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A NEW PROOF OF SINAI'S FORMULA OF ENTROPY OF HYPERBOLIC BILLIARDS. ITS APPLICATION TO LORENTZ GAS AND STADIUM

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§ 1. Introduction

This work is devoted to the dynamical systems with elastic reflections called billiards. In short, a billiard is a closed region ("the vessel") in which a point particle ("the billiard ball") moves freely with elastic reflections at the boundary ("the walls"). Billiards serve as convenient models in various fields of classical physics - see, for example, the review [1]. They are also used in studying the problems of quantum chaos.

If the walls of the vessel are concave, then the billiard is called dispersing. If they are not strictly concave admitting flat directories, then the billiard is semidispersing. Many celebrated models in statistical physics can be reduced to dispersing and semidispersing billiards.Among them are gases of hard spheres, the Lorentz gas and the Rayleigh gas. These billiards possess strong stochastic properties and are very similar to geodesic flows on surfaces of negative curvature. Namely, their main property is the exponential instability of trajectories.In the number of cases they are proved to be ergodic, mixing, K-systems [2,3,4] and B-systems [5]. Close to them are Bunimovich billiards [6,7]. By analogy to the theory of geodesic flows we shall call all the above billiards hyperbolic.

An important parameter of a dynamical system showing the rate of divergence of its trajectories is the measure-theoretic entropy introduced by A.N.Kolmogorov in 1958. The methods of entropy evaluation were being intensively developed during sixties and seventies. In 1970 Ya.G.Sinai [2] obtained a formula of the entropy of two-dimensional dispersing billiards. In 1978 he also extended [8] it to multidimensional semidispersing billiards.In the latter case the formula looks as following:

$$h = \int_{\mathfrak{M}} \operatorname{tr} B(x)\, d\mu(x). \qquad (1)$$

Here $\mathfrak{M}$ denotes the phase space, μ is the Liouville measure and $B(x)$ is the curvature operator of the unstable manifold at the point x (see § 2 for rigorous definitions).

Unfortunately,a complete proof of (1) has not been published. Only the outline of the proof was given by Ya.G.Sinai in preprint [8]. However, due to the singularities of billiard systems the direct proof of (1) in full details seems to be troublesome.

Now we are able to deduce the Sinai formula (1) from the general Katok-Strelcyn theory of hyperbolic maps with singularities [9]. Besides, we formulate and prove an analogue of the formula (1) for Bunimovich billiards.

All the obtained formulae are collected in four theorems.

Theorem 1.Let a semidispersing billiard satisfy the condition G from § 2. Then its entropy (i.e. the entropy of one-time map of the billiard flow) is expressed via the formula (1).

Theorem 2. Let T be the billiard ball map (see § 2 for definition) of the system from Theorem 1. Then its entropy is

$$h(T) = \int_M \ln \det \left(I + \tau(x)\, B(x) \right) d\nu(x). \qquad (2)$$

Theorem 3. Let T be the billiard ball map of two-dimensional Bunimovich billiard satisfying the condition G.Then its entropy is

$$h(T) = \int_M \ln \left| 1 + \tau(x)\, B(x) \right| d\nu(x). \qquad (3)$$

Theorem 4.The entropy of all the systems described in Theorems 1-3 is finite. For billiards in polygons and in polyhedra it equals zero. Otherwise it is strictly positive.

The condition G and all the notations are introduced in § 2. We do not supply the proofs of Theorems 1-4 here. They will be published elsewhere.

We apply our results to several popular models:the Lorentz gas (§ 3), the Bunimovich stadium and its modifications (§ 4). We find the asymptotics of the entropy of these systems under certain "reshaping" of the boundary when the entropy goes to zero.The interest to such problems comes from the attempts to study the laws of transition from the stable state to chaos. Here we prove or correct several conjectures coming from numerical experiments and intuitive arguments [10,11,12,23].

Acknowledgements. I am grateful to Professor Ya.G.Sinai for numerous discussions. My work was started during my staying at the Mathematical Institute of Hungarian Academy of Science. Now I have a pleasure to thank Professors A.Kramli and D.Szasz for their interest to the problem and for their kind hospitality.

§ 2. BILLIARDS: GENERALITIES

A complete and rather detailed introduction to the theory of billiards can be found in [13]. Here we give a summary of the basic results.

Let Q be a bounded open connected region in the euclidean space $\mathbb{R}^d$ or in the d-torus $\mathbb{T}or^d$. We suppose that the boundary ∂Q consists of a finite number of smooth components $\partial Q = \Gamma_1 \cup \dots \cup \Gamma_r$. Every component Γ_i is defined by an equation $\phi_i(x) = 0$ for a smooth (at least of class C^5) function $\phi_i(x)$ with no singular points on Γ_i. Assume also that the set $\bigcup_{i \neq j} \Gamma_i \cap \Gamma_j = \Gamma^*$ is a finite union of submanifolds of dimension $\le d-2$. Denote $n(q)$ for $q \in \partial Q \setminus \Gamma^*$ the inward unit normal vector to ∂Q.

A billiards is a system generated by the uniform motion of a point particle at unit speed inside Q with elastic reflections at the boundary ∂Q.

A billiards is called dispersing (semidispersing) if at any point $q \in \partial Q \setminus \Gamma^*$ the curvature operator $K(q)$ of the surface

∂Q with respect to the inward normal $n(q)$ is positive definite (positive semidefinite), i.e. $K(q) > 0$ (resp., $K(q) \geq 0$).

Dispersing billiards are also called Sinai billiards.

Billiard in a plane region Q $(d=2)$ is called Bunimovich billiard if its boundary ∂Q consists of components of three types:

1) strictly concave components (as in dispersing billiards);
2) straight segments;
3) convex circular arcs such that their complements to the full circles do not intersect other components of ∂Q.

The components of the first type are called dispersing, those of the second type are called neutral and those of the third type are called focussing.

The phase space $\mathfrak{M}$ of the system is the unit tangent bundle over Q, which can be represented as $\bar{Q} \times S^{d-1}$ with natural identification at the boundary (here S^{d-1} is the unit sphere corresponding to the velocity vectors). We represent the points $x \in \mathfrak{M}$ as the pairs (q,v), where $q \in \bar{Q}$ and $v \in S^{d-1}$. Denote π the natural projection of $\mathfrak{M}$ onto $\bar{Q}$.

Billiard system is a flow $\{S^t\}$ in $\mathfrak{M}$. It preserves the Liouville measure $d\mu = c_\mu\, dq\, dv$, where dq, dv are the Lebesgue measures in Q and S^{d-1} respectively, and c_μ is the normalizing factor: $c_\mu = (|Q| \cdot |S^{d-1}|)^{-1}$ (here and on $|\cdot|$ denotes the Lebesgue volume).

Billiards are usually studied with the help of the induced map called the billiard ball map. It is the map T induced by the flow $\{S^t\}$ in the set M called border:

$$M = \{ x = (q,v): q \in \partial Q, (v,n(q)) > 0 \}.$$

For $x \in \mathfrak{M}$ denote $\tau(x)$ the moment of the nearest reflection at ∂Q (in the future). Then one can define T via the formula $T(x) = S^{\tau(x)+0}(x)$ for any $x \in M$. The function $\tau(x)$, $x \in M$ is called the time of first return and T is called first return map.

It is well known that T preserves the measure $d\nu = c_\nu \times$

$(v,n(r))\, dr\, dv$ over M, where dr is the Lebesgue volume in the surface ∂Q and c_ν is the normalizing factor: $c_\nu = (|\partial Q| \; |B^{d-1}|)^{-1}$. Here $|B^{d-1}|$ is the volume of the (d-1)-dimensional unit ball, which is equal to the integral of the function $(v,n(r))$ over the unit semisphere $\{ v: (v,n(r)) > 0 \}$.

In the ergodic theory $\{S^t\}$ is called a special flow (or Ambrose-Kakutani flow) constructed via the automorphism T and the function $\tau(x)$. According to the Abramov formula [14] the entropy of the flow $\{S^t\}$ and that of the map T are related to each other:

$$h(T) = h(\{S^t\}) \cdot \int_M \tau(x)\, d\,\nu(x). \qquad (4)$$

It is also well known that the integral of the function $\tau(x)$ is easily evaluated through c_μ, c_ν :

$$\int_M \tau(x)\, d\,\nu(x) = c_\nu \,/\, c_\mu . \qquad (5)$$

Condition G. There exists an integer $m_0 \geq 1$ such that every point $q \in \partial Q$ unless it belongs to a focussing component of ∂Q, has a neighborhood $U(q)$ such that any billiard trajectory can suffer no more than m_0 successive reflections staying within $U(q)$.

Ya.G.Sinai [15] proved the condition G under rather general assumptions about the boundary ∂Q. For the systems of hard spheres similar conditions were proved in [16,17]. If a system of hard spheres in a vessel admits positions in which one or more of them are "clutched" as in Fig.1 then G is not valid.

The condition G seems to play not a very important role in our proofs of Theorems 1-4. Nonetheless, the problems arising when G hurts are not yet solved.

Now the operator $B(x)$ only remains undefined of all the notations of § 1. It is really a key point of the theory of hyperbolic billiards. Take a point $x=(q,v) \in \mathfrak{M}$ and denote $J(x)$ the hyperplane in $\mathbb{R}^d$ containing the point q and orthogonal to the vector v. Now consider the trajectory $x_t = (q_t, v_t) = S^t x$ of our point x. Along it all the spaces $J(x_t)$ can be identified in a natural way. Indeed, if the time interval (t_1, t_2) contains no moments of reflections, then $J(x_{t_1})$ and $J(x_{t_2})$ are parallel so they can be identified via the shift along the velocity vector $v(t_1)$. If t is a moment of reflection, then $J(x_{t-0})$ and $J(x_{t+0})$ can be identified via the projection along the normal vector $n(q_t)$ (clearly it is an isometry). Applying these rules alternatively we identify all the spaces $J(x_t)$. In the subsequent formulae this idea works implicitly.

Let t be a moment of reflection of the trajectory of our point x. We associate to this reflection the linear operator

$$D = 2\,(v_{t+0}, n(q_t))\; V\, K(q_t)\, V^* \tag{6}$$

acting in the space $J(x_{t+0})$ (hence in all the spaces $J(x_t)$ due to our identification). Here V^* denotes the projection of $J(x_{t+0})$ onto the tangent space $\mathcal{T}_q M$ parallel to the vector v_{t+0} and V^* denotes the projection of $\mathcal{T}_q M$ onto $J(x_{t+0})$ parallel to the normal $n(q_t)$.

Now let $0 \geqslant t_1 > t_2 > \dots$ be the moments of successive reflections of the negative semitrajectory x_t, $t \le 0$. Denote $\tau_i = |t_{i+1} - t_i|$ for $i \ge 1$ and D_i the operator associated with the reflection at the moment t_i. We write down the formula of $B(x)$ as an operator-valued continued fraction:

$$B(x) = \frac{I|}{||t_1|} + \frac{I|}{|D_1} + \frac{I|}{|\tau_1 I} + \frac{I|}{|D_2} + \frac{I|}{|\tau_2 I} \dots \tag{7}$$

(here I means the identity operator in $J(x)$).

In [8,18] the convergence of the fraction (7) was proved provided $|t_i| \to \infty$ as $i \to \infty$. The resulting operator acts in $J(x)$ and is self-adjoint. For semidispersing billiards $D_i \ge 0$ for all i, hence $B(x) \ge 0$. For Bunimovich billiards the space $J(x)$ is one-dimensional, so the operators D_i and $B(x)$ are ordinary real numbers. The reflections at focussing components have negative associated operators D_i , hence the value B(x) can also be negative somewhere in $\mathfrak{M}$. Moreover, it has singularities ($B(x)=\infty$) at some inner points of $\mathfrak{M}$ called conjugate points [1].

Corollary. The Sinai formula (1) is invalid in the case of Bunimovich billiards because of nonintegrable singularities of B(x) in vicinities of the conjugate points.

As for the border M, the function B(x) is positive in dispersing and neutral components and negative in focussing ones. From this fact and the definition of Bunimovich billiards more accurate estimate easily follows:

$$2\,\frac{K(q)}{(v,n(q))} \le B(x) \le \frac{K(q)}{(v,n(q))} \tag{8}$$

for any point $x=(q,v) \in M$ in a focussing component (recall that $K(q) < 0$ here).

Remark. The class of Bunimovich billiards seems to be very narrow. Indeed, much wider classes of plane billiards with focussing components of the boundary are actually hyperbolic. Such classes were constructed in recent papers due to M.Wojtkowski [19] and R.Markarian [20]. Nonetheless we are not able to evaluate their entropy by the formula (3) because the convergence of the continued fraction B(x) is not yet proved there.

All the necessary facts about B(x) are cited above. But for the sake of completeness we briefly discuss its other properties. The operator B(x) allows one to write down an equation of the unstable manifold of the flow $\{S^t\}$ (the exact expression can be found in [21]). The total multiplicity of all nonzero eigenvalues of B(x) is equal to the dimension of the unstable manifold at the point x .

§ 3. THE ENTROPY OF PERIODIC LORENTZ GAS

Here we call the periodic Lorentz gas the billiards in d-torus

($d \geq 2$) with one or more disjoint spherical regions called scatterers, removed. If a universal cover (in $\mathbb{R}^d$) is considered instead of the torus, we obtain a particle moving in the space and reflecting off a periodic configuration of obstacles. Note that this system was introduced by H.Lorentz in 1905 to describe the dynamics of the electron gas in metals. At present it is known to be ergodic and K-system [22,3]. Computer-assisted estimations of its entropy and of the rate of correlation decay were done in [10,11].

Let the centers of scatterers be fixed and its radii decrease to zero. As a limit one obtains a completely integrable flow on the torus which has zero entropy. We are interested in the asymptotics of the entropy decrease as the scatterers shrinks.

First consider the system with a single scatterer of radius R.

Proposition 1. If the radius R is small enough, then the entropy of the induced map T is

$$h(T) = -d(d-1)\ln R + O(1) \tag{9}$$

and the flow entropy is

$$h(\{S^t\}) = -\text{const}\, R^{d-1}\ln R + O(R^{d-1}), \tag{10}$$

where const $= d(d-1)|B^{d-1}|$.

Here and on $O(R^{\alpha})$ denotes a quantity f such that $\limsup_{R\to 0} |R^{-\alpha} f| < \infty$.

Before proving Proposition 1 we make some general remarks.

Remark. The formula (9) for d=2 has been conjectured in [10]. There is also an erroneous conjecture for $d \geq 3$ ($h(T) \sim -d\ln R$) in the same paper.

Remark. Comparing (9) to the well-known Pesin formula [9]

$$h(T) = \int_M \sum\nolimits_+ \chi_i(x)\, k_i(x)\, d\nu(x)$$

(here $\chi_i(x)$ are the Lyapunov exponents, $k_i(x)$ are their multiplicities and the sum is taken over all positive $\chi_i(x)$) we deduce an estimate of the maximal Lyapunov exponent:

$$-d \ln R + O(1) \le \chi_{max} \le -d(d-1) \ln R + O(1) \qquad (11)$$

(note that due to ergodicity all the Lyapunov exponents together with their multiplicities are a.e. constant in M). In [11] χ_{max} was conjectured to be $\sim -[(3d+2)/4] \ln R$. Our estimate (11) shows that this conjecture is wrong.

Conjecture. The spherical scatterers are highly symmetric, so we guess that all the positive Lyapunov exponents are identical here and hence due to (9) they equal $-d \ln R + O(1)$.

In general, the problem of evaluating all the Lyapunov exponents in multidimensional hyperbolic billiards remains open. The physical explanation of its values is unclear, too.

Proof of Proposition 1. For brevity we write $\langle f\rangle$ instead of $\int_M f(x)\, d\nu(x)$ and τ instead of $\tau(x)$. The relation (10) immediately results from (9) and (4),(5) because $\langle\tau\rangle = c_\nu / c_\mu = |B^{d-1}| R^{d-1} + O(1)$.

Now we prove (9). According to (7) the operator $B(x)$ at a point $x = (r,v) \in M$ is expressed as $B(x) = D_1 + (\tau_1 I + \dots)^{-1} = D_1 + \Delta_B$ where D_1 is the operator of type (6) associated with the reflection at the moment t=0. Straightforward calculations show that D_1 has an eigenvalue $2 R^{-1}(v,n(r))^{-1}$ of multiplicity one and an eigenvalue $2 R^{-1}(v,n(r))$ of multiplicity (d-2). Denote $\tilde{D}_1 = R \cdot D_1$ and apply Theorem 2:

$$\begin{aligned} h(T) &= \langle \ln \det (I + \tau R^{-1}\tilde{D}_1 + \tau \Delta_B) \rangle = \\ &= \langle \ln(\tau R^{-1})^{d-1}\rangle + \langle \ln \det (\tilde{D}_1 + R\, \tau^{-1} I + R\, \Delta_B) \rangle = \\ &= (d-1) [- \ln R + \langle\ln \tau\rangle] + \Delta_h . \end{aligned}$$

It is trivial that $\tau(x)$ is bounded away from zero, so $\|\Delta_B\| <$ const for sufficiently small R. This allows one to deduce that a limit

$$\lim_{R \to 0} \Delta_h = H(d) \qquad (12)$$

exists. Moreover, it can be found explicitly: $H(2) = 1$, $H(3) = \ln 4$

and for $d \geq 4$ one obtains $H(d) = \ln 2^{d-1} - (d-3) \cdot \langle \ln(v, n(r)) \rangle =$

$\ln 2^{d-1} - (d-3)\ |S^{d-2}| (\int_0^1 r^{d-2} \ln \sqrt{1-r^2}\ dr)$.

Lemma 1. For all sufficiently small R one has that

$$\ln \langle\tau\rangle - c_\tau \leq \langle\ln \tau\rangle \leq \ln \langle\tau\rangle,$$

where c_τ depends on d only.

The second inequality in Lemma 1 follows from the convexity of the function $\ln x$. To prove the first one we estimate the measure of the set of points $x \in M$ such that $\tau(x) < z$, where z is a variable. We use the periodicity of the scatterers and an obvious geometric observation: a ball of radius R at the distance L off the origin O "shadows" an area $\sim (R/L)^{d-1}$ on the unit sphere centered at O.

The resulting estimate is:

$$\nu\{x \in M: \tau(x) \leq z\} \leq \text{const} \sum_{n=1}^{[z]} [n^d - (n-1)^d]\ (R/n)^{d-1} \leq \text{const}\ z\ R^{d-1}$$

for all positive z. It immediately implies the desired inequality: $\langle\ln \tau\rangle \geq -(d-1) \ln R - \text{const}$ completing the proof of Lemma 1.

All the above estimates result in (9), hence Proposition 1.

Remark. All our formulae but the first inequality in Lemma 1 are contained in [10]. Far stronger conjecture than Lemma 1 comes from numerical experiment [10]: a limit $\lim (\langle\ln \tau\rangle - \ln \langle\tau\rangle)$ exists as $R \to 0$. If it is true and the limit equals P, then the formula (9) can be made even more accurate:

$$h(T) = -d(d-1) \ln R + A + o(1),$$

where $A = H(d) + (d-1)\ [P - \ln |B^{d-1}|]$. Note that an estimate $A = 2 \cdot \ln(2 \pm 0.2)$ comes from numerical results [11].

Now consider the Lorentz gas with several scatterers of radii $R_1, R_2, \ldots, R_m$. Denote for brevity $Z_0 = R_1^{d-1} + R_2^{d-1} + \ldots + R_m^{d-1}$ and $Z_1 = R_1^{d-1} \ln R_1 + \ldots + R_m^{d-1} \ln R_m$.

Proposition 2. If all the radii are small enough, then the entropy of the induced map T is

$$h(T) = -(d-1)\ [\ \ln Z_0 + Z_1 / Z_0\ ] + O(1) \tag{13}$$

and the flow entropy is

$$h(\{S^t\}) = -\text{const}\ [\ Z_0 \ln Z_0 + Z_1\ ] + O(Z_0), \tag{14}$$

where const = $(d-1)|B^{d-1}|$.

The proof of Proposition 2 repeats that of Proposition 1. We shall just point out some differences in formulae.

Now $\langle\tau\rangle = c_\nu / c_\mu \approx |B^{d-1}| Z_0$, so $\ln\langle\tau\rangle = \ln Z_0 + O(1)$. Furthermore the operator D_1 takes the form $D_1 = R_x^{-1}\tilde{D}_1$, where R_x is the radius of the scatterer to which the point x is attached. Therefore

$$h(T) = \langle\ \ln\det[\ R_x^{-1}\tau\ (\ \tilde{D}_1 + R_x\,\tau^{-1} I + R_x\ \Delta_B\ )]\ \rangle =$$

$$= (d-1)\ [\ -\langle\ \ln R_x\ \rangle + \langle\ln\tau\rangle\ ] + \Delta_h\ .$$

Straightforward calculations show that $\langle\ \ln R_x\ \rangle = Z_1 / Z_0$. The rest of the proof is the same as that of Proposition 1.

Finally, let us replace the spherical scatterers with convex scatterers of arbitrary shape. Let all the scatterers shrink homotetically with a common scale parameter ε ($\varepsilon \to 0$). In this case the entropy of the induced map T is equal to

$$h(T) = -d(d-1)\ \ln\varepsilon + O(1)$$

and the flow entropy is

$$h(\{S^t\}) = -\text{const}\ \varepsilon^{d-1}\ln\varepsilon\ + O(\ \varepsilon^{d-1}\ ).$$

The proof is similar to that of Propositions 1,2 and we drop the details.

§ 4. THE ENTROPY OF BUNIMOVICH STADIUM AND RELATED SYSTEMS

L.A.Bunimovich [7] introduced a billiard system in a domain Q bounded by two parallel segments of length $2a$ and by two semi-circles of radii R. The boundary of Q is then a closed C^1-curve but not a C^2-curve. Q is called a stadium-like region or simply stadium. L.A.Bunimovich has proved [7] that the billiards in Q is ergodic. Its entropy was studied in [19,23,24] and numerically estimated in [12].

It is convenient to consider two induced maps for this system. One is the map T introduced in § 2 and the other one denoted by T_* is induced in the set $M_* = \{ x=(q,v) \in M: K(q) \neq 0 \}$, i.e. M_* is the part of the border attached to two semicircles only. The map T_* preserves the measure $d\nu_* = c_{\nu_*}(v,n(r))\, dr\, dv$, where $c_{\nu_*} = (4\pi R)^{-1}$. Denote also $\tau_* = \tau_*(x)$ for $x \in M_*$ the time of first return to M_*.

The formula (4) implies that

$$h(\{S^t\}) = h(T) / \langle\tau\rangle = h(T_*) / \langle\tau_*\rangle_* \tag{15}$$

where $\langle\cdot\rangle$ and $\langle\cdot\rangle_*$ denote the mean values with respect to the measures ν and ν_* correspondingly.

We consider "long narrow" stadiums ($R \ll a$).

Proposition 3. If the ratio R/a is small enough then

$$h(T_*) = \frac{2}{\pi} \ln \frac{a}{R} + O(1) , \tag{16}$$

$$h(T) = \frac{R}{a} \ln \frac{a}{R} + O(\frac{R}{a}) , \tag{17}$$

$$h(\{S^t\}) = \frac{1}{\pi \cdot a} \ln \frac{a}{R} + O(\frac{1}{a}) . \tag{18}$$

Proof. It is easy to count that $\langle\tau\rangle = c_\nu / c_\mu = \pi R + O(R^2)$ and $\langle\tau_*\rangle_* = c_{\nu_*} / c_\mu = 2a + R$. Comparing to (15) we deduce that the expressions (16)-(18) are equivalent, so we have to prove just one of them.

We prove the formula (16). Define natural coordinates (r,ϕ) in M_*: r is the arc length parameter along the semicircles ($0 < r < \pi R$ in one of them and $\pi R < r < 2\pi R$ in the other one) and ϕ is the angle between the velocity vector v and the normal vector $n(r)$, $-\pi/2 < \phi < \pi/2$. Then $(v,n(r)) = \cos\phi$, so we can rewrite (8) as $-2(R\cos\phi)^{-1} \le B(x) \le -(R\cos\phi)^{-1}$. This implies the following estimate:

$$\frac{\tau_*}{R\cos\phi} - 1 \le |1 + \tau_* B(x)| \le \frac{2\tau_*}{R\cos\phi} - 1 . \tag{19}$$

Further we have to split the space M_* into two parts: M_*^0 and M_*^1: the part M_*^1 corresponds to transitions from one of the semicircles to the other and the part M_*^0 corresponds to successive reflections from the same semicircle. Fig.2 shows this splitting in (r,ϕ) coordinates: the part M_*^0 is hatched. Clearly, $\tau_* = 2R\cos\phi$ on M_*^0 and $\tau_* \ge 2a$ on M_*^1. Together with (19) it leads to the following estimates: $0 < \ln|1 + \tau_* B(x)| < \ln 3$ for $x \in M_*^0$ and

$$\ln|1 + \tau_* B(x)| = \ln\tau_* - \ln R - \ln\cos\phi + \Delta(x)$$

with $|\Delta(x)| < \ln 2$ for $x \in M_*^1$. Denote $\langle f\rangle_0$ and $\langle f\rangle_1$ the integrals of a function $f(x)$ with respect to the measure ν_* over the domains M_*^0 and M_*^1 respectively. One can easily compute that $\nu_*(M_*^1) = 2/\pi$. All the above estimates result in the decomposition

$$h(T_*) = \langle\ln\tau_*\rangle_1 - \frac{2}{\pi}\ln R + O(1).$$

The function $\tau_*(x)$ for $x \in M_*^1$ can be approximated by

$$\tau_*(x) = (2a + p(x)) / \sin(r/R - \phi),$$

where $0 \le p(x) \le 2R$. Taking the integral over M_*^1 we obtain $\langle\ln\tau_*\rangle_1 = 2\pi^{-1}\ln a + O(1)$. As a result we come to the formula (16) and accomplish the proof of Proposition 3.

Remark. In the paper [24] another asymptotics has been found as $R/a \to 0$: $h(T) \sim \text{const}\, R$. It contradicts to our (17). The author of [24] used a method for entropy calculation which apparently should now be recognized as wrong.

Remark. One can see from the formulae (16)-(18) that the entropy $h(T)$ only vanishes as $R \to 0$ and $a \sim \text{const}$ (two other entropies tend to ∞). But if one fixes the total stadium area $S = 4aR + \pi R^2$, then the flow entropy $h(\{S^t\})$ will also tend to zero as $R/a \to 0$.

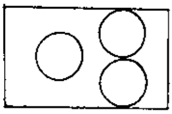

Fig.1

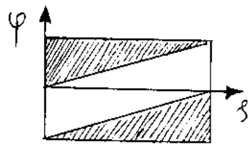

Fig.2

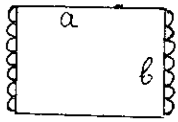

Fig.3

Fig.4

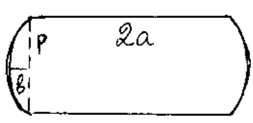

Fig.5

Remark. The formula (18) has the following straightforward consequence. Let us take a rectangle Π with the sides a,b. To two its sides of length b we glue chains of jointed semicircles of the same radii $R \ll b$ (as in Fig.3). The entropy of the billiards in the resulting region Q_R is then equal to

$$h(\{S^t\}) = (\pi a)^{-1} \ln (a/R) + O(1).$$

Note that it tends to infinity (!) as $R \to 0$ (while the region Q_R "tends" to the rectangle Π). The matter is that the trajectories of the billiards in Q_R do not converge to those in Π .

The latter remark suggests an interesting construction.

First supplementary example. Consider an arbitrary polygon P and glue to its sides chains of jointed semicircles of sufficiently small radii R. Denote this region by Q_R^- (see Fig.4a). Take another copy of the polygon P and cut out of it chains of jointed semicircles of the same radii R as above situated along its sides. The rest of the polygon area will be denoted by Q_R^+ (see Fig.4b). Obviously, Q_R^- is a Bunimovich billiards while Q_R^+ is a Sinai billiards.

One can easily count that in both billiards $\langle\tau\rangle = 2|P|/ |\partial P| + O(R)$. Here $|P|$ is the polygon area and $|\partial P|$ is its perimeter length. It is convenient in both cases to split the space M into two parts M^0 and M^1 . For Q_R^- it must be done in the same way as for the case of stadium. In Q_R^+ the part M^0 corresponds to transitions from an arbitrary semicircle to one of its two closest neighbors. The domain M^1 in (r,ϕ) space looks quite different for the two cases Q_R^- and Q_R^+ . But its measure is surprisingly the same: $\nu(M^1) = 2/\pi$. For the billiards Q_R^- the entropy h(T) is evaluated just as for the stadium: $h(T) = -2\,\pi^{-1} \ln R + O(1)$. Some minor modifications should be done to adapt these arguments to the billiards in Q_R^+ (we drop the details). Applying the formula (4) we get the flow entropy of the billiards in Q_R^- and in Q_R^+ :

$$h(\{S^t\}) = -\text{const} \ln R + O(1),$$
where const $= |\partial P|(\pi |P|)^{-1}$.

Our results actually mean that the entropy (hence the rate of divergence of the trajectories) in the Bunimovich billiards and in the Sinai ones has the same asymptotics up to a common coefficient in the principal terms!

Finally, note that one can take any plane region with smooth or piecewise smooth boundary instead of the polygon P.

Second supplementary example. G.M.Zaslavsky [23] considered an interesting variation of the Bunimovich stadium. He replaced two semicircles by two identical circular arcs of the radii R and of the height b (see Fig.5). We assume as in [23] that $b \ll R \ll a$. We use here all the notions introduced for the stadium.Besides, denote 2p the length of the chords connecting the endpoints of our arcs. Now it is easy to count that $\langle\tau\rangle = c_\nu / c_\mu \approx \pi p$ and $\langle\tau_*\rangle_* = c_{\nu_*} / c_\mu \approx \pi a$. Repeating all the steps of the proof of Proposition 3 we come to the relation $h(T_*) = \nu_*(M_*^1) \ln(a/R) + O(1)$. But in the present case $\nu_*(M_*^1) \to 1$ as $b/R \to 0$. Some extra manipulations provide more precise expression: $h(T_*) = \ln(a/R) + \text{const} + o(1)$, where const $= 1 + \ln 2$ (we do not supply the details). Together with the formula (4) this easily yields that $h(T) = (p/a) \times \ln(a/R) + O(p/a)$ and $h(\{S^t\}) = (\pi a)^{-1} \ln(a/R) + O(1/a)$. Our results confirm those obtained in [23] on base of rather intuitive arguments (the formula $h(T_*) \approx \ln(ab/p^2)$ is essentially supplied there).

In conclusion let us make a general remark.

Remark. All our examples in § 3,4 have a common feature: the curvature of the boundary ∂Q tends to infinity while the (mean) free path remain bounded below. Due to this property the rate of convergence of the trajectories is actually determined by the last reflection only.In other words, the behavior of the system approximately is a markovian one, i.e. the "memory" of all preceding reflection but the last one is getting lost. For the systems of this kind our method of entropy evaluation seems quite universal.

However, there is another class of completely different dynamical systems. Take, for instance, the Bunimovich stadium when $a \ll R$ (the stadium approximates a circle). Here the impact of each sole reflection (including the last one) onto the process of the exponential divergence of trajectories becomes negligible. Roughly speaking, the rate of the trajectory divergence is determined by N latest reflections where $N \sim \sqrt{R/a}$. The entropy of this system and of other systems of that kind was estimated by M.Wojtkowski [19,25] who has elaborated rather universal method for estimating the entropy of hyperbolic maps from below.

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