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OPERATOR ORDERING AND SYMBOL REPRESENTATIONS OF ENVELOPING ALGEBRAS

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1. Introduction

Operator symbol representations have their origin in mathematical literature in the theory of pseudo-differential operators and in physical literature in so-called phase space methods of quantum mechanics. Phase space methods in quantum theory make use of symbol representations of the Weyl algebra. A restriction to a pure algebraic view point, i.e. to symbols of algebra elements rather than of operators, allows to extent the class of non-commutative algebras for which symbol representations exist. Furthermore, the twisted product techniques can be extended to subrings of the quotient division ring of the Weyl algebra. The method has been elaborated /1/ for calculations in computer algebra systems and was applied to quantum mechanical problems (see /1/ and ref. therein), to Gröbner bases calculations in non-commutative algebra /2/, and to Lie optics in order to calculate aberrations (see /3/ and ref. therein).

The method can be generalized to enveloping algebras U(L) of Lie algebras L different from the Heisenberg Lie algebra. The method starts from a one-to-one correspondence between the linear space of the enveloping algebra U(L) and that of the symmetric algebra S(L) over L. The algebra S(L) can be equipped with a so-called twisted product so that it becomes isomorphic to U(L).

Operator orderings have been described in mathematical and physical literature by various techniques /4-6/. Symbol representations are powerful tools in this context /4/. A generalization to enveloping algebras U(L) resp. to differential operators over Lie groups will be of theoretical and practical interest.

The ordering problem consists first of all in a description of different operator orderings. Different ordering rules correspond to different bases in U(L). The elements of U(L) are represented by different symbols in dependence of the ordering rule chosen. An appropriated twisted product depends on the ordering rule in a non-trivial way. We generalize the notion of an ordering defining function ϕ. It is possible to calculate ϕ for various ordering rules. This helps to determine a transformation between different types of symbols and allows to use the same fast multiplication algorithm for different orderings. The fast multiplication algorithm for the non-commutative algebra must not be related to twisted product techniques. A very fast algorithm/7/ makes uses of the fast integer arithmetic of special processors. The algebra elements are represented by integers in that case. The representation is related to a given basis. The mentioned transformation allows to use the same fast algorithm also if the algebra elements are given in another basis related to a different ordering.

2. Basic notions and notations

Let L be a finite dimensional Lie algebra over the field K. The Lie algebra L is given by its structure constants c_{ij}^{k} in a basis $\{X_1,X_2,\dots,X_n\}$, $[X_i,X_j]=\sum_{k=1}^{n} c_{ij}^{k}X_k$ for $i,j=1,2,\dots,n$.

We denote by $U(L)$ the enveloping algebra of L over K. One gets $U(L)=T(L)/J$ from the tensor algebra $T(L)$ of L by factorization according to the ideal J generated by the set $\{ X\otimes Y - Y\otimes X -[X,Y] \mid X,Y \in L \}$. The enveloping algebra $U(L)$ has a graded structure $U(L) = \oplus_n U_n$, $U_n = U^n \setminus U^{n-1}$, where $U^n= \{X\in U(L) \mid \deg X \le n \}$. The set $\{X_1^{i_1}X_2^{i_2}\cdots X_n^{i_n}\}$, $(i_1,i_2,\dots,i_n)\in N_0^n$ is a basis of $U(L)$ according to the Poincaré-Birkhoff-Witt theorem. We use the convention $X_1^0X_2^0\cdots X_n^0= 1$, and we denote by capital letters X_k also the elements of L in their embedding $L \xrightarrow{i} U(L)$ in $U(L)$. So, there is no problem to differ non-commutative multiplication of elements $X_1^{i_1}X_2^{i_2}\cdots X_n^{i_n}$ in $U(L)$ from commutative multiplication in the symmetric algebra $S(L)$. The basis elements of $S(L)$ are denoted by small letters $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$. The symmetric algebra $S(L)$ over L can be obtained from the tensor algebra $T(L)$ by factorization $S(L)=T(L)/I$ according to the ideal I generated by the set $\{X\otimes Y - Y\otimes X \mid X,Y \in L\}$. $S(L)$ is isomorphic to the algebra of polynomials $K[x_1,x_2,\dots,x_n]$.

We use multiindex notation:

$$x=(x_1,x_2,\dots,x_n),\ \xi=(\xi_1,\xi_2,\dots,\xi_n),\ \xi x= \sum_{k=1}^{n} \xi_k x_k,\ \alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)\in N^n$$

$$|\alpha|=\sum_{k=1}^{n} \alpha_k,\quad x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},\quad \frac{\partial^{|\alpha|}}{\partial\xi^\alpha} = \frac{\partial^{|\alpha|}}{\partial\xi_1^{\alpha_1}\,\partial\xi_2^{\alpha_2}\cdots\partial\xi_n^{\alpha_n}}.$$

3. Generalized Weyl ordering

A one-to-one correspondence between the linear spaces of $S(L)$ and $U(L)$ is defined by the Gelfand map/8/:

$$\omega : S(L) \longrightarrow U(L)$$

given by

$$\omega(x_{i_1}x_{i_2}\dots x_{i_r}) = \frac{1}{r!} \sum_{\pi} X_{i_{\pi(1)}} X_{i_{\pi(2)}} \cdots X_{i_{\pi(r)}} \tag{1}$$

and by linearity of ω. The sum runs over all permutations of r elements. The r.h.s. of (1) is the complete symmetrization $\langle X_{i_1}X_{i_2}\dots X_{i_r}\rangle$ of $X_{i_1}X_{i_2}\dots X_{i_r}$.

This complete symmetrization will be synonymously called Weyl ordering. The map (1) will be called Weyl quantization due to applications in physics. The element $y \in S(L)$ is called Weyl symbol of $\omega(y) \in U(L)$. The notions Weyl ordering and Weyl quantization are now generalizations of the corresponding notions known for the special case if L is the Heisenberg Lie algebra H_n. H_n is the $(n+1)$-dimensional Lie algebra over R given by $[P_i,Q_j] = \delta_{ij}Z$ with $\delta_{ij}= \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$ $(i,j=1,2,\dots,n)$, where Z is a central element.

<u>Example 1:</u> Let L be the Lie algebra so(3) with the basis $\{X_1,X_2,X_3\}$, $[X_1,X_2]=X_3$, $[X_2,X_3]=X_1$, $[X_3,X_1]=X_3$.

Then

$$\omega(x_1x_2) = \langle X_1X_2\rangle = \tfrac{1}{2}(X_1X_2+X_2X_1).$$

The element $x_1x_2 \in S(L=so(3))$ is the Weyl symbol of

$$\tfrac{1}{2}(X_1X_2+X_2X_1) = X_1X_2 - \tfrac{1}{2} X_3 \quad \in U(so(3)).$$

4. Twisted product for Weyl symbols

We introduce in $S(L)$ a new non-commutative multiplication which makes $S(L)$ isomorphic to $U(L)$. Since ω is bijective we are allowed to define

$$y_1 * y_2 = \omega^{-1}(\omega (y_1)\cdot\omega (y_2)) \tag{2}$$

A key idea in symbol representations is to calculate the non-commutative multiplication between symbols in a direct way without the non-commutative multiplication in $U(L)$.

For the usage of computer algebra systems it is very convenient that the multiplication coincides with the following convolution product /9/. Note that only differentiations occur.

$$(y_1 * y_2)(x) = e^{x\,D(\partial_{x'},\,\partial_{x''})}\; y_1(x')\, y_2(x'')\,\big|_{x'=x''=x} \qquad (3)$$

The differential operator D in the exponential series is formally given by

$$D(\partial_{x'},\partial_{x''}) = \tau(\partial_{x'},\partial_{x''}), \qquad (4)$$

where $\tau(\xi,\eta) = \lambda(\xi,\eta)-\xi-\eta$ is the non-linear part of the exponent in the Baker-Campbell-Hausdorff formula (see for instance /9/)

$$e^{\xi X} \circ e^{\eta X} = e^{\lambda(\xi,\eta)X} \qquad (5)$$

Since we calculate twisted products of polynomials, the formal exponential series of Eq. 3 will be used only up to a finite order. A proof, that the twisted product of Eq. 3 coincides with that defined in Eq. 2 will be given in section 6 together with a justification of the notation used. The operator D will be given by a truncated series.

The twisted product in combination with the linear map ω from Eq. (1) allows to execute tedious calculations in U(L) using now only symbols. This can be done easily in almost all computer algebra systems, since only differentiation and commutative polynomial algebra facilities are required. Of course, D has to be determined. However, if D is calculated once up to an order high enough in dependence of degree of the polynomials, then D can be used in all subsequent calculations. Moreover, useful formulae as applied in a LOGLAN package /10/ for calculations in enveloping algebras and Lie fields can be derived from the twisted product.

5. Orderings defined by a function

In order to apply the twisted product techniques also for orderings different from that defined by the map ω of Eq. 1, we introduce the notion of an ordering defining function ϕ. For this reasons we consider another way to define the map ω. The method is partially known in literature for the special case if L is the Heisenberg Lie algebra (see /1/ and references therein).

Let $\xi x = \sum_{k=1}^{n} \xi_k x_k$ be a general homogeneous element of degree 1 in S(L). Replacing the small x's by capital X's and not touching the parameters ξ we get a linear element in U(L). If this will be done in the exponential series $e^{\xi x}$, so this is just the way to describe Weyl quantization for exponential series

$$\Omega\,(\,e^{\xi x}\,) = e^{\xi X} \qquad (6)$$

From Eq. 6 we derive the Gelfand map of Eq. 1, i.e. the Weyl quantization of polynomials. Formal differentiation $\frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}}\dots\big|_{\xi=0}$ applied on $e^{\xi x}$ gives the monomial $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ whereas the same operator applied on $e^{\xi X}$ gives just the complete symmetrization $\langle X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \rangle = \omega(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) \in U(L)$ of $X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$. That will not wonder, since the Weyl quantization $\omega(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})$ of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ occurs as "coefficient" of $\xi^{\alpha} = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$ in the exponential series $e^{\xi X} = \sum_i \frac{(\xi X)^i}{i!}$, $(|\alpha|=i)$. Differentiation with respect to ξ at $\xi=0$ picks out just that coefficient:

$$\omega(x^{\alpha}) = \frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}}\, \Omega(e^{\xi x})\big|_{\xi=0} = \langle X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \rangle . \qquad (7)$$

So we receive the linear map ω by formal differentiation from Ω as shown in the following diagram.

$$\begin{tikzcd}
e^{\xi x} \arrow[r, "\Omega"] \arrow[d, "\frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}} \cdots |_{\xi=0}"'] & e^{\xi X} \arrow[d, "\frac{\partial^{|\alpha|}}{\partial \xi^{\alpha}} \cdots |_{\xi=0}"] \\
x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \arrow[r, "\omega"] & \langle X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \rangle
\end{tikzcd}$$

complete symmetrization

We refer to $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ as the Weyl symbol of the symmetrized basis element $\langle X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \rangle$. Note that the map ω defines the Weyl basis $\{\langle X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \rangle \mid \alpha \in N^n\}$ of U(L) according to Eq. (1). We introduce an ordering defining function ϕ and generalize the map Ω as follows

$$\Omega_\phi\left(e^{\xi x}\right) = e^{\xi X + \phi(\xi)X} , \tag{8}$$

where $\phi=(\phi_1(\xi),\dots,\phi_n(\xi))$ has to be appropriately chosen. We will not try to characterized the class of allowed functions ϕ. However, ϕ can be explicitly determined for various orderings. The particular case $\phi = 0$ describes Weyl ordering. Formal differentiation to ξ at $\xi=0$ can be used to generate a basis in U(L). The basis differs from the Weyl basis if ϕ is not zero or a constant.

We define the map ω_ϕ for polynomials by

$$\omega_\phi(x^\alpha) = \frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\,\Omega_\phi(e^{\xi x})\,\Big|_{\xi=0} = \langle X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}\rangle_\phi \tag{9}$$

and by linearity of ω.

The method is summarized in the following diagram.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
e^{\xi x} \arrow[r, "\Omega_\phi"] \arrow[d, "\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\cdots\,|_{\xi=0}"'] & e^{\xi X + \phi(\xi)X} \arrow[d, "\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\cdots\,|_{\xi=0}"] \\
x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \arrow[r, "\omega_\phi"] & \langle X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}\rangle_\phi
\end{tikzcd}$$

We now speak of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ as the ϕ-symbol of the ϕ-ordered basis element $\langle X^{\alpha_1}X^{\alpha_2}\cdots X^{\alpha_n}\rangle_\phi$ of U(L) .

Example 2: Standard ordering, Poincaré-Birkhoff-Witt basis.

In many practical cases and with almost all computer algebra systems it seems much easier to work with the Poincaré-Birkhoff-Witt basis $X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}$ of U(L) than with the symmetrized elements of the Weyl basis. The function ϕ for the Poincaré-Birkhoff-Witt basis (PBW-basis) or standard basis can be calculated iteratively using the Baker-Campbell-Hausdorff formula. We will do it step wise so that the origin and nature of ϕ becomes transparent.

Because of $X_i^{\alpha_i} = \frac{\partial^\alpha}{\partial\xi_i^{\alpha_i}}\, e^{\xi_i X_i}|_{\xi_i=0}$ for $i=1,2,\dots,n$, we obtain

$$X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n} = \left(\frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}}\, e^{\xi_1X_1}|_{\xi_1=0}\right)\cdots\left(\frac{\partial^{\alpha_n}}{\partial\xi_n^{\alpha_n}}\, e^{\xi_nX_n}|_{\xi_n=0}\right) . \tag{10}$$

Since the differentiation concerns different parameters this equals to

$$X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\left(e^{\xi_1X_1}\cdots e^{\xi_nX_n}\right)|_{\xi=0} . \tag{11}$$

In order to find the function ϕ such that

$$X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\left(e^{\xi X + \phi(\xi)X}\right)|_{\xi=0} , \tag{12}$$

we iteratively apply the BCH-formula

$$e^{\xi_1X_1}\cdot e^{\xi_2X_2}\cdots e^{\xi_nX_n} = e^{\xi_1X_1+\xi_2X_2+\tau(\underline{\xi}_1,\underline{\xi}_2)X}\, e^{\xi_3X_3}\cdots e^{\xi_nX_n} , \tag{13}$$

where $\underline{\xi}_i=(0,..,0,\xi_i,0,\dots,0)$ denotes the n-tuple with ξ_i in the i-th component. Then the "vector valued" function ϕ is given by

$$\phi(\xi) = \tau^{(1)}+\dots+\tau^{(n-1)} , \tag{14}$$

where $\tau^{(k)}$ is recurrently defined by

$$\tau^{(k)}=\tau(\underline{\xi}_1+\dots+\underline{\xi}_k+\tau^{(1)}+\dots+\tau^{(k-1)},\underline{\xi}_{k+1}) \tag{15}$$

with

$$\tau^{(1)}=\tau(\underline{\xi}_1,\underline{\xi}_2) .$$

The function ϕ is the key to transformations between symbol for different orderings. Let $\phi(\xi)$ be the ordering defining function for the Poincaré-Birkhoff-Witt basis, i.e. for the standard ordering of example 2. The transformation $e^{x'\phi(\partial_x)}\dots|_{x'=x}$ applied on a standard symbol of an element Y of U(L) gives just the Weyl symbol of the same element Y.

$$e^{x'\phi(\partial_x)}\, y_s(x)|_{x'=x} = y_w(x) . \tag{16}$$

The proof starts again from exponentials. Note that $e^{\xi x}$ is the ϕ-symbol of $e^{\xi X+\phi(\xi)X}$ according to Eq. 8. The following identity

$$e^{x'\phi(\partial_x)}\, e^{\xi x}\,|_{x'=x} = e^{\xi x + \phi(\xi)x} \tag{17}$$

may be verified by comparison of coefficients.

The r.h.s. of Eq. 17 is just the Weyl symbol of $e^{\xi X + \phi(\xi)X}$.

Formal differentiation $\frac{\partial^{|\alpha|}}{\partial\xi^\alpha} \cdots \big|_{\xi=0}$ of Eq. 17 gives for the l.h.s. $e^{x'\phi(\partial_x)} x^\alpha \big|_{x'=x}$ where x^α is the ϕ-symbol of $\langle X_1^{\alpha_1} X_2^{\alpha_2} \cdot\cdot X_n^{\alpha_n} \rangle_\phi =$ $X_1^{\alpha_1} X_2^{\alpha_2} \cdot\cdot X_n^{\alpha_n}$ for the standard ordering. The same formal differentiation gives for the r.h.s. of Eq. 17 a polynomial in $x_1,\ldots,x_n$ which is just the Weyl symbol of $\langle X_1^{\alpha_1} X_2^{\alpha_2} \cdot\cdot X_n^{\alpha_n} \rangle_\phi = X_1^{\alpha_1} X_2^{\alpha_2} \cdot\cdot X_n^{\alpha_n}$.

6. Twisted products

The idea to derive the twisted product for polynomials starting from exponential series is the key for very simple proofs. First we show that for Weyl ordering the twisted product of Eq. 2 is given by Eq. 3, where D has to be the nonlinear part of the BCH-formula. So we start with an Ansatz that the twisted product is given by an operator e^{xD} for an appropriate D and we have to demand

$$\Omega(e^{\xi x}) \circ \Omega(e^{\eta x}) = \Omega(e^{x D(\partial_{x'}, \partial_{x''})} e^{\xi x'} e^{\eta x''}\big|_{x'=x''=x}) . \tag{18}$$

Note that the following identity is valid.

$$e^{x D(\partial_{x'}, \partial_{x''})} e^{\xi x'} e^{\eta x''}\big|_{x'=x''=x} = e^{xD(\xi,\eta) + \xi x + \eta x} . \tag{19}$$

This identity may be easily verified by comparison of coefficients in the exponential series. Substitution of Eq. 19 into the r.h.s of Eq. 18 and Weyl quantization (Ω) gives

$$e^{\xi X} e^{\eta X} = e^{(\xi + \eta + D(\xi,\eta))X} . \tag{20}$$

Hence, $D(\xi,\eta)$ has to coincide with the nonlinear part $\tau(\xi,\eta)$ in the BCH-formula.

In order to justify the notation used, we remember that according to the BCH-formula $e^{\xi X} e^{\eta X} = e^{(\xi+\eta+\tau(\xi,\eta))X}$ the nonlinear part $\tau(\xi,\eta)$ defines an analytic map from L x L into L. $\tau(\xi,\eta)$ may be obtained by an absolutely convergent series $\tau(\xi,\eta) = \sum_{n=2}^{\infty} c_n(\xi,\eta)$, where c_n are recurrently defined polynomial maps from L x L into L of degree n. We refer to /11/ for details. Formal differentiation $\frac{\partial^{|\alpha|}\partial^{|\beta|}}{\partial\xi^\alpha\,\partial\eta^\beta} \cdots \big|_{\xi=\eta=0}$ applied on the rewritten Eq. 18

$$\Omega(e^{x D(\partial_{x'}, \partial_{x''})} e^{\xi x'} e^{\eta x''}\big|_{x'=x''=x}) = e^{\xi X} e^{\eta X} \tag{21}$$

gives

$$\omega(x^\alpha * x^\beta) = \omega(x^\alpha)\, \omega(x^\beta) \tag{22}$$

according to Eq. 7. So the twisted product of Eq. 2 coincides with that of Eq. 3 due to linearity of ω. □

7. Conclusions

If the basis of U(L) is given not in the Weyl ordering but in the standard ordering (resp. ϕ-ordering) there exist in principle two procedures for the multiplication of two elements of U(L).

Algorithm I:

a) Replace the elements of U(L) by their symbols according to the ordering rule given. These ϕ-symbols are immediately given by replacing the capital X's by small ones.

b) Transform the ϕ-symbols by $e^{x'\phi(\partial_x)} \ldots\big|_{x'=x}$ into Weyl symbols.

c) Compute the twisted product. The result is given by its Weyl symbol.

d) Transform the Weyl symbol to a ϕ-symbol by $e^{-x'\phi(\partial_x)} \ldots\big|_{x'=x}$.

e) The (commutative) ϕ-symbol written in ϕ-ordering gives just the result by replacing small x's by capital ones.

Algorithm II:

This algorithm differs from Algorithm I by the fact that one defines a twisted product immediately for ϕ-symbols. This may be done by the same techniques and it will be presented in a forthcoming paper.

Note that the transformation between different orderings helps also if another fast multiplication for a given basis of U(L) is available in a computer algebra system.

The Goldberg-Eriksen algorithm for BCH-series is an effective one [12] for the computation of D up to a certain order [13]. Several simplification for the twisted product realization in REDUCE [14] and criteria for the truncation of the series are applicable [13]. Note that the twisted product operator acts on polynomials. So the BCH-series for D and the exponential series e^D are used only up to a certain order in dependence of the degree of the polynomials.

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