C^1(G;R)$.

Let $G_1=\{u\in L \mid u=\tau(v)v,\ v\in G\}$.

Then $w(G_1)=0$.

Proof

It is sufficient to prove that for any subspace L_1 of L with codimension 1 and any smooth function $p(v)\in C^1(L_1;R)$ defined on L_1 $w(T)=0$, where $u_0\perp L_1$, $|u_0|_L=1$, $T=\{u\in L \mid u=p(v)u_0+v,\ v\in L_1\}$. We can assume that $u_0\in H$.

Let us assume that $w(T)>0$. Then, we consider sets $T_\alpha=T+\alpha u_0$. By /16/ $w(T_\alpha)>0$ for all α. Hence, there exists $m\in\mathbb{N}$ such that $w(T_\alpha)>\frac{1}{m}$ for a noncountable set of α. Then $w(L)=+\infty$, i.e. we get a contradiction.

Lemma 13 is proved.

3°. Here, we shall construct the invariant measure for the problem (I), (3),(4). For every $\bar\tau_N=(\bar u_N,\bar v_N)\in X_n$ let
$$\Phi_n(\bar\tau_N)=h\sum_{i=1}^{N}\{F(x_i,u_i^2+v_i^2)-m_0(u_i^2+v_i^2)\}=\int_0^A F_1(s)ds,$$

where
$$F_1(s)=\frac{F(x_i,u_i^2+v_i^2)+\frac{1}{2}m_0(u_i^2+v_i^2)-F(x_{i-1},u_{i-1}^2+v_{i-1}^2)-\frac{1}{2}m_0(u_{i-1}^2+v_{i-1}^2)}{h}(s-x_{i-1})$$
$$+F(x_{i-1},u_{i-1}^2+v_{i-1}^2)+\frac{1}{2}m_0(u_{i-1}^2+v_{i-1}^2)$$

for $x\in[x_{i-1},x_i]$, $i=\overline{1,N}$.

It is clear that $\Phi_n: X_n\to R$ is the continuous map.

Lemma 14

Let $B\subset L$ be a bounded set. Then, for any $\varepsilon>0$ there exist $\delta>0$ and number $M>0$ such that
$$|\Phi_n(\bar\tau_N)-\Phi(u)|<\varepsilon$$
for all $\bar\tau_N\in X_n$, $u\in B$ where $n\geqslant M$,
$$|\bar\tau_N-u|_L<\delta, \qquad \Phi(u)=\int_0^A\{F(x,|u(x)|^2)+\tfrac{1}{2}m_0|u(x)|^2\}dx.$$

Proof

We have
$$|\Phi_n(\bar\tau_N)-\Phi(u)|\leq|\Phi_n(\bar\tau_N)-\Phi(\bar\tau_N)|+|\Phi(\bar\tau_N)-\Phi(u)|, \qquad (13)$$
then
$$|\Phi_n(\bar\tau_N)-\Phi(\bar\tau_N)|\leq C_1h, \qquad (14)$$
where $C_1=\text{const}>0$, and
$$|\Phi(\bar\tau_N)-\Phi(u)|\leq\int_0^A\{|F(x,|\bar\tau_N|^2)-F(x,|u(x)|^2)|+\frac{m_0}{2}\big||\bar\tau_N|^2-|u|^2\big|\}dx$$
$$\leq C_2\int_0^A\big||\bar\tau_N(x)|^2-|u(x)|^2\big|dx\leq C_3|u-\bar\tau_N|_L, \qquad (15)$$

where $C_3=\text{const}>0$.

Lemma 14 follows from (13)-(15).

Lemma 15

$w(\Phi^{-1}(a)) = 0$ for any $a \in R$.

Proof

By (f) for any $u \in S = \{u \in L \mid |u|_L = 1\}$ there exists at most one $r = r(u) > 0$ such that $\phi(r(u)u) = a$ and $r(u) \in C^1(S; R)$ by the theorem on the implicit function. By lemma 13 $w(\Phi^{-1}(a)) = 0$ and lemma is proved.

By theorem 1 and ball $B_R = \{u \in L \mid |u|_L \leq R\}$ is the invariant set for the solutions of (I), (3), (4) and $w(B_R) > 0$ by lemma 6. Let us fix $R > 0$ and consider $B_R = B$ as a new phase space of the problem. By (f) the functional is continuous and bounded on B.

Let $\Omega \in \mathfrak{M}$, $\Omega \subset B$, $\Omega_n = \Omega \cap X_n$,

$$\mu(\Omega) = \int_\Omega e^{\Phi(u)} w(du).$$

Lemma 16

Let Ω be opened. Then $\lim_{n\to\infty} \mu_n(\Omega_n) \geq \mu(\Omega)$.

Proof

We consider measures $\bar{\mu}_n$ such that

$$\bar{\mu}_n(\Omega) = \int_\Omega e^{\Phi(u)} w_n(du)$$

for any $\Omega \subset \mathfrak{M}$. By lemma 14 for every $\varepsilon > 0$ there exists number $n_0 > 0$ such that

$$|\bar{\mu}_n(\Omega) - \mu_n(\Omega_n)| < \varepsilon \tag{16}$$

for all $n \geq n_0$. Let $g_\varepsilon(u)$ be a continuous function on L, $0 \leq g_\varepsilon(u) \leq 1$ and

$$g_\varepsilon(u) = \begin{cases} 1 & \text{if } u \in \Omega,\ dist(u, \partial\Omega) \geq \varepsilon, \\ 0 & \text{if } u \bar{\in} \Omega. \end{cases}$$

We have by lemma 12

$$\lim_{n\to\infty} \bar{\mu}_n(\Omega) \geq \lim_{n\to\infty} \int_\Omega g_\varepsilon(u) e^{\Phi(u)} w_n(du) = \int_\Omega g_\varepsilon(u) e^{\Phi(u)} w(du)$$

and hence

$$\lim_{n\to\infty} \bar{\mu}_n(\Omega) \geq \mu(\Omega).$$

Then, by (16)

$$\lim_{n\to\infty} \mu_n(\Omega_n) + \varepsilon \geq \mu(\Omega)$$

for any $\varepsilon > 0$ and

Lemma 16 is proved.

Corollary 1

Let $\Omega \subset L$ be closed, $\Omega_n = \Omega \cap X_n$.

Then $\lim_{n\to\infty} \mu_n(\Omega_n) \leq \mu(\Omega)$.

Let $S(t)$, $S_n(t): L \to L$ be the evolution operators for the problems (I), (3), (4) and (I0), respectively, i.e. for any solution $u(x,t)$ of (I), (3), (4) $S(\tau)u(x,t) = u(x, t+\tau)$ and by analogy for $S_n(t)$. Let $T_n(t)$ be the operator on X_n to X_n such that for any $\tau_N(\cdot,t)$ lying in X_n for every fixed t which corresponds to some solution $\tau^N(t)$ of (5)-(7) $T_n(\tau)\tau_N(\cdot,t) = \tau_N(\cdot, t+\tau)$. It is clear from theorem I that the operators $S(t)$, $S_n(t)$, $T_n(t)$ have inverse continuous operators for every fixed t.

Lemma 17

Let $t_0, t_1 \in R$, $\Omega(t_0) \in \mathfrak{M}$, $\Omega(t_0) \subset B$, $\Omega(t_1) = S(t_1 - t_0)\Omega(t_0)$ and $\Omega(t_i)$ (i = 0,1) be opened. Then $\mu(\Omega(t_0)) = \mu(\Omega(t_1))$.

Proof

Let $\varepsilon > 0$. By theorem $1(e)$ there exist opened sets $K(t_0) \subset \Omega(t_0)$, $K(t_1) \subset \Omega(t_1)$ such that $\mu(\Omega(t_i) \setminus K(t_i)) < \varepsilon$ (i=0,1), $S(t_1 - t_0)K(t_0) = K(t_1)$ and there exists $d > 0$ for which

$$dist(K(t_i), \partial\Omega(t_i)) > d. \quad (i = 0,1).$$

Let $K_m = [S_m(t_0 - t_1)K(t_1)] \cap K(t_0)$. By theorem I there exists number $m_0 > 0$ such that

$$|T_n(t_1 - t_0)\tau_N - S_n(t_1 - t_0)\tau_N| < \frac{d}{2} \tag{17}$$

for all $\tau_N \in B$, $n \geq m_0$. Let $M_m = \bigcup_{k=m_0} K_k$. Then

$$M_{m_0} \subset M_{m_0+1} \subset M_{m_0+2} \subset \dots \quad , \quad \bigcup_{m \geq m_0} M_m = K(t_0).$$

by theorem 1. Hence, there exists number $m_1 \geq m_0$ such that

$$w(K(t_0) \setminus M_{m_1}) < \varepsilon . \tag{18}$$

Now we will define the map P_m. Let $P_m u = T_{n_1}(t_1 - t_0) u$ for any $u \in M_m \cap X_m$ where $n_1 \geq m_0$ is the minimal number for which $u \in S_{n_1}(t_0 - t_1) K(t_1)$. By (17) $N_m = P_m(M_m \cap X_m) \subset \Omega_{\frac{d}{2}}(t_1) = \{u \in \Omega(t_1) \mid dist(u, \partial\Omega(t_1)) \geq \frac{d}{2}\}$

Then

$$\mu_n(P_n(M_n \cap X_n)) = \mu_n(T_n(t_1 - t_0)[M_n \cap X_n]) =$$

$$= \mu_n(M_n \cap X_n)$$

because $\mu_{n_1}(X_{n_2}) = 0$ for all $n_1 > n_2$. On the other hand, $M_n \cap X_n \supset M_{m_1} \cap X_n$ for $n \geq m_1$. Hence, by lemma 16, corollary 1 and (18)

$$\mu(\Omega(t_0)) - 2\varepsilon \leq \mu(M_{m_1}) \leq \underline{\lim}_{n\to\infty} \mu_n(M_n \cap X_n) \leq$$

$$\leq \overline{\lim}_{n\to\infty} \mu_n(P_n(M_n \cap X_n)) \leq \mu(\Omega(t_1))$$

and then due to the arbitrariness of ε

$$\mu(\Omega(t_0)) \leq \mu(\Omega(t_1)).$$

By analogy

$$\mu(\Omega(t_0)) \geq \mu(\Omega(t_1))$$

and lemma 17 is proved.

Theorem 2

Let $t_0, t_1 \in R, \Omega(t_0) \in \mathfrak{M}$, $\Omega(t_0) \subset B$.

Then $\mu(\Omega(t_0)) = \mu(\Omega(t_1))$.

Proof

As in lemma 17, we can prove that $\mu(\Omega(t_0)) = \mu(\Omega(t_1))$ for any closed $\Omega(t_0)$. By theorem 1 $\Omega(t_1)$ is closed too. Then, for any $\varepsilon > 0$ we can find two sets: an opened K_1 and a closed K_2 such that $K_2 \subset \Omega(t_0) \subset K_1$ and $\mu(K_1 \setminus K_2) < \varepsilon$. Then $S(t_1 - t_0) K_2 \subset \Omega(t_1) \subset S(t_1 - t_0) K_1$ and since $S(t_1 - t_0) K_2$ is closed and $S(t_1 - t_0) K_1$ is opened $\Omega(t_1)$ is measurable. There exist opened sets M_0, M_1 such that $\Omega(t_i) \subset M_i$ and $\mu(M_i \setminus \Omega(t_i)) < \varepsilon$ (i=0,1). Let $N_0 = [S(t_0 - t_1) M_1] \cap M_0$, $N_1 = S(t_1 - t_0) N_0$. Then N_i are opened (i=0,1) and $w(N_i \setminus \Omega(t_i)) < \varepsilon$. By lemma 17 $\mu(N_0) = \mu(N_1)$ and theorem 2 is proved.

4^0. As an application we will establish the Poincare recurrence theorem.

Theorem 3

Let (f) be valid, $\Omega(t_0) \in \mathfrak{M}$, $\Omega(t_0) \subset B$, $\mu(\Omega(t_0)) > 0$. Then, for every $T > 0$ there exists $t > t_0 + T$ such that

$$\mu(\Omega(t_0) \cap \Omega(t)) > 0 .$$

where $\Omega(t) = S(t - t_0)\Omega(t_0)$.

For the proof, see /7,13/.

Theorem 3'

Let (f) be valid. Then, almost all points of L are stable in the Poisson's sense.

By the invariant measure one may investigate also other properties of the dynamical system (see /7,13/).

We note that condition (f) is used for the existence and uniqueness of solutions for the problem (I), (3), (4). If one proves analogous results with replacement of L by H^α, $\alpha \in (0, \frac{1}{2})$, then the invariant measure can be constructed on H^α, $\alpha \in (0, \frac{1}{2})$ and the condition (f) may be changed by an essentially weaker condition.

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