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COMPUTER ALGEBRA GENERATING RELATED 2nd ORDER LINEAR DIFFERENTIAL EQUATION

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1. Introduction

As is known, J.Lioville has already shown that the arbitrary linear ordinary differential equation (LODE) is not integrable in quadrature generally . However, not being reduced, the value of the integrable cases even more increases due to that discovery. This results from their fundamental role in various applications of mathematics, mechanics, physics,etc.

A natural extension of the set of integrable cases is attained by considering LODE resolvable in special functions, thereby the idea of integrability is generalized itself.

The theory of differential equations and applications dispose of a significant number of equations with known solutions. Their accumulation came about irregularly, essentially at the expense of separate equations discovered by one researcher or another at times. Today , an urgent need is observed for regular procedures , admitting use of computer,to constract purposefully differential equations which are resolvable in the sense mentioned.

The following way of multiplying resolvable equations seems to be tempting : to arrive at the case concerned one should take a proper equation and apply one transformation or another of variables. But the basic difficulty of this approach consists just in finding successful substitutions. Such devices are of heuristic character and therefore ineffective.

In connection with the inverse scattering problem method and the KdV equation theory, techniques of multipling integrable equations using the classic first order differential transformation [1,2] and the Infeld - Hull factorization [3] as well have gained wide use. The efficiency of the Kummer - Liouville transformation has been proved in [4,5] which in many cases allows given equations to be integrated by reducing them to equations with constant coefficients. In the present paper the algorithmization of an original procedure for multiplying integrable equations [6] is described which is also associated with the problem of reducibility of the second order LODE. It is shown how computer algebra can be applied to construct specific sequences of LODE solved in terms of a chosen generating equation.

2. Initial correlations

In the works [5-7] an important special case of the Kummer - Liouville problem is considered, namely reduction of the second order LODE to equations with constant coefficients. Later on we shall be based on the following principal results.

For the LODE:

$$y'' + a_1(x)y' + a_0(x)y = 0 \ , \quad a_k(x) \in C^k, \quad (')=d/dx \tag{1}$$

by means of the point local transformation of variables:

$$y = v(x)z \ , \quad dt = u(x)dx \ , \quad v(x), u(x) \in C^2 \tag{2}$$

to be reduced to the LODE with constant coefficients

$$\ddot{z} + b_1\dot{z} + b_0 z = 0 \ , \quad b_1, b_0 = \text{const} \ , \quad (\dot{})=d/dt \tag{3}$$

it is necessary and sufficient that the transformation functions satisfy the relations:

$$(1/2)(u''/u) - (3/4)(u'/u)^2 - (1/4)\delta u^2 = A_0(x) \ , \tag{4}$$

$$v = |u|^{-1/2} \exp((-1/2)\int a_1 dx + (1/2)b_1 \int u \, dx) \tag{5}$$

$$A_0 = a_0 - (1/2)a_1' - (1/4)a_1^2 \ , \quad \delta = b_1^2 - 4b_0.$$

From the form of equation (3) and transformation (2) it is clear that with regard for (5) the fundamental system of solutions of LODE (1) being reducible to the form (3) can be presented as:

$$y_{1,2} = v \exp(r_{1,2} \int u \, dx) = |u|^{-1/2} \exp((-1/2)\int a_1 dx \pm (1/2)(\delta)^{1/2} \int u \, dx), \tag{6}$$

where $r_{1,2}$ are the characteristic roots of equation (3). Equation (4) is closely related to the resolvent equation for LODE (1) reduced to the canonical form $Y'' + A_0(x) Y = 0$:

$$R''' + 4 A_0(x)R' + 2 A_0'(x)R = 0. \tag{7}$$

From this it follows that for the general solution of equation (4) a nonlinear superposition principle is valid with respect to the linearly independent solutions of LODE (1):

$$u = (Ay_2^2 + By_1y_2 + Cy_1^2), \quad B^2 - 4AC = \delta. \tag{8}$$

Correlations (4) and (5) can be reformulated excluding the function u(x). Then we come to the integro-differential equation in v(x):

$$v'' + a_1(x)v' + a_0(x)v^{-3}\exp((-2)\int a_1 dx)[k + b_1 \int v^{-2}\exp(-\int a_1 dx)dx]^{-2}dx, \tag{9}$$

where k=1 if $b_1 = 0$, and k=0 if $b_1 \neq 0$.

On the other hand, supposing u(x) is known, we obtain the LODE for v(x):

$$v'' + a_1(x)v' + [a_0(x) - b_0 u^2(x)] \, v = 0. \tag{10}$$

3. The Kummer-Liouville procedure

In the previous paper of the authors [4] on the basis of relation (4) for u(x) practical aspects were considered of reducing LODE (1) to the preassigned form (3) - the equation with constant coefficients whose general solution is known. Now, applying equation (9) to v(x) we shall be engaged in somewhat the inverse problem : construction of a sequence of LODE, the general solution for each of those is expressed in elementary functions in terms of the fundamental system of solutions of a beforehand chosen generating equation. The latter term is caused by the fact that the basic differential field of the mentioned sequence of LODE is generated by the Picard - Vessiat extension of the field of the initial equation [8].

For more clearness and simplification of calculation (but without loss of generality) we shall consider LODE in the semi-canonical form:

$$y'' + a_{(1)}(x)y = 0 \tag{$a_{(1)}$}$$

(the sense of the notation being introduced will become clear from subsequent).

Let the fundamental system of solutions for equation $(a_{(1)})$ be known. Then it is reducible by a transformation of type (2):

$$y = v_{(1)}(x)\, z\ ,\quad dt = u_{(1)}(x)\, dx, \qquad (11)$$

where the function $u_{(1)}(x)$ is determined by (8):

$$u_{(1)}(x) = (A_{(1)}y^2_{(1)2} + B_{(1)}y_{(1)1}y_{(1)2} + C_{(1)}y^2_{(1)1})^{(-1)},$$
$$B^2_{(1)} - 4A_{(1)}C_{(1)} = \delta_{(1)}, \qquad (12)$$

and $v_{(1)}(x)$ satisfies a LODE of type (10):

$$v'' + [a_{(1)}(x) - b_{0(1)}u^2_{(1)}(x)]\, v = 0. \qquad (13)$$

Let us redenote the dependent variable in (13):

$$y'' + a_{(2)}(x)y = 0\ ,$$
$$a_{(2)}(x) = a_{(1)}(x) - b_{0(1)}u^2_{(1)}(x). \qquad (a_{(2)})$$

In view of (6) one can write the fundamental system of solutions for (13) and, hence, for the LODE $(a_{(2)})$:

$$y_{(2)1,2}(x) = v_{(1)1,2}(x) =$$
$$= |u_{(1)}(x)|^{-1/2}\exp(\pm(1/2)b_{1(1)}\int u_{(1)}(x)\,dx),\quad b_{1(1)} \neq 0;$$
$$y_{(2)1}(x) = v_{(1)1}(x) = |u_{(1)}(x)|^{-1/2}, \qquad (14)$$
$$y_{(2)2}(x) = v_{(1)2}(x) = |u_{(1)}(x)|^{-1/2}\int u_{(1)}(x)\,dx),\quad b_{1(1)} = 0.$$

Considering the problem of reducibility of the LODE $(a_{(2)})$ and the accompanying equation in $v_{(2)}(x)$ we come to the LODE:

$$y'' + a_{(3)}(x)y = 0,$$
$$a_{(3)}(x) = a_{(1)}(x) - b_{0(1)}u^2_{(1)}(x) - b_{0(2)}u^2_{(2)}(x), \qquad (a_{(3)})$$

and so on.

Having executed n-1 of these steps, we obtain a LODE of the "n-th generation":

$$y'' + a_{(n)}(x)y = 0,$$

$$a_{(n)}(x) = a_{(1)}(x) - \sum_{k=1}^{n-1} b_{0(k)}u^2_{(k)}(x). \qquad (a_{(n)})$$

The outlined procedure which is immediately related to the problem of reducibility of LODE to form (3) by transformation (2) will be called the Kummer-Liouville (K-L) procedure, and the initial LODE of the first generation $(a_{(1)})$ generating.

In the infinite sequence of LODE:

$$(a_{(1)}) \text{------>} (a_{(2)}) \text{------>.......------>} (a_{(n)}) \qquad (a)$$

motion not only straight but also backwards is admitted, so that the fundamental system of solutions of each its member can be expressed in terms of those of others and, in the end, in terms of the solutions of the generating equation. In this connection, one should keep in mind that according to expressions (12)-(14), actually, the sequence of 4-parameter families of LODE arises as far as we are free in choosing numeric values of b_1, b_0 (or δ) and any two constants from A, B and C:

$$y'' + [a_{(n)}(x) - (1/4)(b^2_{1(n)} - \delta_{(n)})(A_{(n)}y^2_{(n)2} + B_{(n)}y_{(n)1}y_{(n)2} +$$
$$+ C_{(n)}y^2_{(n)1})^{(-2)}]y = 0,\quad (1/4)(b^2_{(n)1} - \delta_{(n)}) = b_{0(n)}. \qquad (a_{n+1})$$

Besides, the coefficients of LODE from the next generation contain solutions of the corresponding equation from the previous one, the form of which principally depends on combination of the signs of the parameters b_1 and δ setting results of integration in (6) and (14). In all, there are 9 possible variants: 3 cases for b_1 ($b_1^2 > 0$; $b_1 = iw$, $b_1^2 = -w^2 < 0$; $b_1 = 0$) with 3 cases for δ ($\delta>0$; $\delta<0$; $\delta=0$) but one of them ($b_1=0$, $\delta=0$) does not lead to a new equation since then $a_{(n+1)}(x) \equiv a_{(n)}(x)$.

The relations of the corresponding functions u(x) and fundamental systems of LODE of neighboring generations $(a_{(n)})$ and $(a_{(n+1)})$ are shown in the table below. For greater obviousness the subscript (n) has been omitted in notation of the solutions $y_{(n)1,2}(x)$ and parameters b_1, δ, α, β, A, B, and C. The important special cases of the variants $\delta>0$ and $\delta=0$ are also given.

Table

Transformation function $u_{(n)}(x)$	Fundamental system of solutions $y_{(n+1)1,2}$		
	$b_1=\mathrm{Re}\,b_1\neq 0$ 1)	$b_1=\mathrm{Im}\,b_1=iw$ 2)	$b_1=0$ 3)
1. $\delta=(\alpha_1\beta_2-\alpha_2\beta_1^2)>0$ $u_{(n)}(x)=(1)$	(1,1)	(1,2)	(1,3)
2. $\delta=B^2-4AC<0$ $u_{(n)}(x)=(2)$	(2,1)	(2,2)	(2,3)
3. $\delta=0$ $u_{(n)}(x)=(3)$	(3,1)	(3,2)	(3,3)
4. $\delta=\alpha^2>0$ $u_{(n)}(x)=(4)$	(4,1)	(4,2)	(4,3)
5. $\delta=0$ $u_{(n)}(x)=(5)$	(5,1)	(5,2)	(5,3)

Notation:

$$E(x) = ((\alpha_1 y_2 + \beta_1 y_1)(\alpha_2 y_2 + \beta_2 y_1))^{1/2}$$

$$P(x) = |(\alpha_1 y_2 + \beta_1 y_1)/(\alpha_2 y_2 + \beta_2 y_1)|$$

$$F(x) = (Ay_2^2 + By_1y_2 + Cy_1^2)^{1/2}$$

$$Q(x) = \mathrm{atan}((2Ay_2 + By_1)/((-\delta)^{1/2}y_1))$$

$$R(x) = (\alpha y_2 + \beta y_1)/y_1$$

$$G(x) = (y_1(\alpha y_2 + \beta y_1))^{1/2}$$

$$S(x) = \log|R(x)|$$

1. $\delta = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 > 0$,

$$u_{(n)}(x) = (\alpha_1 y_2 + \beta_1 y_1)^{-1}(\alpha_2 y_2 + \beta_2 y_1)^{-1}; \qquad (1)$$

1) $b_1 = \mathrm{Re}\, b_1 \neq 0$,

$$y_{(n+1)1,2}(x) = E(x)[P(x)]^{(\pm b_1/(2\delta^{1/2}))}; \qquad (1,1)$$

2) $b_1 = \mathrm{Im}\, b_1 = iw$,

$$y_{(n+1)1,2}(x) = E(x)\,{\cos \atop \sin}\{(w/(2\delta))\log P(x)\}; \qquad (1,2)$$

3) $b_1 = 0$,

$$y_{(n+1)1,2}(x) = \left\{ \begin{matrix} E(x) \\ E(x)\log P(x) \end{matrix} \right\}; \qquad (1,3)$$

2. $\delta = B^2 - 4AC < 0$,

$$u_{(n)}(x) = (Ay_2^2 + By_1y_2 + Cy_1^2)^{-1}; \qquad (2)$$

1) $b_1 = \mathrm{Re}\, b_1 \neq 0$,

$$y_{(n+1)1,2}(x) = F(x)\,{\cosh \atop \sinh}\{b_1/(-\delta)^{1/2}Q(x)\}; \qquad (2,1)$$

2) $b_1 = \mathrm{Im}\, b_1 = iw$,

$$y_{(n+1)1,2}(x) = F(x)\,{\cos \atop \sin}\{(w/(-\delta)^{1/2})Q(x)\}; \qquad (2,2)$$

3) $b_1 = 0$,

$$y_{(n+1)1,2}(x) = \left\{ \begin{matrix} F(x) \\ F(x)Q(x) \end{matrix} \right\}; \qquad (2,3)$$

3. $\delta = 0$,

$$u_{(n)}(x) = (\alpha\, y_2 + \beta\, y_1)^{-2}; \qquad (3)$$

1) $b_1 = \mathrm{Re}\, b_1 \neq 0$,

$$y_{(n+1)1,2}(x) = (\alpha y_2 + \beta y_1)\,{\cosh \atop \sinh}\{b_1/(2\alpha R(x))\}; \qquad (3,1)$$

2) $b_1 = \mathrm{Im}\, b_1 = iw$,

$$y_{(n+1)1,2}(x) = (\alpha y_2 + \beta y_1)\,{\cos \atop \sin}\{(w/(2\alpha R(x))\}; \qquad (3,2)$$

3) $b_1 = 0$,

$a_{(n+1)} = a_{(n)},\ y_{(n+1)} = y_{(n)}$;

4. $\delta = \alpha^2 > 0$,

$$u_{(n)}(x) = y_1^{-1}(\alpha y_2 + \beta y_1)^{-1}; \tag{4}$$

1) $b_1 = \text{Re } b_1 \neq 0$,

$$y_{(n+1)1,2}(x) = G(x)\,|R(x)|^{(\pm b_1/(2\alpha))}; \tag{4,1}$$

2) $b_1 = \text{Im } b_1 = iw$,

$$y_{(n+1)1,2}(x) = G(x) \begin{matrix}\cos\\ \sin\end{matrix} \{(w/(2\alpha))S(x)\}; \tag{4,2}$$

3) $b_1 = 0$,

$$y_{(n+1)1,2}(x) = \left\{ \begin{matrix} G(x) \\ G(x)S(x) \end{matrix} \right\}; \tag{4,3}$$

5. $\delta = 0$,

$$u_{(n)}(x) = y_1^{-2}; \tag{5}$$

1) $b_1 = \text{Re } b_1 \neq 0$,

$$y_{(n+1)1,2}(x) = y_1 \begin{matrix}\cosh\\ \sinh\end{matrix} \{(b_1 y_2)/(2\, y_1)\}; \tag{5,1}$$

2) $b_1 = \text{Im } b_1 = iw$,

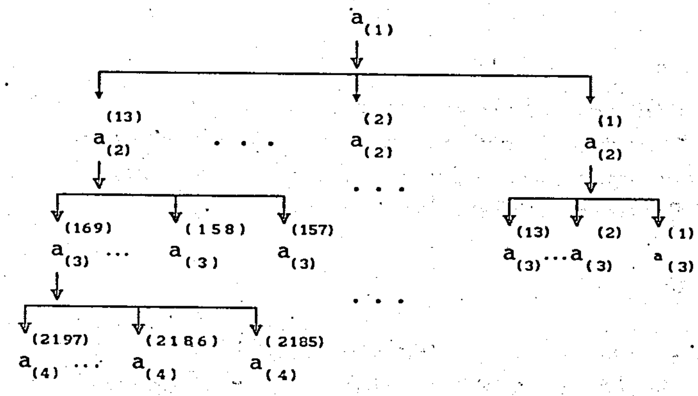
$$y_{(n+1)1,2}(x) = y_1 \begin{matrix}\cos\\ \sin\end{matrix} \{(w\, y_2)/(2\, y_1)\}; \tag{5,2}$$

3) $b_1 = 0$,

$$a_{(n+1)} = a_{(n)},\ y_{(n+1)} = y_{(n)} \tag{5,3}$$

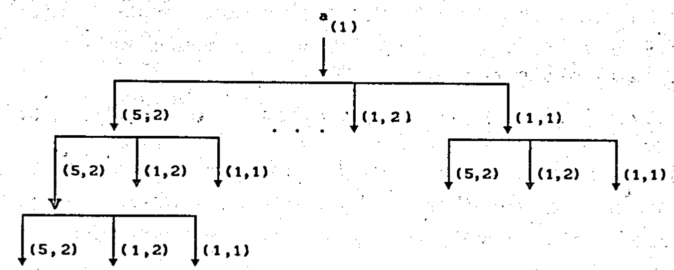
Thus, the K-L procedure generates 8 equations with different coefficients (or 13 ones including the mentioned special cases) from each chosen LODE , i.e. the number of LODE is growing from generation

to generation according to geometric progression with the ratio 8 (or 13) which is illustrated by the following scheme - "tree":



(D)

Here the superscripts are used to distinguish the equations within the same generation: each coefficient has its own "history" of obtaining , i.e., the path covered from the root to the given node of the tree (graph). In this standpoint the tree looks as follows (the first digit in the parentheses indicates a row number, and the second one a column number in the table; the the cases (3,3) and (5,3) are omitted):



Of some interest is also the direct connection between group properties of equations of the sequence (a). So, considering the problem of reducibility of the LODE $(a_{(n)})$ and $(a_{(n+1)})$ and corresponding relations (4) and (5) one can show that the LODE

$(a_{(n+1)})$ is reducible to the form $((-1/4)b^2_{1(n)})$ (i.e., to the equation with constant coefficients $\ddot{z} - (1/4)b^2_{1(n)}\, z = 0$) by the transformation :

$$y = |u_{(n)}|^{-1/2} z, \quad dt = u_{(n)}(x)\, dx.$$

In conclusion note that along with sequences of integrable LODE , the K-L procedure permits constructing analogous sequences of integrable associated equations: third order linear (resolvent) (7) , nonlinear (4) and integro-differential (9).

4.The basic sequence

As an example of application of the K-L procedure we consider the sequence (we call it basic) being generated by the equation $y''=0$ whose fundamental role is known [9]. The second generation of this sequence are represented by LODE of the form:

$$y'' + (\,k/(Ax^2 + Bx + C)^2\,)y = 0\ , \quad k=\text{const},$$

studied in detail in [6] where the complete collection of their solutions is offered.

Below two instances are given of LODE treated by another technique in [10] in connection with the Schrödinger radial equation for a particle in a central field . The LODE are found to be of the third generation of the basic sequence.

<u>Example</u> 1. $y''+ [1/(4x^2) + 1/(x^2(\alpha \log x + \beta)^4)]y = 0\ , \ \alpha,\beta=\text{const}.$

The equation is obtained by means of the K-L procedure along the path on the tree (D) with the parameters (the digits under the arrows indicate , as before , the corresponding numbers of rows and columns of the above table):

$$(0)\ \xrightarrow[(4,3)]{\delta_{(1)}>0;\, b_{1(1)}=0} \left(\frac{1}{4x^2}\right) \xrightarrow[(3,2)]{\delta_{(2)}=0;\, b_{1(2)}=iw_{(2)}} \left(\frac{1}{4x^2} + \frac{1}{x^2(\alpha \log x+\beta)^4}\right)$$

and has the fundamental system of solutions

$$y_{(3)1,2} = x^{1/2}(\alpha\cdot\log x + \beta)\ {\cos \atop \sin}\ (1/(\alpha(\alpha \log x + \beta))).$$

<u>Example</u> 2. $y'' - [m(m+1)/(x^2) + R^{-4}(x)]y = 0\ ,$

$$R(x) = \alpha\, x^{(m+1)} - (\beta/(2m+1))x^{-m}\ , \quad m \neq 1/2\ ,$$

(for m=-1/2 see example 1).

For this equation there are a corresponding path:

$$(0)\ \xrightarrow[(4,1)]{\delta_{(1)}>0;\, b^2_{1(1)}>0} \left(-\frac{m(m+1)}{(x^2)}\right) \xrightarrow[(3,1)]{\delta_{(2)}=0;\, b^2_{1(2)}>0} \left(-\frac{m(m+1)}{(x^2)} - R^{-4}(x)\right)$$

and a fundamental system of solutions

$$y_{(3)1,2} = R(x)\ {\cosh \atop \sinh}\ \left(\frac{(2m+1)x^{(m+1)}}{\beta R(x)}\right).$$

If the term with $R^{-4}(x)$ in the equation is opposite in sign then the hyperbolic functions should be replaced with the corresponding trigonometric ones, which conforms to the variant $\delta_{(2)}=0$, $b_{1(2)}=iw_{(2)}$ (i.e., (3,2)) on the second step of the K-L procedure.

<u>Example</u> 3. The equations with constant coefficient:

$$y''\pm \lambda^2 y = 0$$

belong to the second generation of the basic sequence and are obtained along the paths

$$\delta_{(1)} = 0,\ b_1 = 2\lambda i \quad \text{and} \quad \delta_{(1)} = 0,\ b_1 = 2\lambda$$

(i.e., (5,2) and (5,1) respectively). Among LODE generated in turn by these equations, the following ones are of interest for example.

The equation :

$$y'' + [\lambda^2 - b_{0(2)}(p \sin^2(\lambda x) + q \sin(\lambda x)\cos(\lambda x) + r \cos^2(\lambda x))^{-2}]y = 0,$$

$$p,q,r = \text{const}, \quad \lambda^2(q^2 - pr) = \delta_{(2)} < 0,$$

has the fundamental system of solutions:

$$y_{(3)1,2} = \cos(\lambda x)(\,Q(x))^{1/2}\ {\cosh \atop \sinh}\ \left(\frac{b_{1(2)}}{(-\delta_{(2)})^{1/2}}\, S(x)\right)\ ,$$

$$Q(x) = p \tan^2(\lambda x) + q \tan(\lambda x) + r\ ,$$

$$S(x) = \operatorname{atan}\left(\frac{\lambda}{(-\delta_{(2)})^{1/2}}\right)(2p\cdot\tan(\lambda x) + q),$$

if the numeric coefficients are such that:

$$b^2_{1(2)} = \delta_{(2)} - 4b_{0(2)} > 0.$$

For $b_{1(2)} = 0$ we have:

$$y_{(3)1}(x) = \cos(\lambda x)(Q(x))^{1/2}, \quad y_{(3)2}(x) = R(x)\, y_{(3)1}(x).$$

Analogous formulae are valid for the LODE:

$$y'' + [\lambda^2 + b_{0(2)}(p \sinh^2(\lambda x) + q \sinh(\lambda x)\cosh(\lambda x) + r \cosh^2(\lambda x))^{-2}]y = 0,$$

as well, but with replacing the trigonometric functions by the corresponding hyperbolic ones in the expressions for Q(x) and S(x).

Example 4 (see [11]). $y'' + [1/(4x^2) \sum_{k=0}^{n} \prod_{s=0}^{k} (\log_s^{-2} x)]y = 0$,

where $\log_0 x = 1$, $\log_s x = \underbrace{\log \log \ldots \log}_{s \text{ times}} x$.

The equation belongs to the (n+2)-th generation of the basic sequence and is obtained for $\delta_{(k)}>0$ 6 $b_{1(k)}=0$, $k=2,\ldots,(n+1)$, i.e., the path (4,3)---->(4,3)----->(4,3) corresponds to it.

The LODE is reduced to $\ddot{z} = 0$ by the transformation:

$$y = (p(x))^{1/2} z, \quad dt = (1/p(x))\, dx, \quad p(x) = x \prod_{s=0}^{n} (\log_s x)$$

and has the fundamental system of solutions:

$$y_{(n+2)1}(x) = (p(x))^{1/2}, \quad y_{(n+2)2}(x) = y_{(n+2)1}(x) \prod_{s=0}^{n+1} (\log_{s+1} x).$$

5. Algorithm description

The application of the recursion formulas relating (by means of elementary functions) the coefficients and solutions of the neighboring generations (see the table) allows one to make the calculation process fully controlled and selective . On the basis of the K-L procedure the authors have developed an algorithm for generating the sequence (a) with an arbitrary given number of generations starting from the given coefficient $a_1(x)$ and linearly independent solutions $y_1(x)$, $y_2(x)$ of the generating equation.

Using another algorithm one can obtain a LODE with an integrable

potential (a coefficient $a_{(n)}(x)$) of any structure (from those possible within the procedure) having chosen a certain path from the root to the desirable node of the tree (D) . To this, a needed sequence of number pairs is set where each one corresponds to the chosen variant of transferring to the next generation according to the row and column numbers in the table respectively .The description of the algorithm is outlined below.

<u>Algorithm</u> for generating a sequence of 2-nd order integrable LODEs of the preassigned structures

<u>Given</u> a 2-nd order LODE $y'' + a(x)y = 0$ with known fundamental system of solutions $y_1(x)$, $y_2(x)$ and two arrays $\{r_i\}, \{c_i\}$, $i = 1 \div n$, where n is the number of the desirable generation (not counting the initial one). The sequence of the pairs (r_i, c_i), $i=1 \div n$, determines the path along the tree (D) leading to the needed LODE $(a_{(n)})$:
$y'' + a_{(n)}(x)y = 0$.

<u>G1</u>. $a_{(1)} := a(x)$; $y_{(1)1} := y_1(x)$; $y_{(1)2} := y_2(x)$;
$i := 1$;

<u>G2</u>. Determine $u_{(i)}(x)$ according to the expression in the r_i-th row of the table with $y_{(i)1}(x)$ and $y_{(i)2}(x)$.

<u>G3</u>. $a_{(i+1)} = a_{(i)} - (1/4)(b^2_{1(i)} - \delta_{(i)})\, u^2_{(i)}$.

<u>G4</u>. Determine $y_{(i+1)1}(x)$ and $y_{(i+1)2}(x)$ by the formulas at the intersection of the r_i2th row and c_i-th column of the table with $y_{(i)1}(x)$ and $y_{(i)2}(x)$.

<u>G5</u>. $i := i+1$;

<u>If</u> $i \le n$ <u>then</u> go to <u>G2</u>

<u>else</u> return $a_{(n+1)}(x)$, $y_{(n+1)1}(x)$ and $y_{(n+1)2}(x)$.

The algorithm mentioned has been implemented in the REDUCE computer algebra system on IBM PC compatible computers.

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