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SYMMETRY OF THE HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS AND SEPARATION OF VARIABLES

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1. Introduction and stating of the problem

The method of separation of variables in solving linear partial differential equations has been already known in XIX th Century. From that time on different aspects of this problem arise many times in different contexts. One of the most widely explored one is connected with the solutions space symmetries that occur when solving LPDE (See e.g.[1-4] and papers quoted therein.) Unfortunately the general setting of the problem is missing hitherto. Even in the low-dimensional cases (two variables) there are no general theorems but rather a collection of computable examples (see e.g.[4]). In this paper we are formulating and proving, as far as we know for the first time, general assertions concerning the interrelations between the symmetry of the equations and the separation of variables. Partly these results have been manifested in numerous examples treated independently up to now.

Let us introduce some notions and necessary notations. The homogeneous partial differential equation:

$$\hat{Q}_m(x_1,\dots x_n)\Psi = 0, \quad x_i \in \mathbb{R}, \ \forall i \tag{1}$$

will be the main object of our considerations, where the operator $\hat{Q}$ in some given coordinate system $\mathcal{Y}$ has the general form

$$\hat{Q}_m(x_1,x_2,\ldots x_n)=\sum_{k=0}^{m}\ \sum_{i_k=1}^{n} f_{i_1 i_2 \ldots i_k}(x_1,\ldots,x_n)\partial_{x_{i_1}\ldots x_{i_k}}. \qquad (2)$$

<u>Definition</u> 1 A linear operator $\hat{S}: W^m_{loc}(\mathbb{R}^n) \longrightarrow W^m_{loc}(\mathbb{R}^n)$ is called symmetry for the set of solutions of equation (1) if $[\hat{S},\hat{Q}]=\hat{P}\hat{Q}$, $\hat{P}$ being some operator with $im(\hat{P}) \in ker(\hat{Q})$. In what follows we shall consider differential symmetry operators only. All assertions should be only of local validity excluding some zero-measure sets from definition domain of the coefficients.

It is noteworthy that in physical literature the operators $\hat{S}$ are known as dynamical symmetry (See e.g[5]), since $\hat{S}$ and $\hat{Q}$ commute on the solution of Eq.(1). However in the greatest part of examples this notion tacitly presumes that all symmetries lie into the enveloping algebra of some Lie algebra. But sometimes it may happen (see Sec.4 and the example therein) that the symmetry operators $\hat{S}$ doesn't belong to the enveloping of the first order (Lie) vector fields algebra , so an operator of this type should provide a nontrivial generalization of the standard symmetry. Moreover, these $\hat{S}$'s, when apply for the Schroedinger operator $\hat{Q}= i\partial/\partial t-\hat{H}$, would lead to dynamical invariant which might have no obvious group- theoretical meaning. In order to argue this an important factorization of the enveloping algebra is needed. From this end assume that the class of functions to which belong the solutions of Eq.(1) is some $L^p(\mathbb{R}^n,\sigma)$ the measure σ being unfixed. The enveloping vector field algebra $\{V\}^{\sigma}$ symmetric with respect to the measure σ should differ from the enveloping algebra $\{V\}^{\mu}$ $(\mu\neq\sigma)$ by terms induced (in local coordinates $\mathfrak{X}$) by the replacement $\partial \longrightarrow \partial+\ \partial log(\sigma/\mu)$. Henceforth all higher-order formal symmetries have to be necessarily tested for with respect to this equivalence.(See e.g.[6] where this factorization might change substantially some of the announced results.)

For sake of brevity let us denote by x,y,z the first three variables x_1,x_2,x_3 respectively. Consider the elements of the equivalence class $\hat{\mathcal{Q}}=\{\hat{Q},\hat{Q}'|\hat{Q}=e^{-R}\hat{Q}'e^{R}\}$, R being a real function. This factorization corresponds to trivial $\mathbb{R}_+$ gauge invariance of Eq.(1).

<u>Definition</u> 2. One says that the variable x is linearly

splitted from the rest of variables in the coordinate system $\mathcal{Y}$ if there exists function R and representative $\hat{Q}^R \in Q$ such that

$$\hat{Q}^R_m = \Phi(\cdot)(\hat{Q}^R_k(x) + \hat{Q}^R_l(y,z,..)), \quad k \le m, \quad l \le m; \quad max(k,l) = m \tag{3}$$

and

$$\Psi(\cdot) = \Psi_1(x)\Psi_2(y,z,...).$$

Here $\Phi(\cdot)$ is an arbitrary function depending on all variables. It is an elementary exercise to check that $\hat{Q}^R_k$ is a symmetry in the sense of Definition 1.

Problem 1. Let for Eq.(1) be known the diffeomorphism φ: $(x_1,x_2,...,\Psi) \longrightarrow (u,v,..e^{R(u,v,..)}\Psi(u,v,...))$ such that a complete separation of variables is valid in the new coordinates. What are the symmetries for this case and is it possible to find the solution of Eq.(1) making use of them?

Problem 2. Whether the splitting of variables may have place all symmetries {S} being known ?

Problem 3. Let the set of symmetries {S} of Eq (1) be known and suppose that the linear splitting of variables is holding in some coordinates.Is it possible to find these coordinates (i.e. the diffeomorphism φ) ?

The structure of the rest of the paper is the following. In Sec.2 we state and prove some general lemmas for n-dimensional m^{th}order LPDE admitting a separation of variables. Sec.3 contains a constructive way of checking the stipulations which the coefficients of the LPDE (for m=2,n=2) and the symmetry operator must to comply with. Moreover,an algorithm for finding the diffeomorphism φ is proposed too. In Sec.4 we deal with two examples: the two- dimensional Schroedinger's equation with barrier potential provides an illustration for general method and notions introduced, while in the second example we demonstrate the symmetry operator $\hat{S}$ not belonging to the linear enveloping of the first order symmetries.

2. General theorems

First of all let us consider the complete splitting of three variables in order to get some hint for the general case. According to the formula (3) each element $\hat{Q}^R$ can recast into the form[1]

$$\hat{Q}^R(x,y,z)= \Phi(x,y,z)\{\hat{Q}_{m_1}(x)+ \alpha_1(y,z)(\hat{Q}_{m_2}(y)+\hat{Q}_{m_3}(z))\}. \qquad (4)$$

When seeking for a complete splitting of variables one sees by direct check that the operators $\hat{S}_1=\hat{Q}_{m_1}$ and $\hat{S}_2=\hat{Q}_{m_2}(y)+f(y)\hat{Q}_{m_1}(x)$ are symmetries together with the representation $\alpha_1(y,z)=[f(y)+\varphi(z)]^{-1}$; f and φ being arbitrary functions.

Note that the operator $\hat{S}_3= \hat{Q}_{m_3}(z)+\varphi(z)\hat{Q}_{m_1}(x)$, although being a symmetry, in fact coincides with $\hat{S}_2$, since the operator algebra (according to the Definition 1) is defined *mod*(Q_m), i.e. on the solutions of (1). Note also that restrictions on the function α_1 results due to nonzero integration constant coming still at the first step of x-splitting.

Now it is not difficult to anticipate the general formula for n-variables complete splitting. The following algebraic lemma holds:

__Lemma 1__ The general n-variables additively splitted operator $\hat{Q}$ can be represented in the form

$$\hat{Q}=\Phi(\cdot)\{ \hat{Q}_{m_1}(x_1)+ \alpha_2(x_2,..x_n)(\hat{Q}_{m_2}(x_2)+\alpha_3(x_3,...,x_n)(\hat{Q}_{m_3}+ \ldots$$
$$+\alpha_{n-1}(x_{n-1},x_n)(\hat{Q}_{m_{n-1}}(x_{n-1})+ \hat{Q}_{m_n}(x_n) \quad))..\}, \qquad (5)$$

where the coefficients α_k depend on n-k+1 arbitrary functions $\{\rho_l^1(x_l)\}$ and recurrently on the next $\{\alpha_m\}$ m>k by the following way

$$1/\alpha_k= \quad \rho_k^k(x_k)+\alpha_{k+1}(\rho_{k+1}^k(x_{k+1})+\alpha_{k+2}(\rho_{k+2}^k(x_{k+2})+\ldots$$
$$\alpha_{n-1}(\rho_{n-1}^k(x_{n-1})+\rho_n^k(x_n))..). \qquad (6)$$

The proof is inductive and by direct inspection after solving some elementary functional equations. Everywhere the union of the subsets $\mathcal{U}_m=\{x|\alpha_m=0\}$ where the division by α's fails[6] are to be excluded from the domain of definition.

Now all symmetry operators for the equation (1) admitting full separation of variables are constructed by the following

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[1] *Whenever possible we shall omit superscript R or replace it by tilde.*

Lemma 2 . The Eq.(1) admitting a complete separation of variables, in the sense of Definition 2, is weakly invariant in the sense of Definition 1 and possesses only n-1 commuting symmetries $\{S_i\}_{i=1}^{n-1}$ given by the following simple recurrence :

$$\hat{S}_{k+1} = \hat{Q}_{k+1} + \sum_{1}^{k} \rho_{k+1}^{i+1}(x_{k+1})\hat{S}_1 \quad ;k=0,..,n-2. \tag{5'}$$

The proof of this formula is a little bit tedious but straightforward one. The completeness of the set $\{\hat{S}\}$ of commuting differential symmetries defined on the functions $W^m_{loc} \ni f:\mathbb{R}\backslash\cup\mathcal{U}_m \longrightarrow \mathbb{R}$ results from the local integrability of (1)[6] .

These two lemmas give one constructive answer to the first part of Problem 1, excluding perhaps the cases when the completeness of commutative symmetries breaks down. Moreover the solutions of Eq (1) appear as eigenfunctions of the symmetry algebra. In the next section we shall demonstrate that the presence of symmetry is only a necessary condition for the variable splitting, whilst the existence of a restricted class of symmetries gives the sufficient condition too.

3. Construction theorem for two dimensions

The common features and technical difficulties arising in the general situation emerge still in the simplest case m=2,n=2 we shall discuss in some details here.

Let $\hat{Q}(x,y)\Psi=0$ be a homogeneous second-order real PDE that admits splitting of variables in some coordinates system $\mathcal{Y}$ with coordinate functions y_1,y_2 and fixed $\mathbb{R}_+$transformation [2] (i.e. when the so called R-separation holds[4]). In order to cover the generic case we shall always assume that the operator $\hat{Q}$ is not factorisable, which by definition means that any decompositions in the form $\hat{Q}= \hat{Q}_a \cdot \hat{Q}_b$ should have a zero-order multiplier .

According to the Lemma 1 the general form of the equation (1) doesn't contain unknown structure function α, namely

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[2] *Note that up to now R remains an arbitrary (real) function. But in fact it will be fixed by taking the operators Q_{m_1} in Eq.(5) in standard form.*

$$\Phi(y_1,y_2)\cdot(\hat{Q}_{m_1}(y_1)+\hat{Q}_{m_2}(y_2))\Psi = 0,\ max(m_1,m_2)=m, \tag{7}$$

where the last condition on *m*'s implies that one of the operators $\hat{Q}$'s in (7) must to be a second order <u>symmetry</u> <u>operator</u> .

Here some comments are in order. For parabolic type LPDE (namely when one of the m's equals unity) every separation of variables generates some first order symmetry operator that can be represented in the standard form $\partial_u+f(u)$ reducing this way the separation of variables to problem simpler than the generic one $m_1= m_2= 2$ we are going to deal with.

<u>Assertion</u> <u>1</u>. If in some coordinate system $\mathcal{Y}$ there exists a separation of variables in the sense of Lemma 1 the corresponding second-order symmetry operator $\hat{S}$ being known, then:

i) there should exist a diffeo $\mathfrak{F}$ [3] that transforms the symmetry $\hat{S}$ into the following canonical form:

$$\mathfrak{F}^{*}S\ \mathfrak{F} = \sigma\partial_{uu}+f_1(u),\ (\sigma=\pm1), \tag{8}$$

the sign σ depending on whether the diffeo $\mathcal{F}$ is an orientation preserving map or not (See e.g.[7] ,u stands for y_1 or y_2;

ii) the same $\mathcal{F}$ should transform the operator $\hat{Q}^R(x,y)$ into the form given by Eq(7).

The proof is constructive, allowing an 'explicite' finding of the $\mathcal{Y}$-coordinates. Beginning with a general form for symmetry operator $\hat{S}$ in $\mathcal{X}$-variables:

$$\hat{S}=a\partial_{xx}+2b\partial_{xy}+c\partial_{yy}+d\partial_x+e\partial_y+f \tag{9}$$

and performing the R-transformation $\hat{S} \longrightarrow e^R\hat{S}e^{-R}$ we obtain for the modified $\hat{S}$

$$\hat{\tilde{S}}= \hat{S}+(-2a\partial_xR-2b\partial_yR)\partial_x+(-2b\partial_xR-2c\partial_yR)\partial_y-d\partial_xR-e\partial_yR-$$

$$- c\partial_{yy}R-2b\partial_{xy}R -a\partial_{xx}R+a(\partial_xR)^2+c(\partial_yR)^2+2b\partial_xR\partial_yR. \tag{10}$$

Going back in Eq.(8) to $\mathcal{X}$-coordinate system and comparing with (10) we get a set of equations for v's ,R and f_2 to be determined:

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3 *In fact alvays in our constructions a little bit more general diffeomorphism (or shortly diffeos)* $\mathfrak{F} = \varphi\ (\varphi_1(u),\varphi_2(v))$ *are to be used, where* $\{\ \varphi_i\}:R^1_{loc}\longrightarrow R^1_{loc}$ *stand for local one-dimensional diffeos.*

$$a=\sigma(v_x)^2/\Delta^2\ ,\ b=-\sigma v_x v_y/\Delta^2,\ c=\ \sigma(v_y)^2/\Delta^2, \tag{11}$$

$$d-2a\partial_x R-2b\partial_y R=\sigma\{\ \frac{v_y}{\Delta}\ \partial_x(v_x/\Delta)-\ \frac{v_x}{\Delta}\ \partial_y(v_y/\Delta)\}, \tag{12}$$

$$e-2b\partial_x R-2c\partial_y R=\sigma\{\ \frac{v_x}{\Delta}\ \partial_y(v_x/\Delta)-\ \frac{v_y}{\Delta}\ \partial_x(v_y/\Delta)\}, \tag{13}$$

$$f_2=\ f-d\partial_x R-e\partial_y R+c(\partial_y R)^2-c\partial_{yy}R-2b\partial_{xy}R-a\partial_{xx}R+a(\partial_x R)^2+2b\partial_x R\partial_y R$$
$$\equiv\ e^R(\hat{S}\cdot e^{-R}), \tag{14}$$

the quantity Δ being the Jacobian $u_x v_y-u_y v_x$, $\sigma=\pm1$. First we find that $b^2-ac=0$ (<u>parabolicity condition</u>). Second for the components v_x, v_y Eqs (11) gives $v_x=\pm\Delta\varepsilon\sqrt{\sigma a}$, $v_y=\pm\ \Delta\theta\sqrt{\sigma c}$, where the signs θ,ε are to be fixed by the relation containing b. Comparing (12),(15) we obtain a *necessary* condition for existing of diffeo φ

$$e+\varepsilon\theta(c/a)^{1/2}d=\sigma\{v_x/\Delta\partial_y(v_x/\Delta)-v_y/\Delta\partial_x(v_x/\Delta)\}+$$
$$+\varepsilon\theta\sigma(c/a)^{1/2}\ \{v_y/\Delta\partial_x(v_y/\Delta)-v_x/\Delta\partial_y(v_y/\Delta)\}. \tag{15}$$

Here the replacement of the expressions in the curly brackets by the solution of Eqs (11) leads to a *PDE for the coefficients* of the symmetry operator $\hat{S}$. Moreover, starting with R satisfying say (12) R have to be selected in such a way to get the right u-dependence of the function f_2 , namely:

$$u=\ F(f_2). \tag{16}$$

Furthermore from Eqs.(11),(15) ,using the definition of Δ, we find

$$F'(f_2)=[\varepsilon(\sigma a)^{1/2}\partial_x f_2-\theta(\sigma c)^{1/2}\partial_y f_2]^{-1}, \tag{17}$$

which immediately leads to another *necessary* condition

$$\begin{vmatrix}\partial_x F' & \partial_y F' \\ \partial_x f_2 & \partial_y f_2\end{vmatrix}=0 \tag{18}$$

for the validity of representation (16). This condition is also to be imposed on the solution R of Eq (12) or (13) since f_2 and F' has been still determined via Eqs.(14) and (17) respectively. Note that when Eq (17) holds the determinant of the system

$$v_y=\varepsilon(\sigma a)^{1/2}(u_x v_y-u_y v_x);\quad v_x=\theta(\sigma c)^{1/2}(u_x v_y-u_y v_x)$$

identically vanishes hence no matter which one to take for finding the coordinate function v(x,y). Now in order to reconstruct the coordinates functions u, v in diffeo φ it is also necessary to demand the total separation of variables, which according to formula (7), implies that the operator:

$$\hat{S}_2 = \frac{1}{\Phi}\hat{Q}^R - \hat{S}^R \qquad (19)$$

must have the standard form too (See Eq (8)):

$$\hat{S}_2 = \beta\partial_{vv} + f_3(v), (\beta=\pm 1). \qquad (20)$$

Proceeding in complete analogy with the previous case instead of Eqs (11)-(14) one gets the following system of equations:

$$a+qA=\beta(u_y)^2/\Delta^2, \quad 2b+2qB=-\beta\frac{2\,u_x u_y}{\Delta^2}, \quad c+qC=\beta(u_x)^2/\Delta^2, \qquad (11')$$

$$d+qD-(2a+2Aq)\partial_x R-(2b+2qB)\partial_y R=\beta\{u_y/\Delta\partial_x(u_y/\Delta)-u_x/\Delta\partial_y(u_y/\Delta)\}, \qquad (12')$$

$$e+qE-(2b+2qB)\partial_x R-(2c+2qC)\partial_y R=\beta\{u_x/\Delta\partial_y(u_x/\Delta)-u_y/\Delta\partial_x(u_x/\Delta)\}, \qquad (13')$$

$$f_3=f+qF-(d+qD)\partial_x R-(e+qE)\partial_y R+(c+qC)(\partial_y R)^2-(c+qC)\partial_{yy}R-(2b+2qB)\partial_{xy}R-$$

$$-(a+qA)\partial_{xx}R+(a+qA)(\partial_x R)^2+(2b+2qB)\partial_x R\partial_y R\equiv$$

$$\equiv e^R\Big[(\hat{S}+q\hat{Q})e^{-R}\Big]. \qquad (14')$$

Note that the 'parabolic condition' emerges again:

$$(c+qC)(a+qA)=(b+qB)^2. \qquad (21)$$

Here some explanations are in order. The capital letters A,B,..F stand for the coefficients of the $\hat{Q}_2$ in $\mathfrak{X}$-coordinates. The function q replaces the arbitrary factor 1/Φ. Proceeding within the general scheme previously sketched in Sec.3 one obtains the equations:

$$u_x/\Delta=\gamma[(\beta(c+qC)]^{1/2}; \quad u_y/\Delta=\delta[\beta(a+qA)]^{1/2}; \quad \gamma,\delta=\pm 1 \qquad (22)$$

and a *necessary* self-consistency condition for the coefficients

$$e+qE+\gamma\delta[(c+qC)/(a+qA)]^{1/2}(d+qD)=$$

$$\beta[u_x/\Delta\partial_y(u_x/\Delta)-u_y/\Delta\partial_x(u_x/\Delta)]+$$

$$\gamma\delta\beta[(c+qC)/(a+qA)]^{1/2}[u_y/\Delta\partial_x(u_y/\Delta)-u_x/\Delta\partial_y(u_y/\Delta)]. \qquad (23)$$

Let us remind again that $f_3=f_3(v)$, or equivalently $v = \Phi(f_3(x,y))$.Then after replacement derivatives of v and u into the Jacobian $\Delta=u_x v_y-u_y v_x$ one gets

$$\Phi'=[\gamma\sqrt{\beta(c+qC)}\partial_y f_3-\delta\sqrt{\beta(a+qA)}\partial_x f_3]^{-1}, \qquad (17')$$

which in addition to Eq.(18) gives another *necessary* condition:

$$\begin{vmatrix} \partial_x f_3 & \partial_y f_3 \\ \partial_x \Phi' & \partial_y \Phi' \end{vmatrix} = 0 . \qquad (18')$$

Here we recognize the requirement $\Phi'=\Phi'(f_3)$. This way from the definition of Δ we obtain two equations for the coordinate maps:

$$u_x=\gamma\sqrt{\beta(c+qC)}\ (u_x v_y - u_y v_x),$$

$$v_x= - \theta\sqrt{\sigma c}\ [u_y v_x - u_x v_y],$$

which together with definitions of F,Φ,f_2 give

$$v_x/u_x= \theta(\sigma c)^{1/2}[\gamma\sqrt{(\beta(c+qC)}\ ]^{-1/2} \longrightarrow$$

$$\frac{\partial_x f_3[\varepsilon(\sigma a)^{1/2}\partial_x f_2-\theta(\sigma c)^{1/2}\partial_y f_2]}{\partial_x f_2[\gamma\sqrt{\beta(c+qC)}\partial_y f_3- \delta\sqrt{\beta(a+qA)}\partial_x f_3]} = \frac{\theta(\sigma c)^{1/2}}{\gamma\sqrt{\beta(c+qC)}} . \qquad (24)$$

Let us resume this construction in a condensed form: the separation of variables gives the general form of the operators Q_1^R and Q_2^R as well as the symmetry S^R in $\mathcal{Y}$-system. The pullback $\mathcal{F}^* Q_i^R \mathcal{F}$ and $\mathcal{F}^* S \mathcal{F}$ compared with a prescribed form of the *operators* $\hat{S}$ *and* $\hat{Q}_i$*'s give the* necessary conditions *for* variables splitting: *there should exist* a *symmetry operator* $\hat{S}$ *related with a diffeo* φ, satisfying the parabolicity condition and equations (15), (18), (23), (18'), (24), together with specific gauge fixing imposed on function R (Eqs (12) and (12')).

For general 2^{nd}-order LPDE the realization of all steps performed in the theorem is a problem of the same complexity as the solution of initial equation, but our analysis in principle gives an answers to the Problem 2 Sec.1. Moreover the following corollary gives the *necessary* and *sufficient* condition for splitting of variables:

Corollary Let the set $\{\hat{P}_i\}$ span a basis of vector fields in some domain *D* not containing $\cup\mathcal{U}_m$. Then the splitting of variables in the sense of definition 1 holds iff the conditions imposed by the Assertion 1 could be satisfied by the choice of constants $\{a_i\}$ appearing in decomposition of symmetry operator $\hat{S}$:

$$\hat{S} = \sum a_i \hat{P}_i + q\hat{Q} .$$

This corollary simply follows from the constructive proof of our assertion giving a positive reply to the third problem stated in Sec.1.

A generalization of the method for higher dimensions ($n>2$) does not involve additional difficulties, while its implementation for the higher order ($m>2$) LPDE, leads in general case to new difficulties.

The results announced in Sec 3 can be easily extended (in some sense minimally) to the complex case too. Namely we shall consider a 2-dimensional second-order LPDE with complex coefficients :

$$\hat{Q}_2(x,y)\Psi=0$$

looking for complex solutions Ψ. Again if there exists splitting of variables and consequently - a symmetry $\hat{S}$ with canonical form

$$e^{i\theta(u)}\partial_{uu}+[\varphi_1+i\varphi_2]$$

one can repeat all steps by Sec.3, taking into account that $\hat{S}$ is now parameterized as follows

$$\hat{S}=(a_1+ia_2)\partial_{xx}+2(b_1+ib_2)\partial_{xy}+(c_1+ic_2)\partial_{yy}+(d_1+id_2)\partial_x+$$
$$+(e_1+ie_2)\partial_y+l_1+il_2.$$

Besides that the $\mathbb{R}_+$ gauge have to be replaced by $\mathbb{C}$-gauge, since function R becomes complex one, whilst the coordinate part of the diffeos φ remains real one.

4 Examples

I) Consider one simple example illustrating the notions and general constructions developed in Secs.2,3 :one dimensional non-stationary Schroedinger equation with centrifugal barrier potential

$$(i\partial_t+\partial_{xx}-A/x^2)\Psi=0 \qquad ,\hat{Q}\Psi=0,\ A=const. \tag{25}$$

In this simple example we have to deal with the complex equation case (Sec.3) the class of basis vector fields P_1 being first order one (due to the parabolicity of equation (25). In this section we shall only re-obtain and interpret the main results of [4] in our notations . In fact starting with the operators

$$K_{-2}=\partial_t,\ K_2=-t^2\partial_t-tx\partial_x-t/2+ix^2/4,\ K_0=2t\partial_t+x\partial_x+1/2,$$

according to the general prescription, the symmetry operator $S_1\equiv\hat{S}$ should have the form

$$\hat{S}=(a+2ct-bt^2)\partial_t+(cx-btx)\partial_x+(ibx^2/4-bt/2+c/2)$$

the unknown constant a,b,c to be defined by the self consistency conditions imposed in Sec.3.

Reminding the $\mathbb{C}$- invariance of the Eq.(1) we introduce the 'tilded' function Ψ and operator $\hat{Q}$ (In what follows we shall omit the tilde whenever possible)

$$e^{R}\hat{Q}e^{-R}e^{R}\Psi = 0 = \hat{\tilde{Q}}\tilde{\Psi}; \quad \hat{\tilde{Q}} \equiv e^{R}\hat{Q}e^{-R}, \ \tilde{\Psi} \equiv e^{R}\Psi.$$

The operator S transforms to the form

$$\hat{S} \Rightarrow e^{R}\hat{S}e^{-R} = (a+2ct-bt^2)\partial_t + (cx-btx)\partial_x + (ibx^2/4-bt/2+c/2- \\ -(a+2ct-bt^2)\partial_t R - (cx-btx)\partial_x R. \quad (26)$$

Let us introduce the coordinate part of the diffeomorphism φ, namely $v=v(t)$, $u=u(t,x)$[4]. Then, one obtains

$$\hat{S} = e^{i\Theta(v(t))}(u_t/\Delta)\partial_x - e^{i\Theta(v(t))}(u_x/\Delta)\partial_t + \ell(v(t)) \equiv \\ = e^{i\Theta}(u_t/\Delta)\partial_x - e^{i\Theta}(u_x/\Delta)\partial_t + f(t), \quad (26')$$

Δ being the quantity $\Delta = -u_x v_t$.

Combining (26) and (26′) we have

$$e^{i\Theta}(u_t/\Delta) = cx-btx, \quad (27)$$

$$e^{i\Theta}(u_x/\Delta) = a+2ct-bt^2, \quad (28)$$

$$f(t) = ibx^2/4-bt/2+c/2-(a+2ct-bt^2)\partial_t R-(cx-btx)\partial_x R. \quad (29)$$

Now eliminating v_t from (27): $v_t = e^{i\Theta}/(a+2ct-bt^2)$ the operator $\hat{Q}$ changes its form

$$\hat{Q} = e^{R}\hat{Q}e^{-R} = i\partial_t + \partial_{xx} - A/x^2 - i\partial_t R - 2\partial_x R\partial_x + (\partial_x R)^2 - \partial_{xx}R,$$

while for the symmetry S_2 from one side we have to have

$$\tilde{S}_2 = q(t,x)\tilde{Q} - \tilde{S} = \\ = q\partial_{xx} + (iq-a-2ct+bt^2)\partial_t + (btx-cx-2q\partial_x R)\partial_x + q(\partial_x R)^2 - \\ -q\partial_{xx}R - iq\partial_t R - qA/x^2 - ibx^2/4 + bt/2 - c/2 + (a+2ct-bt^2)\partial_t R + (cx-btx)\partial_x R.$$

From other side $\hat{S}_2$ is to be of the form

$$\hat{S}_2 = e^{i\alpha(u)}\partial_{uu} + \zeta_2(u),$$

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[4] *Remind that for the second-order parabolic equations the symmetry operator S is of the form*

$$S = e^{i\Theta(v)}\partial_v + \ell(v).$$

ζ_2 being an appropriate function. I.e. one obtains:

$$\hat{S}_2=(e^{i\alpha(u)}/u_x^2)\partial_{xx}+((e^{i\alpha}/u\ )\partial_x(1/u_x)\ +\zeta_2(u).$$

Comparing the both expressions we get the equalities:

$$e^{i\alpha}/u_x^2=q, \tag{30}$$

$$iq-a-2ct+bt^2=0, \tag{31}$$

$$btx-cx-2q\partial_x R\ =(e^{i\alpha}/u_x)\partial_x(1/u_x), \tag{32}$$

$$\zeta_2=q(\partial_x R)^2-q\partial_{xx}R-iq\partial_t R-qA/x^2-ibx^2/4+bt/2-c/2+(a+2ct-$$
$$-bt^2)\partial_t R+(cx-btx)\partial_x R. \tag{33}$$

The simplest choice of R satisfying Eq.(29) is $R=x^2\varphi(t)$

1) Consider first a trivial splitting: $R=0 \Leftrightarrow \varphi=0$. From Eq (29) one has that $f(t) = ibx^2/4-bt/2+c/2$ which leads to $b=0$.

The choice $a=1$, $b=0$, $c=0$ satisfying all relations (27)-(33), leads to $S=\partial_t$, $v=t$, $u=x$.

For the less restrictive choice: $a=0$,$\Theta=0$ one has obviously $u_t/u_x= -x/2t, u=kx\beta(t)$, where $\beta(t)= t^{-1/2}$. From (31) we deduce that $q= -2ict$. Substituting into (30) we obtain $e^{i\alpha}t/k^2= -2ict$. Take $c=1$, then $\alpha=-\pi/2$, $k=1/\sqrt{2}$ and for v,u we have

$$v=\ln(t)/2, u=2^{-1/2}xt^{-1/2},$$

Eqs (32),(33) being identically satisfied.Then: $t=e^{2v}$, $x=2^{1/2}ut^{1/2}$. Going back to the old notations namely $e^{2v}\Rightarrow v$, $2^{1/2}u \Rightarrow u$ the final form of the new variables becomes $t=v$, $x=uv^{1/2}$.

2) $\varphi(t)\neq 0$. In this case for $c=0$ we have

$$ib/4-(a-bt^2)\partial_t\varphi+2bt\varphi=0. \tag{34}$$

Now from (31) and (30) one obtains $q=i(bt^2-a)$, $e^{i\alpha}/u_x^2=i(bt^2-a)$, having solution of the form $u=x\gamma(t)$, $\alpha=sgn(bt^2-a)\pi/2$, $\Theta=0$. Then $\gamma(t)= (sgn(bt^2-a)(bt^2-a))^{-1/2}$. For the unknown function ζ_2 we find:

$$\zeta_2=4q\varphi^2x^2-2q\varphi-iqx^2\varphi-qA/x^2+bt/2.$$

In order to satisfy the requirement $\zeta_2=\zeta_2(u)$ we are choosing $\varphi=-ibt/(4bt^2-4a)$. Then for the functions $\zeta_2=\zeta_2(u)$, the Eq. (34) being satisfied identically.

Now replacing t by v in the coordinate expressions:

$$u=x(sgn(bt^2-a)(bt^2-a))^{-1/2}, v_t=1/(a-bt^2),\ R=-ibtx^2/(4(bt^2-a)),$$

which define the diffeomorphism φ. we re-obtain all coordinate systems listed in the Miller's textbook:

i) $t=v$, $x=u(sgn(bv^2-a)(bt^2-a))^{1/2}$, $R=-isgn(bv^2-a)bu^2v/4$, $(R\neq 0)$, $\hat{S}= aK_{-2}+ bK_2$

ii) $t=v$, $x=u$, $R=0$, $\hat{S}= K_{-2}$

iii) $t=v$, $x=uv^{1/2}$, $R=0$, $\hat{S}= K_0$.

II. As we announced in the Introduction it may happen that the second order symmetry operator $\hat{S}$ (even in the case of 2-nd order LPDE with two variables) should not belong to the enveloping algebra of the first order symmetry operators. One of the simplest example is the two-dimensional oscillator with 'imaginary frequencies':

$$\hat{Q}\Psi=0,\ \hat{Q}=\partial_{uu}+u^2+\partial_{vv}+v^2.$$

We shall seek for the first order symmetry operators $\hat{S}$ satisfying the commutation relations

$$[\hat{S},\hat{Q}]=p\hat{Q}, \qquad (35)$$

the arbitrary p being necessarily a function. Writing $\hat{S}$ in the form $\hat{S}= a\partial_u+b\partial_v+c$ and replacing into (35) it is elementary to find the following set of equations for the unknown functions a,b,c,p:

$$2\partial_u a=p, \qquad (36)$$

$$2\partial_v b=p, \qquad (37)$$

$$\partial_u b+\partial_v a=0, \qquad (38)$$

$$\partial_{uu}a+\partial_{vv}a+2\partial_u c=0, \qquad (39)$$

$$\partial_{uu}b+\partial_{vv}b+2\partial_v c=0, \qquad (40)$$

$$\partial_{uu}c+\partial_{vv}c-2au-2bv=p(u^2+v^2). \qquad (41)$$

From (36) and (37),(38) we have $\partial_u a=\partial_v b$ and $\partial_v a=-\partial_u b$ together with the integrability conditions

$$\partial_{uu}a+\partial_{vv}a=0;\quad \partial_{uu}b+\partial_{vv}b=0, \qquad (42)$$

which imply $\partial_u c=0$, $\partial_v c=0 \Rightarrow c=const$. Without solving the equations for a and b it is obvious that if the first-order symmetry operators $\hat{S}$ exist they should be of the form $a\partial_u+b\partial_v+const$, and the second order symmetry operator corresponding to the splitting of variables

$$\partial_{uu}+u^2 \qquad (43)$$

(or in the same footing $\partial_{vv}+ v^2$) *will not* belong to the enveloping algebra of such $\hat{S}$, since otherwise, instead of (43), it should have the form

$$\hat{S} = \sum A_{ij}\partial_{ij} + \sum B_j \partial_j + const.$$

These two operators equals only on the space of functions $L^p(R^2,\mu)$ where the measure μ ought to be *appropriately choosen* after the structure of coefficients $\{A_{ij}\},\{B_j\}$ has been restored solving the system of Eqns.(36)-(41).

In other terms for the right choice of the symmetries one always needs an accurate definition of the solution space contained in the range of $\hat{Q}$.

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