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ERROR ESTIMATES FOR DISCRETIZATION IN TIME TO LINEAR HOMOGENEOUS PARABOLIC EQUATIONS WITH NONSMOOTH INITIAL DATA

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1. **Introduction.** Let X be a Banach space with norm $\| \ \|$. Let A be a sectorial operator in X with the domain $D(A)$. We consider this abstract homogeneous parabolic problem

$$(1.1) \qquad \begin{aligned} u'(t) + Au(t) &= 0 \\ u(0) &= v \in X. \end{aligned}$$

It is well known that there exists a unique solution of (1.1) and it can be described in this way

$$(1.2) \qquad u(t) = T(t)v = (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t} (\lambda + A)^{-1} v \, d\lambda,$$

where Γ is a curve in $\rho(-A)$ (the resolvent set of $-A$) such that $\arg \lambda \to \pm \varphi$ as $|\lambda| \to \infty$ for any fixed $\varphi \in (\pi/2, \pi)$.

In this paper we give error estimates for discretization in time (Rothe's method, backward Euler's method) to the problem (1.1). We are interested here in the case when no regularity assumptions are assumed for the initial element $v \in X$. The main results are formulated in the Theorem 2.

If anybody is interested in error estimates for the semidiscrete Galerkin method applied to our problem (or to nonhomogeneous problem) in Hilbert spaces we refer the reader for example to [1-3], [5-6],....

In [6] one can found error estimates for completely discrete schemes applied to this simple problem

$$u' - \Delta u = f$$
$$(1.3) \qquad u = 0 \qquad \text{on } \partial\Omega \times \langle 0,\infty)$$
$$u(x,0) = v(x) \in L_2(\Omega).$$

For the homogeneous case ($f = 0$) and for semidiscretization in time only, there is the error estimate derived in time steps t_n ($t_n = n\,\Delta t$, Δt is a time step) and it is

$$(1.4) \qquad C(\Delta t\ t_n^{-1})^p\ \|v\|_{L_2(\Omega)},$$

where C is an absolute constant and p is the order of time discretization.

Remark 1. The same technique as in [6] may be applied to our problem if X is a Hilbert space and A is selfadjoint positive definite in X.

Remark 2. For the backward Euler method this estimate follows from (1.4)

$$(1.5) \qquad C\,\Delta t\ t_n^{-1}\ \|v\|_{L_2(\Omega)}.$$

The same result is proved in [4] for a weak solution of an more abstract homogeneous parabolic problem in a Hilbert space.

Remark 3. In the following C denotes the positive generic constant independent of t, T.

The contribution of our paper is following:

i) we work in a Banach space only;

ii) we deal with more general operator A as in [4], [6];

iii) we us an another proof technique;

iv) we showed the continuity between backward Euler's method and the theory of semigroups.

2. Preparatory lemmas. In this paragraph we prove a few assertions from real and complex analysis.

Lemma 1. *If* $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda < 0$ *and* $t,\tau > 0$ *then*

$$\left|(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\right| \leqslant |\lambda|^2\ |\mathrm{Re}\lambda|^{-2}\ \left|(1-\tau\mathrm{Re}\lambda)^{-t/\tau} - e^{\mathrm{Re}\lambda t}\right|.$$

Proof : For any fixed t,τ,λ we denote

$$f(\lambda) = e^{-\lambda t}\ (1-\tau\lambda)^{-t/\tau}.$$

It is easy to see that

$$f'(\lambda) = t\tau\lambda\ e^{-\lambda t}(1-\tau\lambda)^{-t/\tau-1}.$$

So we can write

$$f(\lambda) - f(0) = \int_0^1 \lambda f'(\theta\lambda)\ d\theta =$$

$$= t\lambda^2\tau \int_0^1 e^{-\theta\lambda t}(1-\tau\theta\lambda)^{-t/\tau-1}\ \theta\ d\theta.$$

Using this we get

$$|f(\lambda) - f(0)| \leqslant t|\lambda|^2\tau \int_0^1 e^{-\theta\mathrm{Re}\lambda t}\ |1-\tau\theta\lambda|^{-t/\tau-1}\ \theta\ d\theta \leqslant$$

$$\leqslant t|\lambda|^2\tau \int_0^1 e^{-\theta\mathrm{Re}\lambda t}\ (1-\tau\theta\mathrm{Re}\lambda)^{-t/\tau-1}\ \theta\ d\theta =$$

$$= |\lambda|^2\ |\mathrm{Re}\lambda|^{-2}\ (f(\mathrm{Re}\lambda) - f(0)).$$

The rest of the proof follows from

$$|(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}| = e^{\mathrm{Re}\lambda t}\ |f(\lambda) - f(0)|. \qquad \square$$

Lemma 2. *Let δ, t, τ be positive real numbers. Then*

$$\int_{\delta}^{\infty} \left[(1+\tau y)^{-t/\tau} - e^{-yt}\right] y^{-1}\, dy \leqslant \tau t^{-1}.$$

Proof : Let us denote

$$\mathrm{Ei}(\alpha) = \int_{\alpha}^{\infty} e^{-z} z^{-1}\, dz \qquad \forall \alpha > 0.$$

One can prove that

$$\alpha e^{\alpha} \mathrm{Ei}(\alpha) \leqslant 1 \qquad \forall \alpha > 0.$$

Using

$$\frac{d}{dy}\left[e^{yt}(1+\tau y)^{-t/\tau} - 1\right] = t\tau y e^{yt}(1+\tau y)^{-t/\tau-1}$$

and integrating by parts we have

$$\int_{\delta}^{\infty} \left[(1+\tau y)^{-t/\tau} - e^{-yt}\right] y^{-1}\, dy =$$

$$= \int_{\delta}^{\infty} e^{-yt} y^{-1} \int_{0}^{y} t\tau s e^{st}(1+\tau s)^{-t/\tau-1}\, ds\, dy =$$

$$= \int_{\delta}^{\infty} e^{-st} s^{-1} ds \int_{0}^{\delta} t\tau s e^{st}(1+\tau s)^{-t/\tau-1}\, ds +$$

$$+ \int_{\delta}^{\infty} \int_{y}^{\infty} e^{-st} s^{-1} ds\; t\tau y e^{yt}(1+\tau y)^{-t/\tau-1}\, dy \leqslant$$

$$\leqslant \mathrm{Ei}(\delta t)\delta t e^{\delta t} \tau \int_{0}^{\delta} (1+\tau s)^{-t/\tau-1}\, ds +$$

$$+ \int_{\delta}^{\infty} \mathrm{Ei}(yt) y t e^{yt} \tau (1+\tau y)^{-t/\tau-1}\, dy \leqslant \int_{0}^{\infty} \tau(1+\tau s)^{-t/\tau-1}\, ds = \tau t^{-1}. \square$$

Lemma 3. *Let $\lambda \in \mathbb{C}$ where $|\lambda| \leqslant \delta$. Then for $\tau \leqslant (2\delta)^{-1}$ there exists a positive constant $C = C(\delta)$ such that*

$$\left|(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\right| \leqslant e^{Ct}\tau t^{-1} \qquad \forall t > 0.$$

Proof : Let us fix $t > 0$ and $0 < \tau < (2\delta)^{-1}$. If we denote

$$f(\lambda) = e^{-\lambda t}(1-\tau\lambda)^{-t/\tau},$$

then analogously as in Lemma 1 we estimate

$$|f(\lambda) - f(0)| \leqslant t\tau|\lambda|^2\int_0^1 e^{-\theta \mathrm{Re}\lambda t}|1-\tau\theta\lambda|^{-t/\tau-1}\theta\, d\theta \leqslant$$

$$\leqslant t\tau|\lambda|^2\int_0^1 e^{\theta t(|\lambda|-\mathrm{Re}\lambda)}e^{-\theta|\lambda|t}(1-\tau|\lambda|\theta)^{-t/\tau-1}\theta\, d\theta \leqslant$$

$$\leqslant e^{2\delta t}\,(f(|\lambda|) - f(0)).$$

In virtue of

$$f(x) - f(0) = \int_0^x t\tau y e^{-yt}(1-\tau y)^{-t/\tau-1}\, dy$$

and using

$$tye^{-yt} \leqslant 1 \qquad \forall y, t \in \mathbb{R}$$

one can obtain

$$f(|\lambda|) - f(0) \leqslant \int_0^{|\lambda|} (1-\tau y)^{-t/\tau-1}\, dy \leqslant \tau t^{-1}\,(1-\tau|\lambda|)^{-t/\tau} \leqslant \tau t^{-1}e^{2\delta t}.$$

The rest of the proof follows from

$$\left|(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\right| = e^{\mathrm{Re}\lambda t}\,|f(\lambda)-f(0)|. \qquad \square$$

3. Main results. Solving the problem (1.1) by discretization in time we get such elliptic problems

$$(u_i - u_{i-1})\tau^{-1} + Au_i = 0 \tag{3.1}$$
$$u_0 = v,$$

where τ is a time step; $u_i = u(i\tau)$; $i = 1,2,\dots$.

It is easy to see that

$$u_i = (I + \tau A)^{-i} v.$$

The main idea of Rothe's method is following. If we know all the u_i ($i = 1,2,\dots$) then we construct the Rothe function (as a approximate solution of (1.1)) in this way

$$u_n(t) = u_{i-1} + (t-t_{i-1})\tau^{-1}(u_i-u_{i-1}) \tag{3.2}$$

where $t \in \langle t_{i-1}, t_i)$.

Our approach is based on another definition of approximate solution which cuts $u_n(t)$ in the time steps $i\tau$ (i.e. in u_i). Let Γ be the curve taken from (1.1). Le us define the operator $T_\tau(t)$: $X \to X$ (for every $t > 0$, $0 < \tau < \tau_0$) in this way

$$T_\tau(t) = (2\pi i)^{-1} \int_\Gamma (1-\tau\lambda)^{-t/\tau}(\lambda+A)^{-1}\, d\lambda. \tag{3.3}$$

One can prove that the integral in (3.3) is absolutely convergent for every fixed t,τ. The parameter τ corresponds with the time step in (3.1). For any fixed $\tau > 0$, $T_\tau(t)v$ is said to be the approximate solution of (1.1) in our sense. In order to show relation between $u_n(t)$ and $T_\tau(t)v$ we prove

$$T_\tau(i\tau) = (I+\tau A)^{-i} \tag{3.4}$$

for $i = 1,2,\dots$.

In fact we can write

$$I = T(0) = T_\tau(0) = (2\pi i)^{-1} \int_\Gamma (\lambda+A)^{-1}\, d\lambda =$$

$$= (2\pi i)^{-1} \int_\Gamma (1-\tau\lambda)^{-i}(\lambda+A)^{-1}(1-\tau\lambda)^{i}\, d\lambda = \tag{3.5}$$

$$= (2\pi i)^{-1} \int_\Gamma (1-\tau\lambda)^{-i}(\lambda+A)^{-1} \sum_{k=0}^{i} \left[{i \atop k} \right](-1)^k\tau^k\lambda^k\, d\lambda.$$

Using $(\lambda+A)(\lambda+A)^{-1} = I$ we get $A(\lambda+A)^{-1} = -\lambda(\lambda+A)^{-1}$ and in the end one can prove

$$A^k(\lambda+A)^{-1} = (-1)^k\lambda^k(\lambda+A)^{-1} \tag{3.6}$$

for any $k = 1,2,\ldots$.

From (3.5), (3.6) we deduce

$$I = (2\pi i)^{-1}\int_\Gamma (1-\tau\lambda)^{-i}(\lambda+A)^{-1}(I+\tau A)^i \, d\lambda. \tag{3.7}$$

The operator $(I+\tau A)^{-i}$ is linear and bounded for every $i = 1,2,\ldots$ and so (3.7) yields (3.4). Really in fact we have

$$(I+\tau A)^{-i} = (I+\tau A)^{-i}(2\pi i)^{-1}\int_\Gamma (1-\tau\lambda)^{-i}(\lambda+A)^{-1}(I+\tau A)^i \, d\lambda =$$

$$= (2\pi i)^{-1}\int_\Gamma (1-\tau\lambda)^{-i}(\lambda+A)^{-1} \, d\lambda = T_\tau(i\tau).$$

So we can say that $T_\tau(t)$ is a fractional power of $(I+\tau A)^{-1}$, more precisely

$$T_\tau(t) = (I+\tau A)^{-t/\tau}. \tag{3.8}$$

Theorem 1. *The family $\{T_\tau(t)\}_{t\geqslant 0}$ is a semigroup.*

Proof : It is easy to see that $T_\tau(0) = T(0) = I$. Using Cauchy integral theorem we may shift the path of integration in (3.3) for a small distance to the right without changing the value of the integral. The new curve we denote by Γ'. In virtue of the resolvent identity

$$(\lambda+A)^{-1} - (\mu+A)^{-1} = (\mu-\lambda)(\lambda+A)^{-1}(\mu+A)^{-1}$$

we can write

$$T_\tau(t)T_\tau(s) = (2\pi i)^{-2}\int_\Gamma\int_{\Gamma'} (\lambda+A)^{-1}(\mu+A)^{-1}(1-\tau\lambda)^{-t/\tau}(1-\tau\mu)^{-s/\tau} \, d\mu d\lambda =$$

$$= (2\pi i)^{-2}\int_\Gamma\int_{\Gamma'} (\mu-\lambda)^{-1}\left[(\lambda+A)^{-1}-(\mu+A)^{-1}\right](1-\tau\lambda)^{-t/\tau}(1-\tau\mu)^{-s/\tau} \, d\mu d\lambda .$$

One can prove

$$(2\pi i)^{-1}\int_{\Gamma'} (1-\tau\mu)^{-s/\tau}(\mu-\lambda)^{-1}\, d\mu = (1-\tau\lambda)^{-s/\tau}$$

and

$$(2\pi i)^{-1}\int_{\Gamma} (1-\tau\lambda)^{-t/\tau}(\lambda-\mu)^{-1}\, d\lambda = 0.$$

From these facts we deduce

$$T_\tau(t)T_\tau(s) = (2\pi i)^{-1}\int_{\Gamma} (\lambda+A)^{-1}(1-\tau\lambda)^{-(t+s)/\tau}\, d\lambda. \qquad \square$$

Theorem 2. *(i) If* Re $\sigma(-A) \leqslant -\delta_0$ $(\delta_0 > 0)$*, then there exists* $\tau_0 > 0$ *such that*

$$t\|T_\tau(t) - T(t)\| \leqslant C\tau \qquad \forall\tau < \tau_0,\ \forall t > 0.$$

(ii) If Re $\sigma(-A) \leqslant \delta_0$ $(\delta_0 > 0)$, *then there exists* $\tau_0 > 0$ *such that*

$$t\|T_\tau(t) - T(t)\| \leqslant Ce^{Ct}\tau \qquad \forall\tau < \tau_0,\ \forall t > 0.$$

The constants C, τ_0 *depend only on the* $\sigma(A)$..

Proof: (i) Without lost of generality we can suppose that the curve Γ is described in this way

(3.9) $$\lambda \in \Gamma \Leftrightarrow \lambda = -\delta - s\cos\varphi \pm i\, s\sin\varphi$$

where $s \in \langle 0,\infty)$, $\varphi \in (0,\pi/2)$, $\delta = \delta(\delta_0) > 0$.

Let us divide Γ into Γ_1 and Γ_2 such that

(3.10) $$\lambda \in \Gamma_1 \Leftrightarrow \lambda \in \Gamma \wedge \operatorname{Im}\lambda \geqslant 0$$

$$\lambda \in \Gamma_2 \Leftrightarrow \lambda \in \Gamma \wedge \operatorname{Im}\lambda < 0.$$

We give the proof of (i) only for Γ_1. The second case can be proved analogously. Using Lemma 1 one can see that

$$V(\Gamma_1) = \left\|(2\pi i)^{-1}\int_{\Gamma_1} (\lambda+A)^{-1}\left[(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\right] d\lambda\ \right\| \leqslant$$

$$\leqslant C\int_{\Gamma_1} |\lambda|^{-1}\left|(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\right| d\lambda \leqslant$$

$$\leqslant C\int_{\Gamma_1} |\lambda|^2|\mathrm{Re}\lambda|^{-3}\left|(1-\tau\mathrm{Re}\lambda)^{-t/\tau} - e^{\mathrm{Re}\lambda t}\right| d\lambda .$$

For $\lambda \in \Gamma_1$ we have $|\lambda|^2|\mathrm{Re}\lambda|^{-2} \leqslant C(\varphi,\delta)$ and in virtue of (3.9), (3.10) we deduce

$$V(\Gamma_1) \leqslant C\int_0^{\infty} \left[\left[1 + \tau(\delta + s\cos\varphi)\right]^{-t/\tau} - e^{-(\delta + s\cos\varphi)t}\right] \times$$

$$\times (\delta + s\cos\varphi)^{-1} ds = C\int_{\delta}^{\infty} \left[1 + \tau y)^{-t/\tau} - e^{-yt}\right] y^{-1} dy.$$

The rest of the proof is a consequence of Lemma 2.

(ii) Let us divide Γ into Γ_1 and Γ_2 in this way

$$\lambda \in \Gamma_1 \Leftrightarrow \lambda \in \Gamma \wedge \mathrm{Re}\lambda \leqslant -\delta_0$$

$$\lambda \in \Gamma_2 \Leftrightarrow \lambda \in \Gamma \wedge \lambda \notin \Gamma_1.$$

We omit the proof for Γ_1 because it goes in the same way as in (i). In the case when $\lambda \in \Gamma_2$ there exists $\delta > 0$, $\delta = \delta(\delta_0,\varphi)$ such that

$$|\lambda| \leqslant \delta \qquad \text{if } \lambda \in \Gamma_2.$$

So we have

$$V(\Gamma_2) = \left\|(2\pi i)^{-1}\int_{\Gamma_2} (\lambda+A)^{-1}\left[(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\right] d\lambda \right\| \leqslant$$

$$\leqslant C\int_{\Gamma_2} |\lambda|^{-1}\left|(1-\tau\lambda)^{-t/\tau} - e^{\lambda t}\right| d\lambda .$$

Using Lemma 3 one can see that

$$V(\Gamma_2) \leqslant C\, e^{Ct}\tau t^{-1},$$

where the constant C depends on $\sigma(A)$ i.e. $C = C(\delta_0,\varphi)$, from which we conclude (ii). □

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