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N.N.Bogolubov

ON AN ARITHMETICAL THEOREM AND ITS APPLICATION ON THE THEORY OF ALMOST PERIODIC FUNCTIONS

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In the present paper we come back to the problem considered in our paper [x)] in 1939.

We shall proceed to prove the following arithmetical theorem:

Theorem I

Given a finite system of integer, pairwise, nonequal numbers

$$n_1, \dots n_m. \tag{I}$$

Let us denote

$$M = \max_j |n_j|; \quad N = 8M.$$

Then, we can number the natural sequence of N numbers: 0,1,...N-1 so that the derived sequence

$$\alpha_1, \dots \alpha_N$$

has the property.

Any integer q satisfying the inequalities

$$|n| < M \tag{2}$$

$$2\pi R\left(n\frac{\alpha_K}{N}\right) < 1 - \frac{N^2}{m^2(q+1)}, \qquad K = 1, \dots q,$$

in which q is any positive integer such that

$$1 - \frac{N^2}{m^2(q+1)} > 0, \tag{3}$$

can be represented as a combination

$$n = n_p + n_z - n_s - n_t \tag{4}$$

of numbers of the system (I). Here

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x) On some arithmetical properties of the almost periodic functions, Zapisky kafedry mat. fiz. (Academia Nauk in Ukraina), 1939.

$$R(x) = |x - E(x)| \qquad (5)$$

$E(x)$ is the nearest to x integer.

Proof

Let us denote by p, τ, s, t the number of different combinations which can be used to represent n in the form (4).

If n cannot be represented in this form, we put

$$f(n) = 0.$$

Introduce the function

$$\delta(n) = \begin{cases} 1, & n = n_1, \dots n_m \\ 0, & \end{cases}.$$

Then, it is obvious that

$$f(n) = \sum_{s=1}^{m} \sum_{t=1}^{m} \sum_{p=1}^{m} \delta(n + n_s + n_t - n_p). \qquad (6)$$

Let ρ be a primitive root of the Nth order of 1:

$$\rho = e^{\frac{2\pi i}{N}}.$$

Then

$$\sum_{\alpha=0}^{N-1} \rho^{K\alpha} = \begin{cases} N, & K = 0 \ (\mathrm{mod}\ N) \\ 0, & K \neq 0 \ (\mathrm{mod}\ N) \end{cases}$$

and hence

$$\delta(n) = \frac{1}{N} \sum_{\alpha \neq 0}^{N-1} \left(\sum_{K=-\frac{N}{2}}^{N/2} \rho^{(K-n)\alpha} \delta(K) \right) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \left\{ \sum_{K=1}^{m} \rho^{(n_K - n)\alpha} \right\}$$

for

$$-\frac{N}{2} < n < \frac{N}{2}.$$

Note here that

$$-3M \leq n_s + n_t - n_p \leq 3M$$

and hence for

$$|n| < M$$

the inequality

$$-\frac{N}{2} < n + n_s + n_t - n_p < \frac{N}{2}$$

is valid. Consequently,

$$\delta(n + n_s + n_t - n_p) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \left\{ \sum_{K=1}^{m} \rho^{(n_K - n - n_s - n_t + n_p)\alpha} \right\}.$$

Thus

$$f(n) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \sum_{K=1}^{m} \sum_{s=1}^{m} \sum_{t=1}^{m} \sum_{p=1}^{m} \rho^{(n_K - n - n_s - n_t + n_p)\alpha} = \qquad (7)$$

$$= \frac{1}{N} \sum_{\alpha=0}^{N-1} \rho^{-n\alpha} \left| \sum_{K=1}^{m} \rho^{(n_K \alpha)} \right|^4.$$

We see also that

$$\sum_{\alpha=0}^{N-1} \left| \sum_{K=1}^{m} \rho^{n_K \alpha} \right|^2 = \sum_{\alpha=0}^{N-1} \sum_{K=1}^{m} \sum_{\tau=1}^{m} \rho^{(n_K - n_\tau)\alpha} = Nm \qquad (8)$$

Arrange now the quantities

$$\left| \sum_{K=1}^{m} \rho^{n_K \alpha} \right|^2 \qquad \alpha = 0, 1, \dots N-1$$

in the order of decreasing. We get the sequence

$$A_1 \geq A_2 \geq \cdots \geq A_N.$$

Denote by α_q the values of α corresponding to A_q

$$A_q = \left| \sum_{K=1}^{m} \rho^{n_K \alpha_q} \right|^2.$$

Thus, we have numbered the sequence 0,1,2,...N-1 into $\alpha_1, \dots, \alpha_N$.

We see that

$$A_q \leq m^2 = \left| \sum_{K=1}^{m} \rho^{n_K \alpha} \right|^2_{\alpha=0}$$

and consequently

$$A_1 = m^2, \qquad \alpha_1 = 0.$$

From (7) we have

$$f(n) = \frac{1}{N}\sum_{K=1}^{N} \rho^{-n\alpha_K} A_K^2; \qquad |n| < M$$

with (8)

$$\sum_{K=1}^{N} A_K = Nm$$

and

$$A_K \leqq \frac{Nm}{K}.$$

Let now n and q be integers satisfying conditions (2) and (3) of our theorem.

It is clear that

$$N f(n) \geqq \sum_{K=1}^{q} A_K^2 \rho^{-n\alpha_K} - \sum_{K=q-1}^{N} A_K^2 \geqq$$

$$\geqq \sum_{K=1}^{q} A_K^2 \left\{1-(1-\rho^{-n\alpha_K}\right\} - \frac{N^2m^2}{q+1} \geqq \sum_{K=1}^{q} A_K^2 - \sum_{K=1}^{q} |1-\rho^{-n\alpha_K}| A_K^2 - \frac{N^2m^2}{q+1} =$$

$$= \sum_{K=1}^{q} A_K^2 \left\{1-|1-\rho^{-n\alpha_K}|\right\} - \frac{N^2m^2}{q+1}.$$

On the other hand

$$|1-\rho^{-n\alpha_K}| = |1-e^{-2\pi i \frac{n\alpha_K}{N}}| = |1-exp\left\{2\pi i\left(E\left(\frac{n\alpha_K}{N}\right) - \frac{n\alpha_K}{N}\right)\right\}| \leqq$$

$$\leqq 2\pi R\left(\frac{n\alpha_K}{N}\right) < 1 - \frac{N^2}{m^2(q+1)}$$

and therefore

$$\sum_{K=1}^{q} A_K^2 \left\{1-|1-\rho^{-n\alpha_K}|\right\} \geqq \frac{N^2}{m^2(q+1)} \sum_{K=1}^{q} A_K^2 > \frac{N^2m^4}{m^2(q+1)}.$$

Thus

$$f(n) > \frac{N^2m^2}{q+1} - \frac{N^2m^2}{q+1} = 0$$

and our theorem is proved.

Theorem II

Given an infinite sequence of pairwise nonequal integers

$$n_1, \ldots, n_m, \ldots$$

such that

$$\left|\frac{n_m}{m}\right| \leqq G = Const.$$

Then, there is a sequence of numbers

$$\lambda_1, \lambda_2, \ldots \qquad 0 \leqq \lambda_K \leqq 1$$

having the following property: any number n satisfying the inequalities

$$2\pi R(n\lambda_K) < 1 - \frac{64G^2}{q+1}, \quad (K = 1, \ldots q). \tag{9}$$

in which q is any positive integer such that

$$1 - \frac{64G^2}{q+1} > 0,$$

can be represented in the form

$$n = n' + n'' - n''' - n'''', \tag{10}$$

where n', n'', n''', n'''' are elements of the sequence $n', \ldots n_m, \ldots$

Proof

Based on the theorem (I) for any m we find the relevant set

$$\alpha_1^{(m)} \ldots \alpha_N^{(m)},$$

where

$$N_m = 8M_m > 8Gm.$$

Assume that

$$\lambda_K^{(m)} = \frac{\alpha_K^{(m)}}{N_m}.$$

Then, it is obvious that

$$0 \leqq \lambda_K^{(m)} \leqq 1.$$

Therefore, from sequence $m \to \infty$ we can choose such a subsequence ν that $\lambda_K^{(\nu)}$ tend $\nu \to \infty$ to definite limits

$$\lambda_K^{(\nu)} \to \lambda_K, \quad \nu \to \infty; \qquad 0 \leqq \lambda_K \leqq 1 .$$

Let n be any integer satisfying the condition (9). Then, we find in the sequence ν a number ν_o such that for

$$\nu > \nu_o$$

the following inequality is valid:

$$| 2\pi n (\lambda_K - \lambda_K^{(\nu)})| < 1 - \frac{64G^2}{q+1} - 2\pi R(n\lambda_K)$$

$$K = 1, \dots q, \quad M_\nu \geqq |n|$$

Consequently:

$$2\pi R(n\lambda_K^{(\nu)}) \leqq 2\pi R(n\lambda_K) + 2\pi | n(\lambda_K - \lambda_K^{(\nu)})| <$$

$$< 1 - \frac{64G^2}{q+1} < 1 \ \frac{N_\nu^2}{\nu^2(q+1)}$$

and hence on the basis of the theorem (I) we see that n can be represented in the form (IO).

Thus, the theorem (II) is proved.

Now let us pass to application of this theorem to the theory of almost periodic functions.

It should be noted that in accordance with the definition introduced by H. Bohr, some continuous function $f(t)$ determined on the whole real axis is called almost periodic if to each $\varepsilon > 0$ one can make correspond L_ε such that in any interval on the real axis of length L_ε one can find ε almost a period, i.e. such that τ_ε

$$| f(t + \tau_\varepsilon) - f(t)| \leqq \varepsilon .$$

$$-\infty < t < \infty$$

On the basis of this definition H. Bohr proved the theorem of homogeneous approximation, namely: for any almost periodic function $f(t)$ one can make correspond to $\varepsilon > 0$ such numbers (complex numbers) $A_1, \dots A_N$ and such real $\nu_1, \dots \nu_N$ that

$$| f(t) - \sum_{K=1}^{N} A_K e^{i\nu_K t} | \leqq \varepsilon .$$

$$-\infty < t < \infty$$

It should be reminded that long before H. Bohr P. Bohl, a famous scientist, with the aim to generalise the notion of periodicity had introduced the notion of quasi-periodic functions (further generalisation of which are almost periodic functions in the sense of H. Bohr). According to Bohl's definition, the continuous function $f(t)$ given on the whole real axis is called the quasiperiodic one if there are such linearly independent real numbers

$$\omega_1, \dots, \omega_m$$

that to any $\varepsilon > 0$ one can make correspond η_ε so that any τ satisfying the inequalities

$$R(\tau\omega_K) \leqq \eta_\varepsilon, \quad K = 1, \dots, m \tag{11}$$

is ε - almost period for $f(t)$.

It follows from this definition that any quasi-periodic function can be represented in the form

$$f(t) = F(\omega_1 t, \dots, \omega_m t), \tag{12}$$

where $F(x_1, \dots, x_m)$ is the continuous periodic function with period 1.

Indeed we assume that

$$F(x_1, \dots, x_m) = \lim f(\tau)$$

for any sequence τ for which

$$\omega_j \tau \to x_j \tau \ (mod\ 1).$$

This limit does exis as any two sequences τ_1, τ_2 of τ have the property

$$\omega_K(\tau_1 - \tau_2) \to 0 \ (mod 1)$$

i.e.

$$R(\omega_K(\tau_1 - \tau_2)) \to 0 ,$$

from where by definition

$$f(\tau_1) - f(\tau_2) \to 0.$$

In the same way the continuity $F(x_1, \dots x_m)$ is proved.
Taking

$$x_\kappa = \omega_\kappa t$$

we see that

$$\omega_\kappa (\tau - t) \to 0 \ (\mathrm{mod}\ 1)$$

and

$$R(\omega_\kappa(\tau - t)) \to 0.$$

As

$$F(x_1, \dots, x_m)$$

is homogeneously approximated by trigonometric sums of

$$e^{i2\pi(n_1 x_1 + \dots + n x_m)},$$

it is clear that the quasi-periodic function $f(t)$ is homogeneously approximated by trigonometric sums

$$e^{i\lambda_\kappa t},$$

where λ_κ are linear sums of fundamental frequencies.

As is seen, the problem of the homogeneous trigonometric approximation of Boyle's quasi-periodic functions is solved simply since the definition itself of their almost periods explicitly contains the fundamental frequencies.

The situation with Bohr's almost periodic functions is different, and the proof of Bohr's fundamental theory is rather complicated. We should like to show, on the basis of Theorem II, that in the theory of almost periodic functions one can also introduce "fundamental frequencies", thus making the proof of the theorem of trigonometric approximation very simple.

For this purpose let us prove the following theorem:

Theorem III

Let $L > 0$ and $\mathfrak{G}$ be a set of points on the real axis such that in any its interval of length L there is the point $\mathfrak{G}$. Then, for any $\eta > 0$ and sufficiently small $\rho > 0$ there exist lineraly independent $\omega_1, \dots, \omega_s$ having the property:

for any τ satisfying the inequalities

$$R(\tau\omega_j) \leq \rho \qquad j = 1, \dots, s \tag{13}$$

one can find such elements $\tau_1, \tau_2, \tau_3, \tau_4$ of the set $\mathfrak{G}$ that

$$|\tau - \tau_1 - \tau_2 + \tau_3 + \tau_4| \leq \eta. \tag{14}$$

Proof

Let us consider the intervals

$$(2mL, \ 2mL + L), \quad m = 0, 1, \dots$$

and denote by $\tau^{(m)}$ any point $\mathfrak{G}$ from this interval. It is obvious that

$$\left|\frac{\tau^{(m)}}{m}\right| \leq 2L, \qquad \tau^{(m+1)} - \tau^{(m)} \geq L.$$

Now we fix the integer K and some $\delta > 0$ so that

$$K > \frac{1}{L}, \ K \frac{5}{2\eta}, \quad 0 < \delta < \frac{1}{4\pi}.$$

Choose one more integer positive q, satisfying the inequality

$$\frac{1}{2\pi}\left(1 - \frac{64G^2}{q+1}\right) > 2\delta, \qquad G = (2L+1)K.$$

Consider the sequence of integers

$$n_m = E(K\tau^{(m)})$$

One can easily verify that this sequence satisfies the conditions of Theorem (II.) Indeed

$$\left|\frac{n_m}{m}\right| \leq \left|\frac{\tau K}{m}\right| + \frac{1}{2m} < 2LK + K = G$$

$$|n_m - n_\tau| \geq |K\tau^{(m)} - K\tau^{(\tau)} - R(K\tau^{(m)}) - R(K\tau^{(\tau)}) \geq$$

$$\geq |K(\tau^{(m)} - \tau^{(\tau)}| - 1 > \frac{1}{KL} - 1 > 0$$

for $m \neq \tau$.

Based on this theorem let us consider the corresponding sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_q, \ldots \qquad 0 \leqq \lambda_j \leq 1$$

and put

$$\nu_0 = K, \ \nu_1 = K\lambda_1, \ldots \nu_q = K\lambda_q \tag{15}$$

Now let τ be some real number satisfying the inequalities

$$R(\tau\nu_K) \leqq \delta, \qquad K = 0, 1, \ldots, q$$

i.e.,

$$R(\tau K) \leqq \delta, \quad R(\tau K\lambda_1) \leqq \delta, \ldots \ R(\tau K\lambda_q) \leqq \delta.$$

Hence, it follows that

$$R\{E(\tau K)\lambda_1\} \leqq 2\delta, \ldots R\{E(\tau K)\lambda_q\} \leq 2\delta$$

and therefore

$$2\pi R\{E(\tau K)\lambda_j\} \leqq 4\pi\delta < 1 - \frac{64G^2}{q+1}$$

Thus, on the basis of Theorem (II) we can verify that there exist such values n_1, n_2, n_3, n_4 of index m for

$$n = E(\tau K) = E(\tau^{(n_1)}K) + E(\tau^{(n_2)}K) - E(\tau^{(n_3)}K) - E(\tau^{(n_4)})$$

Consequently

$$|\tau K - \tau^{(n_1)}K - \tau^{(n_2)}K + \tau^{(n_3)}K + \tau^{(n_4)}K| \leqq \frac{5}{2}$$

and

$$|\tau - \tau^{(n_1)} - \tau^{(n_2)} + \tau^{(n_3)} + \tau^{(n_4)}| \leqq \frac{5}{2K} < \eta$$

So if

$$R(\tau\nu_K) \leqq \delta, \qquad K = 0, 1, \ldots, \tag{16}$$

then

$$|\tau - \tau_1 - \tau_2 + \tau_3 + \tau_4| < \eta. \tag{17}$$

where $\tau_1, \tau_2, \tau_3, \tau_4$ are numbers of the set $\mathscr{E}$.

It should be noted now that for the set of numbers

$$\nu_0, \ \nu_1, \ldots, \nu_q$$

one can always put into correspondence the system of linearly independent numbers

$$\omega_1, \ldots, \omega_s \qquad (S \leqq q+1)$$

so that the numbers ν_j will be sets of the combination of ω_j

$$\nu_j = \sum_{K=1}^{s} N_{j,K}\,\omega_K$$

with integers $N_{j,K}$.

Therefore, we have

$$R(\nu_j\tau) \leqq \sum_{K=1}^{s} |N_{j,K}| R(\omega_K\tau)$$

Let

$$M \geqq \sum_{K=1}^{s} |N_{j,K}| \qquad j = 1, \ldots, s$$

Assume

$$\rho = \frac{\delta}{M}$$

Then, the inequalities

$$R(\omega_K\tau) \leqq \rho; \qquad K = 1, \ldots, s$$

result in (16) resulting in (17), which proves our theorem.

Now let us come back to the theory of almost periodic functions. We take some almost periodic function $f(t)$. Due to its continuity to any $\varepsilon > 0$ one can make correspond $\delta(\varepsilon)$ such that from the inequality

$$|t' - t''| \leqq \delta(\varepsilon)$$

there follows

$$|f(t') - f(t'')| \leqq \varepsilon.$$

Now let us turn to theorem III and take in it as $\mathscr{E}$ the set of $\varepsilon/8$ almost periods $f(t)$.

Put

$$\eta = \delta\left(\frac{\varepsilon}{2}\right)$$

and find a sufficiently small ρ.

Then, we can verify the presence of such linearly independent $\omega_1, \ldots, \omega_s$ that for any τ satisfying the inequalities

$$R(\omega_j\tau) \leqq \rho \tag{18}$$

one can find in $\mathscr{E}$ such $\tau_1, \tau_2, \tau_3, \tau_4$ that

$$|\tau - \tau_1 - \tau_2 + \tau_3 + \tau_4| \leq \delta(\tfrac{\varepsilon}{2})$$

As these $\tau_1, \tau_2, \tau_3, \tau_4$ are almost periods with $\varepsilon/8$, then obviously quantity

$$\tau' = \tau_1 + \tau_2 - \tau_3 - \tau_4$$

will be $\varepsilon/2$ almost period for $f(t)$. As

$$|\tau - \tau'| \leq \delta(\varepsilon/2)$$

then we see that τ will be ε almost period

$$|f(t+\tau) - f(t) \leq \varepsilon, \quad -\infty < t < \infty \tag{19}$$

So for every $\varepsilon > 0$ one can find such ρ and such linearly independent

$$\omega_1, \ldots, \omega_s$$

that inequalities (18) result in (19), i.e. that τ is almost period. Now we see that Bohr's definition can be reformulated by analogy with Bohl's definition.

The difference lies in that for Bohl's quasi-periodic functions the frequencies $\omega_1, \ldots, \omega_s$ are fixed and for almost periodic ones their number s depends on ε and can tend to infinity as $\varepsilon \to 0$. Nevertheless, even for almost periodic Bohr's functions one can easily construct for each $\varepsilon > 0$ the corresponding continuous function

$$F_\varepsilon(x_1, \ldots, x_s) \tag{20}$$

periodic with period 1 with respect to x_j $(j = 1, \ldots, S)$ so that

$$|f(t) - F_\varepsilon(\omega_1\tau, \ldots, \omega_s t)| \leq \varepsilon \tag{21}$$

Hence, there immediately follows Bohl's theory of trigonometric approximation.

Let us show how $F_\varepsilon(x_1, \ldots, x_s)$ can be constructed.

Consider the function

$$\theta_\rho(x) = \begin{cases} 1 - \dfrac{R(x)}{\rho}, & \text{if } R(x) \leq \rho \\ 0, & \text{if } R(x) \geq \rho \end{cases} \tag{22}$$

for some

$$0 \leq \theta_\rho(x) \leq \frac{1}{2}, \quad |\theta'_\rho(x)| \leq \frac{1}{\rho},$$
$$\theta_\rho(x+1) = \theta_\rho(x), \quad \int_0^1 \theta_\rho(x)\,dx = \rho \tag{23}$$

Assume

$$\Phi(x_1, \ldots, x_j) = \theta_\rho(x_1) \ldots \theta_\rho(x) \tag{24}$$

It is clear that

$$\Phi > 0$$

if and only if

$$R(x_j) < \rho, \quad j = 1, \ldots \rho$$

Otherwise

$$\Phi = 0.$$

Consequently, if

$$\Phi(\omega_1\tau, \ldots, \omega_s\tau) > 0,$$

then

$$|f(t+\tau) - f(t)| \leq \varepsilon, \quad -\infty < t < \infty.$$

Therefore

$$\left| \frac{\frac{1}{T}\int_0^T \Phi(\omega_1\tau, \ldots, \omega_s\tau) f(t+\tau) - f(t)\,d\tau}{\frac{1}{T}\int_0^T \Phi(\omega_1\tau, \ldots \omega_s\tau)\,d\tau} \right| \leq \varepsilon$$

or

$$\left| f(t) - \frac{\frac{1}{T}\int_{-t}^{T-t} \Phi\{\omega_1(\tau - t), \ldots \omega_s(\tau - t)\} f(\tau)\,d\tau}{\frac{1}{T}\int_0^T \Phi(\omega_1\tau, \ldots \omega_s\tau)\,d\tau} \right| \leq \varepsilon.$$

Hence

$$\left| f(t) - \frac{\frac{1}{T}\int_0^T \Phi\{\omega_1\tau - \omega_1 t, \dots, \omega_s\tau - \omega_s t\} f(\tau)\,d\tau}{\frac{1}{T}\int_0^T \Phi(\omega_1\tau, \dots, \omega_s\tau)} \right| \leq \varepsilon + C_\rho \left|\frac{t}{T}\right| ; \quad (25)$$

$$C_\rho = Const.$$

From (22) and (23) we can conclude that there exists a limit

$$F_\varepsilon(x_1, \dots, x_s) = \lim_{T\to\infty} \frac{\frac{1}{T}\int_0^T \Phi(\omega_1\tau - x_1, \dots \omega_s\tau - x_s) f(\tau)\,d\tau}{\frac{1}{T}\int_0^T \Phi(\omega_1\tau, \dots, \omega_s\tau)\,d\tau} \quad (26)$$

which is the continuous function $x_1, \dots, x_s$ with period 1. From (25) there immediately follows (21).

Remark

Here are some explanatory remarks to the problem of existence of the limit (26).

Let us take the Fourier expansion

$$\theta_\rho(x) = \sum_{(n)} h_\rho(n) e^{inx}; \quad (27)$$

$$h_\rho(n) = \int_0^1 \theta_\rho(x) e^{-i2\pi nx} dx = \int_0^1 (1 - \frac{x}{\rho}) e^{-i2\pi nx} dx + \int_{1-\rho}^1 (1 - \frac{(1-x)}{\rho}) e^{-i2\pi nx} dx$$

and note that owing to (23) it will be absolutely convergent

$$\sum_{(n)} |h_\rho(n)| = \sum_{(n)} \left| \frac{1}{n2\pi} \int_0^1 \theta'_\rho(x) e^{-i2\pi nx} dx \right| \leq \frac{1}{2\pi} \sqrt{\sum_{(n)} \frac{1}{n^2}} \sqrt{\int_0^1 |\theta'_\rho(x)|^2 dx} \leq \frac{1}{2\pi\rho} \sqrt{\sum_{(n)} \frac{1}{n^2}}.$$

Therefore, we have an absolutely convergent series

$$\Phi(x_1, \dots x_s) = \sum_{(m)} H_\rho(m) e^{i(mx)}, \quad (28)$$

where

$$(m) = (n_1, \dots n_s), \quad H_\rho(m) = h_\rho(n_1) \dots h_\rho(n_s), \quad (mx) = n_1 x_1 + \dots + n_s x_s. \quad (29)$$

As $\omega_1, \dots \omega_s$ are linearly independent, then

$$\frac{1}{T}\int_0^T \Phi(\omega_1\tau, \dots, \omega_s\tau)\,d\tau \underset{T\to\infty}{\to} H_\rho(0) = h_\rho^s(0)\ \rho^1. \quad (30)$$

Then, we have

$$\frac{1}{T}\int_0^T (\omega_1\tau - x_1, \dots, \omega_s\tau - x_s) f(\tau)\,d\tau =$$

$$= \sum_{(m)} H_\rho(m) C^{-i(mx)} \frac{1}{T}\int_0^T f(\tau) e^{i2\pi(m\omega)\tau} d\tau; \quad (m\omega) = n_1\omega_1 + \dots + n_s\omega_s.$$

Considering that

$$\left| \frac{1}{T}\int_0^T f(\tau) e^{i2\pi(m\omega)\tau} d\tau \right| \leq M = max|f(t)|$$

we see that from the sequence $T \to \infty$ we can choose such a subsequence $T \to \infty$ that all quantities

$$\frac{1}{T'}\int_0^{T'} f(\tau) e^{i(m\omega)\tau} d\tau$$

tend to the limits. Put

$$\lim_{T'\to\infty} \frac{1}{T'}\int_0^{T'} f(\tau) e^{i(m\omega)\tau} d\tau = \mathcal{H}(m). \quad (31)$$

Then, from (26) and (28) we get

$$F_\varepsilon(x_1, \dots, x_s) = \sum_{(m)} \frac{H_\rho(m)}{H_\rho(0)} \mathcal{H}(m) e^{-i(mx)2\pi} \quad (32)$$

and

$$\left| \frac{H_\rho(m)}{H_\rho(0)} \right| \leq 1, \quad \sum_{(m)} \left| \frac{H_\rho(m)}{H_\rho(0)} \mathcal{H}(m) \right| < \infty. \quad (33)$$

Now let us consider the problem of limits (31). We shall show that they exist not only for a specifically chosen sequence $T' \to \infty$ but for any $T \to \infty$. Note that it follows from relations (25), (26), (32), and (33) that for any $\varepsilon > 0$ there exists N_ε such that

$$|f(t)-f_{N_\varepsilon}(t)|\leqq 2\varepsilon \,. \tag{34}$$

where

$$f_{N_\varepsilon}(t)=\sum_{(|m|<N_\varepsilon)}\frac{H_\rho(m)}{H_\rho(0)}\,\mathcal{U}(m)\,e^{-i(m\omega)2\pi\tau} \tag{35}$$

Here the symbol $(|m|<N_\varepsilon)$ denotes inequalities $|m_1|<N_\varepsilon \dots |m_s|<N_\varepsilon$. As $f_{N_\varepsilon}(f)$ is a finite trigonometric sum, then for any real λ there exists a limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f_{N_\varepsilon}(t)\,e^{i\lambda t}\,dt \,.$$

We can find for ε, λ such Ω that

$$\left|\frac{1}{T'}\int_0^{T'} f_{N_\varepsilon}(\tau)e^{i\lambda\tau}d\tau-\frac{1}{T}\int_0^{T} f_{N_\varepsilon}(\tau)e^{i\lambda\tau}d\tau\right|<\varepsilon$$

for all $T'\ T$ satisfying the inequalities

$$T>\Omega\,,\quad T'>\Omega\,. \tag{36}$$

Hence, from (34) it follows that

$$\left|\frac{1}{T'}\int_0^{T'} f(\tau)e^{i\lambda\tau}d\tau-\frac{1}{T}\int_0^{T} f(\tau)e^{i\lambda\tau}\right|<5\varepsilon$$

for all T, T' from inequality (30).

As ε can be whatever small, we see that there exist limits

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\tau)e^{i\lambda\tau}d\tau$$

for any fixed λ

Thus, instead of (31) we can write down

$$\mathcal{U}(m)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\tau)e^{i(m\omega)2\pi\tau}\,d\tau \,. \tag{37}$$

Note that owing to (34) and (35) in the trigonometric approximation obtained the contribution comes only from those $2\pi(m\omega)$ which are true frequencies of the almost periodic function $f(t)$,

i.e. for which

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\tau)e^{i(m\omega)2\pi\tau}\,d\tau\neq 0\,.$$

As concerns the factors of convergence $\dfrac{H_\rho(m)}{H_\rho(0)}T$, then one can easily see that at fixed m

$$\frac{H_\rho(m)}{H_\rho(0)}\to 1 \quad \text{by} \quad \varepsilon\to 0\,.$$

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