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On the Stationary Solutions of a Nonlinear Equation of Schrodinger Type with Some Self-Consistent Potentials

Soliton solutions of Schrödinger equations have been investigated: a) with potential satysfying a nonhomogeneous Boussinesq equation and b) with saturable nonlinearity $(1-e^{-\psi^2})\psi$. It is shown that upper bounds on the soliton amplitude arise.

The problem of stability of nonhomogeneous stationary equations of Schrödinger type with various nonlinearities is considered. It is shown that in two- and three-dimensional geometry, possible stationary solutions are unstable with respect to scale transformation.

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1. Introduction

In connection with intensive development of the soliton theory [1], up to now an unabated interest of theorists working in various fields of physics is attracted to the search for two- and three-dimensional stationary solutions of diverse nonlinear wave equations. From the point of view of plasma turbulence theory, nonlinear field theory and many other applications it is important to investigate such solutions for a nonlinear Schrödinger equation for Klein-Gordon equations and sine-Gordon equation.

For equations in which a nonlinear term increases with the increase of the function sough for, it has been shown* that there are no stable stationary solutions in a non-one-dimensional geometry. In this connection it is interesting to consider various models with more complicated nonlinearities including saturable ones. As the latter we shall take nonlinear terms remaining bounded at the unrestricted growth of unknown function. In section 2 the stationary solutions of Schrödinger equation with the potential satisfying a nonhomogeneous Boussinesq equation are considered. In section 3 the potential is taken in the form $U = 1 - \exp(-\alpha|\psi|^2)$, where ψ is the function to be determined, α is a constant. Finally, in section 4 a fairly general case of $F(|\psi|^2)$ type nonlinearity is studied, and in section 5 brief conclusions are drawn.

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*The Schrödinger and Klein-Gordon equations with cubic nonlinearity have been considered in refs. [2] and [3], respectively, and in ref. [4] sine-Gordon equation has been investigated.

2. *A Model of Coupled Schrödinger and Boussinesq Equations*

In the previous works /5/ of one of the authors the system of coupled Schrödinger and Boussinesq equations has been proposed for the investigation of Langmuir solitons in plasmas*

$$i\frac{\partial\psi}{\partial t}+\psi_{xx}-U\psi=0,$$
$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\epsilon}{3}\frac{\partial^4}{\partial x^4}\right)U-\epsilon\frac{\partial^2}{\partial x^2}(U^2)=\frac{\partial^2}{\partial x^2}(|\psi|^2). \tag{2.1}$$

Here, ψ is the envelope of electric field, U is the disturbance of plasma density, and the dimensionless variables x,t have been determined in ref. /5/. This system well describes nearsonic Langmuir solitons removing the divergence inherent to linear wave equation for the potential /5/. In the linear approximation in a small parameter $\epsilon=\frac{4}{3}\frac{m_e}{m_i}$ the two-parametric set of solutions $\psi^{(0)}(\lambda,M)$ of the system of equations

$$i\frac{\partial\psi^{(0)}}{\partial t}+\psi^{(0)}_{xx}-U^{(0)}\psi^{(0)}=0,$$
$$U^{(0)}_{tt}-U^{(0)}_{xx}=(|\psi^{(0)}|^2)_{xx} \tag{2.2}$$

splits into two solutions

$$\psi=\psi^{(0)}+\psi_{1,2}\ .$$

Here

$$\psi^{(0)}=\frac{\gamma S/\sqrt{2}}{\mathrm{ch}\left(\frac{\gamma}{\sqrt{2}}\psi_m\xi\right)},\quad U^{(0)}=-\gamma^2(\psi^{(0)})^2,$$

$\xi=x-Mt$, , the parameter $M=\frac{v}{c_s}$ is the soliton velocity (c_s is the sound velocity), $\gamma^2=1/(1-M^2)$. The energy level in the soliton rest frame ξ is*

$$\lambda^{(0)}=-\frac{\gamma^2}{2}(\psi^{(0)}_m)^2,\quad S=\int_{-\infty}^{\infty}\psi^2\,dx,\quad \frac{dS}{dt}=0.$$

The functions ψ_1 and ψ_2 are given by

$$\psi_1=\psi^{(0)}\frac{\epsilon}{3}\gamma^6S^2\left(\frac{1}{6}+\mathrm{sech}^2\kappa\xi\right),$$
$$\psi_2=-\frac{\epsilon}{2}\gamma^4\psi^3_m\mathrm{th}^2(\kappa\xi)\left(\ell n\frac{\mathrm{ch}\kappa\xi-1}{\mathrm{ch}\kappa\xi+1}+\frac{4}{3}\mathrm{sech}\kappa\xi\right),$$

where $\kappa=\sqrt{-\lambda}$. The first correction repeats the form of the zeroth approximation $\psi^{(0)}$, i.e., it is an one-humped function of ξ, the second one is a two-humped function and is equal to zero in the point $\xi=0$.

An analysis of the λ level corrections proportional to ϵ results in

$$\lambda_1=0,\quad \lambda_2=-\frac{\epsilon}{3}\gamma^6\psi^4_m,$$

i.e., the energy level of the second solution is lower, and hence it is more realizable.

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*Analogous system of Schrödinger and Korteweg-de Vries (KdV) equations has been studied by the Japanese authors /6/.

*Here and below we understand the term energy level as an eigenvalue of the stationary Schrödinger equation $\psi_{\xi\xi}+\lambda\psi-U\psi=0$, i.e., the function ψ is transformed as $\psi\to\psi\exp(-i\lambda t)$.

In [5] and [6] the exact analytical solution of the system (2.1) has been also obtained, and it is an one-parameter set analogous to the soliton type solutions of the homogeneous KdV and Boussinesq equations. However, in contrast to the latter cases, the solitons of the system (2.1) remain subsonic ones, and their level λ is negative. The solution has the form

$$\psi_3 = \sqrt{48\epsilon\lambda^2}\ \mathrm{th}(\sqrt{-\lambda}\ \xi)\ \mathrm{sech}(\sqrt{-\lambda}\ \xi)\ , \qquad (2.3)$$

$$U_3 = 6\lambda\ \mathrm{sech}^2(\sqrt{-\lambda}\ \xi)\ , \qquad (2.4)$$

and the connection between parameters λ and M is

$$\frac{1}{\gamma^2} = 1 - M^2 = -\frac{20}{3}\epsilon\lambda\ . \qquad (2.5)$$

The forms of the solutions $\psi_3^2(\xi)$ and $\psi_2(\xi)$ are identical (both are two-humped) that apparently indicates the continuous transition ψ_2 to ψ_3 with the enhancement of the role of the terms proportional to ϵ, that takes place at $M \to 1$.

If the results of this model to extrapolate to the case $M \to 0$, then Eq. (2.5) tells us that there are no solitons with the energy greater than $|\lambda_{cr}| = \frac{3}{20}\frac{1}{\epsilon}$.

3. *Soliton Solutions of the One-Dimensional Schrödinger Equation with a Saturable Nonlinearity*

In the paper [5] it has been already pointed out that the system (2.1) is not quite adequate in the study of standing Langmuir solitons. The equation of the form

$$\psi_{xx} - (\lambda^2 - \frac{1}{\epsilon})\psi - \frac{1}{\epsilon}\psi\ \exp(-\epsilon|\psi|^2) = 0 \qquad (3.1)$$

is more realistic *. Here we pass from λ introduced in section 2 to the positively definite quantity $\lambda^2 (\lambda \to -\lambda^2)$. We shall look for the soliton type solutions of Eq. (3.1) by using the conventional procedure of the integration in the phase plane. The first integral of Eq. (3.1) has the form

$$\psi_x^2 = (\lambda^2 - \frac{1}{\epsilon})\psi^2 - \frac{1}{\epsilon^2}\exp(-\epsilon\psi^2) + C.$$

We find the constant C from the condition

$$\psi \to 0$$

as

$$x \to \pm\infty\ .$$

Then in the point of the maximum of $\psi(x)$ from the condition $\psi_x = 0$ we have

$$\lambda^2 = \frac{\exp(-\epsilon\psi_m^2) - 1 + \epsilon\psi_m^2}{\epsilon^2\psi_m^2}\ ,$$

and

$$\frac{\partial\psi}{\partial x} = \left[(\lambda^2 - \frac{1}{\epsilon})\psi^2 - \frac{1}{\epsilon^2}\exp(-\epsilon\psi^2) + \frac{1}{\epsilon^2}\right]^{1/2}.$$

From the last formula we get the condition of the existence of a soliton solution

$$\lambda^2 - \frac{1}{\epsilon} > 0\ . \qquad (3.2)$$

that is, as in section 2, we have again a constraint on the value $|\lambda|$ for the existence of stationary solutions.

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* Equation (3.1) can be strictly obtained from the system of the hydrodynamical equations of plasma theory.

4. *On the Stability of the Nonhomogeneous Stationary Solutions to Nonlinear Equations of Schrödinger Type*

At first, for the simplicity, we consider a stationary equation of the Langmuir turbulence in spherical symmetry

$$-\lambda^2\psi + \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} - \frac{2}{r^2}\psi + \frac{\psi}{\epsilon}(1-e^{-\epsilon|\psi|^2}) = 0 . \quad (4.1)$$

This equation can be obtained by the variation of the Lagrangian

$$\mathfrak{L} = -\int_0^\infty dr\{[(r\psi)_r]^2 + 2\psi^2 - \frac{r^2}{2}(\psi^4 - 2\lambda^2\psi^2) -$$

$$-(\frac{r^2}{\epsilon})[\psi^2(1-\lambda^2\epsilon) + (\frac{1}{\epsilon})\exp(-\epsilon\psi^2) - (\frac{1}{\epsilon})]\}. \quad (4.2)$$

When the first variation $\delta\mathfrak{L}$ is equal to zero, we have Eq. (4.1). The sign of the second variation $\delta^2\mathfrak{L}$ defines the stability of solution. When $\delta^2\mathfrak{L} < 0$, the solutions to Eq. (4.1) will be stable. We verify the stability of solutions of Eq. (4.1) with respect to a scale transformation of the form $r \to \alpha r$, $\psi_\alpha = \psi(\alpha r, \lambda)$. Let $\psi(r,\lambda)$ be a solution of Eq. (4.1), i.e., $\delta\mathfrak{L}[\psi(r,\lambda)] = 0$. We consider

$$\mathfrak{L}_\alpha = -\int_0^\infty dr\{[(r\psi_\alpha)_r]^2 + 2\psi_\alpha^2 - r^2 f(\psi_\alpha,\lambda)\} = -S_\alpha^{(1)} + S_\alpha^{(2)},$$

where

$$S_\alpha^{(1)} = \int_0^\infty dr\{[(r\psi_\alpha)_r]^2 + 2\psi_\alpha^2\}, \quad S_\alpha^{(2)} = \int_0^\infty r^2 f(\psi_\alpha,\lambda)\,dr .$$

The functionals $S^{(1)}$ and $S^{(2)}$ transform as

$$S_\alpha^{(1)} = \frac{S^{(1)}}{\alpha}, \quad S_\alpha^{(2)} = \frac{S^{(2)}}{\alpha^3} .$$

From the condition $\delta\mathfrak{L} = 0$ we have

$$\frac{d\mathfrak{L}}{d\alpha}\Big|_{\alpha=1} = S^{(1)} - 3S^{(2)} = 0 ,$$

or

$$S^{(2)} = \frac{1}{3}S^{(1)} \geq 0. \quad (4.3.)$$

Finding of $\delta^2\mathfrak{L}$ gives us

$$\frac{d^2\mathfrak{L}}{d\alpha^2}\Big|_{\alpha=1} = -2S^{(1)} + 12S^{(2)} = 2S^{(1)} \geq 0 . \quad (4.4)$$

This means that if the solutions to Eq. (4.3) exist, then they are unstable, in any case with respect to the considered scale transformation.

Analogous arguments are true for a more general equation of the stationary plasma turbulence

$$\mathrm{div}(-\lambda^2\nabla\phi - \nabla\nabla^2\phi + F(|\nabla\phi|^2)\nabla\phi) = 0 . \quad (4.5)$$

This equation can be obtained by the variation of the Lagrangian

$$\mathfrak{L} = -\int(\Delta\phi)^2 d\vec{r} + \int d\vec{r}(\int_0^{(\nabla\phi)^2} F(\xi)d\xi - \lambda^2(\nabla\phi)^2) . \quad (4.6)$$

The electric field is determined as usually from the equation

$$\vec{E} = -\nabla\phi .$$

The transformation

$$r \to \alpha r, \qquad \phi_\alpha = \alpha\phi(\alpha r, \lambda)$$

gives us the relation (4.4).

The same procedure can be used for the study of equations describing the propagation of intense electromagnetic wave in a nonlinear medium with saturation. For example, in cylindrical symmetry we have /7/ (see also ref. /8/)

$$-\lambda^2 \psi + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial \psi}{\partial \rho} + \psi [1 - \exp(-\psi^2)] = 0 . \quad (4.7)$$

It is easy to obatin a Lagrangian whose variation gives Eq. (4.7) and to find the second variation of this Lagrangian with respect to the above-mentioned scale transformation.

Note that the problem of stability of stationary solutions to the nonlinear Klein-Gordon equation with saturable nonlinearity, e.g., of (4.1) type or $\psi^3/(1+\psi^2)$ type (see ref. [8]) can be investigated in an analogous way.

5. *Conclusions*

We formulate briefly the results of this work. One may conclude now that the Langmuir wave collapse predicted by Zakharov and described by Schrödinger equation with cubic nonlinearity or that of the form (2.2) will proceed up to a singularity in the solution. This is confirmed by the nonexistence of stationary states in non-one-dimensional geometry and also by the absence of forces which can put hindrance to the collapse if it starts.

Moreover, an equation of the type (4.1) with cubic nonlinearity has either collapsing or divergent solutions.

It is natural that the behaviour of the solutions of Schrödinger equation with saturable nonlinearity is more complicated. As the analysis of stationary solutions in plane geometry shows, even in this case the upper bouds on the soliton amplitude (level) arise. This means that the plane collapse can go on only up to the definite value of amplitude. Then the packet will apparently break. Naturally an analogous situation is expected for two- and three-dimensional solutions *.

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* An indication of such a possibility is in the numerical experiment [9]. The computations [10,11] could not find this phenomenon.

The fact that the stable stationary solutions in non-one-dimensional geometry do not exist for the whole class of nonlinear wave equations (including ones with saturable nonlinearities) and even for some systems of two interacting scalar fields (this is confirmed by the above reasoning and the computer experiments [9,10]) is hardly accidental. Most likely a general phenomenon becomes apparent here. Probably it consists in the impossibility of constructing stationary quasi-particle (soliton) solutions by means of only scalar nonlinear fields (also see ref. [12]). Therefore the idea of consideration of models of interacting scalar and spinor fields seems to be very interesting in constructing stable stationary quasi-particle states. As far as we know such models have been independently proposed in ref. [13] and [14].

It is obvious that subsequent detailed investigation on the basis of scalar-spinor models of stationary quasi-particle solutions, their stability and interactions, if the latter will be found, is of great value.

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